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ABOUT DIFFERENTIAL SANDWICH THEOREMS INVOLVING A MULTIPLIER TRANSFORMATION AND RUSCHEWEYH DERIVATIVE

ALB LUPAȘ ALINA

Abstract. In this work we study a new operator $IR_{\lambda,l}^{m,n}$ defined as the Hadamard product of the multiplier transformation $I(m,\lambda,l)$ and Ruscheweyh derivative R^n, given by $IR_{\lambda,l}^{m,n} : \mathcal{A} \to \mathcal{A}$, $IR_{\lambda,l}^{m,n} f(z) = (I(m,\lambda,l) * R^n) f(z)$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + ..., z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. The purpose of this paper is to derive certain subordination and superordination results involving the operator $IR_{\lambda,l}^{m,n}$ and we establish differential sandwich-type theorems.

1. Introduction

Let $\mathcal{H}(U)$ be the class of analytic function in the open unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}(a,n)$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots$.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$ and $\mathcal{A} = \mathcal{A}_1$.

Let the functions f and g be analytic in U. We say that the function f is subordinate to g, written $f \prec g$, if there exists a Schwarz function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be an univalent function in U. If p is analytic in U and satisfies the second order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad \text{for } z \in U, \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of U.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h analytic in U. If p and $\psi\left(p(z), zp'(z), z^2p''(z); z\right)$ are univalent and if p satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \qquad z \in U, \tag{2}$$

then p is a solution of the differential superordination (2) (if f is subordinate to F, then F is called to be superordinate to f). An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (2). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (2) is said to be the best subordinant.

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Miller and Mocanu [6] obtained conditions h, q and ψ for which the following implication holds

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

For two functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ analytic in the open unit disc U, the Hadamard product (or convolution) of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by $(z) * g(z) = (f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j$.

Definition 1. [5] *For $f \in \mathcal{A}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the multiplier transformation $I(m, \lambda, l) f(z)$ is defined by the following infinite series*

$$I(m, \lambda, l) f(z) := z + \sum_{j=2}^{\infty} \left(\frac{1 + \lambda(j-1) + l}{1+l} \right)^m a_j z^j.$$

Remark 1. *We have*

$$(l+1) I(m+1, \lambda, l) f(z) = (l + 1 - \lambda) I(m, \lambda, l) f(z) + \lambda z (I(m, \lambda, l) f(z))', \quad z \in U.$$

Remark 2. *For $l = 0$, $\lambda \geq 0$, the operator $D_\lambda^m = I(m, \lambda, 0)$ was introduced and studied by Al-Oboudi , which reduced to the Sălăgean differential operator $S^m = I(m, 1, 0)$ for $\lambda = 1$.*

Definition 2. *(Ruscheweyh [8]) For $f \in \mathcal{A}$ and $n \in \mathbb{N}$, the Ruscheweyh derivative R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$\begin{aligned} R^0 f(z) &= f(z), R^1 f(z) = zf'(z), ... \\ (n+1) R^{n+1} f(z) &= z (R^n f(z))' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 3. *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$ for $z \in U$.*

Definition 3. *([2]) Let $\lambda, l \geq 0$ and $n, m \in \mathbb{N}$. Denote by $IR_{\lambda,l}^{m,n} : \mathcal{A} \to \mathcal{A}$ the operator given by the Hadamard product of the multiplier transformation $I(m, \lambda, l)$ and the Ruscheweyh derivative R^n,*

$$IR_{\lambda,l}^{m,n} f(z) = (I(m, \lambda, l) * R^n) f(z),$$

for any $z \in U$ and each nonnegative integers m, n.

Remark 4. *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then*
$IR_{\lambda,l}^{m,n} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, $z \in U$.

Using simple computation one obtains the next result.

Proposition 1. [1]*For $m, n \in \mathbb{N}$ and $\lambda \geq 0$ we have*

$$IR_{\lambda,l}^{m+1,n} f(z) = \frac{1 + l - \lambda}{l+1} IR_{\lambda,l}^{m,n} f(z) + \frac{\lambda}{l+1} z \left(IR_{\lambda,l}^{m,n} f(z) \right)' \tag{3}$$

The purpose of this paper is to derive the several subordination and superordination results involving a differential operator. Furthermore, we studied the results of Selvaraj and Karthikeyan [10], Shanmugam, Ramachandran, Darus and Sivasubramanian [11] and Srivastava and Lashin [12].

In order to prove our subordination and superordination results, we make use of the following known results.

Definition 4. [7] *Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \backslash E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.*

Lemma 1. [7] *Let the function q be univalent in the unit disc U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that Q is starlike univalent in U and $Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in U$. If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$, then $p(z) \prec q(z)$ and q is the best dominant.*

Lemma 2. [4] *Let the function q be convex univalent in the open unit disc U and ν and ϕ be analytic in a domain D containing $q(U)$. Suppose that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) > 0$ for $z \in U$ and $\psi(z) = zq'(z)\phi(q(z))$ is starlike univalent in U. If $p(z) \in \mathcal{H}[q(0),1] \cap Q$, with $p(U) \subseteq D$ and $\nu(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and $\nu(q(z)) + zq'(z)\phi(q(z)) \prec \nu(p(z)) + zp'(z)\phi(p(z))$, then $q(z) \prec p(z)$ and q is the best subordinant.*

2. Main results

We begin with the following

Theorem 1. *Let $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} \in \mathcal{H}(U)$ and let the function $q(z)$ be analytic and univalent in U such that $q(z) \neq 0$, for all $z \in U$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let*

$$Re\left(\frac{\xi}{\beta}q(z) + \frac{2\mu}{\beta}q^2(z) + 1 + z\frac{q''(z)}{q'(z)} - z\frac{q'(z)}{q(z)}\right) > 0, \tag{4}$$

for $\alpha, \xi, \beta, \mu \in \mathbb{C}$, $\beta \neq 0$, $z \in U$ and

$$\psi_{\lambda,l}^{m,n}(\alpha,\xi,\mu,\beta;z) := \alpha - \xi\left(\frac{1+l}{\lambda} - 1\right) + \mu\left(\frac{1+l}{\lambda} - 1\right)^2 + \tag{5}$$

$$\frac{l+1}{\lambda}\left(\xi + 2\mu - \beta - \frac{2\mu(1+l)}{\lambda}\right)\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)} + \mu\left(\frac{l+1}{\lambda}\right)^2\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^2$$

$$+\beta\frac{l+1}{\lambda}\frac{IR_{\lambda,l}^{m+2,n}f(z) - \left(1 - \frac{\lambda}{l+1}\right)IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z) - \left(1 - \frac{\lambda}{l+1}\right)IR_{\lambda,l}^{m,n}f(z)}.$$

If q satisfies the following subordination

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha + \xi q(z) + \mu(q(z))^2 + \beta\frac{zq'(z)}{q(z)}, \tag{6}$$

for $\alpha, \xi, \beta, \mu \in \mathbb{C}$, $\beta \neq 0$, then

$$\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} \prec q(z), \tag{7}$$

and q is the best dominant.

Proof. Let the function p be defined by $p(z) := \frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. We have $p'(z) = \frac{l+1}{\lambda}\frac{\left(IR_{\lambda,l}^{m+1,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} - \frac{l+1}{\lambda}\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \cdot \frac{\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$.

By using the identity (3), we obtain

$$\frac{zp'(z)}{p(z)} = \frac{l+1}{\lambda}\left[\frac{IR_{\lambda,l}^{m+2,n}f(z) - \left(1 - \frac{\lambda}{l+1}\right)IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z) - \left(1 - \frac{\lambda}{l+1}\right)IR_{\lambda,l}^{m,n}f(z)} - \frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right]. \tag{8}$$

By setting $\theta(w) := \alpha + \xi w + \mu w^2$ and $\phi(w) := \frac{\beta}{w}$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z) = zq'(z)\phi(q(z)) = \beta\frac{zq'(z)}{q(z)}$ and $h(z) = \theta(q(z)) + Q(z) = \alpha + \xi q(z) + \mu(q(z))^2 + \beta\frac{zq'(z)}{q(z)}$, we find that $Q(z)$ is starlike univalent in U.

We have $h'(z) = \xi q'(z) + 2\mu q(z) q'(z) + \beta\frac{q'(z)}{q(z)} + \beta z\frac{q''(z)}{q(z)} - \beta z\left(\frac{q'(z)}{q(z)}\right)^2$ and $\frac{zh'(z)}{Q(z)} = \frac{\xi}{\beta}q(z) + \frac{2\mu}{\beta}q^2(z) + 1 + z\frac{q''(z)}{q(z)} - z\frac{q'(z)}{q(z)}$.

We deduce that $Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(\frac{\xi}{\beta}q(z) + \frac{2\mu}{\beta}q^2(z) + 1 + z\frac{q''(z)}{q(z)} - z\frac{q'(z)}{q(z)}\right) > 0$.

By using (8), we obtain $\alpha + \xi p(z) + \mu(p(z))^2 + \beta\frac{zp'(z)}{p(z)} =$

$\alpha - \xi\left(\frac{1+l}{\lambda} - 1\right) + \mu\left(\frac{1+l}{\lambda} - 1\right)^2 + \frac{l+1}{\lambda}\left(\xi + 2\mu - \beta - \frac{2\mu(1+l)}{\lambda}\right)\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)} +$

$\mu\left(\frac{l+1}{\lambda}\right)^2\left(\frac{IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^2 + \beta\frac{l+1}{\lambda}\frac{IR_{\lambda,l}^{m+2,n}f(z) - \left(1-\frac{\lambda}{l+1}\right)IR_{\lambda,l}^{m+1,n}f(z)}{IR_{\lambda,l}^{m+1,n}f(z) - \left(1-\frac{\lambda}{l+1}\right)IR_{\lambda,l}^{m,n}f(z)}$.

By using (6), we have $\alpha + \xi p(z) + \mu(p(z))^2 + \beta\frac{zp'(z)}{p(z)} \prec \alpha + \xi q(z) + \mu(q(z))^2 + \beta\frac{zq'(z)}{q(z)}$.

By an application of Lemma 1, we have $p(z) \prec q(z)$, $z \in U$, i.e. $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} \prec q(z)$, $z \in U$ and q is the best dominant. □

Corollary 1. *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that* (4) *holds. If $f \in \mathcal{A}$ and*

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha + \xi\frac{1+Az}{1+Bz} + \mu\left(\frac{1+Az}{1+Bz}\right)^2 + \frac{\beta(A-B)z}{(1+Az)(1+Bz)},$$

for $\alpha, \beta, \mu, \xi \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), *then $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} \prec \frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is the best dominant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 1 we get the corollary. □

Corollary 2. *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that* (4) *holds. If $f \in \mathcal{A}$ and*

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \prec \alpha + \xi\left(\frac{1+z}{1-z}\right)^\gamma + \mu\left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\beta\gamma z}{1-z^2},$$

for $\alpha, \beta, \mu, \xi \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta \neq 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5), *then $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} \prec \left(\frac{1+z}{1-z}\right)^\gamma$, and $\left(\frac{1+z}{1-z}\right)^\gamma$ is the best dominant.*

Proof. Corollary follows by using Theorem 1 for $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $0 < \gamma \leq 1$. □

Theorem 2. *Let q be analytic and univalent in U such that $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U. Assume that*

$$Re\left(\frac{\xi}{\beta}q(z)q'(z) + \frac{2\mu}{\beta}q^2(z)q'(z)\right) > 0, \text{ for } \xi, \beta, \mu \in \mathbb{C},\ \beta \neq 0. \tag{9}$$

If $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)} \in \mathcal{H}[q(0),1] \cap Q$ and $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ is univalent in U, where $\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$ is as defined in (5), *then*

$$\alpha + \xi q(z) + \mu(q(z))^2 + \frac{\beta zq'(z)}{q(z)} \prec \psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z) \tag{10}$$

implies

$$q(z) \prec \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}, \quad z \in U, \tag{11}$$

and q is the best subordinant.

Proof. Let the function p be defined by $p(z) := \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$.

By setting $\nu(w) := \alpha + \xi w + \mu w^2$ and $\phi(w) := \frac{\beta}{w}$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{q'(z)q(z)[\xi+2\mu q(z)]}{\beta}$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) =$ $Re\left(\frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z)\right) > 0$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$.

By using (8) and (10) we obtain $\alpha + \xi q(z) + \mu(q(z))^2 + \frac{\beta z q'(z)}{q(z)} \prec \alpha + \xi p(z) + \mu(p(z))^2 + \frac{\beta z p'(z)}{p(z)}$.

Using Lemma 2, we have $q(z) \prec p(z) = \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$, $z \in U$,and q is the best subordinant. □

Corollary 3. *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that* (9) *holds. If $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and*

$$\alpha + \xi \frac{1 + Az}{1 + Bz} + \mu\left(\frac{1 + Az}{1 + Bz}\right)^2 + \frac{\beta(A - B)z}{(1 + Az)(1 + Bz)} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z),$$

for $\alpha, \beta, \xi, \mu \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5)*, then $\frac{1+Az}{1+Bz} \prec \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$, and $\frac{1+Az}{1+Bz}$ is the best subordinant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2 we get the corollary. □

Corollary 4. *Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that* (9) *holds. If $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and*

$$\alpha + \xi\left(\frac{1 + z}{1 - z}\right)^{\gamma} + \mu\left(\frac{1 + z}{1 - z}\right)^{2\gamma} + \frac{2\beta\gamma z}{1 - z^2} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z),$$

for $\alpha, \beta, \mu, \xi \in \mathbb{C}$, $\beta \neq 0$, $0 < \gamma \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (5)*, then $\left(\frac{1+z}{1-z}\right)^{\gamma} \prec \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$, and $\left(\frac{1+z}{1-z}\right)^{\gamma}$ is the best subordinant.*

Proof. For $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$, $0 < \gamma \leq 1$ in Theorem 2 we get the corollary. □

Combining Theorem 1 and Theorem 2, we state the following sandwich theorem.

Theorem 3. *Let q_1 and q_2 be analytic and univalent in U such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, for all $z \in U$, with $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_2'(z)}{q_2(z)}$ being starlike univalent. Suppose that q_1 satisfies* (4) *and q_2 satisfies* (9)*. If $f \in \mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and $\psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z)$ is as defined in* (5) *univalent in U, then*

$$\alpha + \xi q_1(z) + \mu(q_1(z))^2 + \frac{\beta z q_1'(z)}{q_1(z)} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z) \prec \alpha + \xi q_2(z) + \mu(q_2(z))^2 + \frac{\beta z q_2'(z)}{q_2(z)},$$

for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $\beta\neq 0$, *implies*

$$q_1(z)\prec\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec q_2(z),$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1(z)=\frac{1+A_1z}{1+B_1z}$, $q_2(z)=\frac{1+A_2z}{1+B_2z}$, where $-1\le B_2<B_1<A_1<A_2\le 1$, we have the following corollary.

Corollary 5. *Let* $m,n\in\mathbb{N}$, $\lambda,l\ge 0$. *Assume that* (4) *and* (9) *hold. If* $f\in\mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ *and*

$$\alpha+\xi\frac{1+A_1z}{1+B_1z}+\mu\left(\frac{1+A_1z}{1+B_1z}\right)^2+\frac{\beta(A_1-B_1)z}{(1+A_1z)(1+B_1z)}\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$$

$$\prec\alpha+\xi\frac{1+A_2z}{1+B_2z}+\mu\left(\frac{1+A_2z}{1+B_2z}\right)^2+\frac{\beta(A_2-B_2)z}{(1+A_2z)(1+B_2z)},$$

for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $\beta\neq 0$, $-1\le B_2\le B_1<A_1\le A_2\le 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in* (5), *then*

$$\frac{1+A_1z}{1+B_1z}\prec\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec\frac{1+A_2z}{1+B_2z},$$

hence $\frac{1+A_1z}{1+B_1z}$ *and* $\frac{1+A_2z}{1+B_2z}$ *are the best subordinant and the best dominant, respectively.*

For $q_1(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0<\gamma_1<\gamma_2\le 1$, we have the following corollary.

Corollary 6. *Let* $m,n\in\mathbb{N}$, $\lambda,l\ge 0$. *Assume that* (4) *and* (9) *hold. If* $f\in\mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ *and*

$$\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma_1}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma_1}+\frac{2\beta\gamma_1z}{1-z^2}\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta,\mu;z)$$

$$\prec\alpha+\xi\left(\frac{1+z}{1-z}\right)^{\gamma_2}+\mu\left(\frac{1+z}{1-z}\right)^{2\gamma_2}+\frac{2\beta\gamma_2z}{1-z^2},$$

for $\alpha,\beta,\mu,\xi\in\mathbb{C}$, $\beta\neq 0$, $0<\gamma_1<\gamma_2\le 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in* (5), *then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma_1}\prec\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\gamma_2},$$

hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ *and* $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ *are the best subordinant and the best dominant, respectively.*

We have also

Theorem 4. *Let* $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}(U)$, $f\in\mathcal{A}$, $z\in U$, $m,n\in\mathbb{N}$, $\lambda,l\ge 0$ *and let the function* $q(z)$ *be convex and univalent in* U *such that* $q(0)=1$, $z\in U$. *Assume that*

$$Re\left(\frac{\alpha+\beta}{\beta}+z\frac{q''(z)}{q'(z)}\right)>0, \tag{12}$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, *and*

$$\psi_{\lambda,l}^{m,n}(\alpha, \beta; z) := \beta \left(\frac{l+1}{\lambda}\right)^2 \frac{IR_{\lambda,l}^{m+2,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)} - \tag{13}$$

$$\beta \left(\frac{l+1}{\lambda}\right)^2 \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^2 + \alpha \frac{l+1}{\lambda} \frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)} - \alpha \left(\frac{1+l}{\lambda} - 1\right).$$

If q *satisfies the following subordination*

$$\psi_{\lambda,l}^{m,n}(\alpha, \beta; z) \prec \alpha q(z) + \beta z q'(z), \tag{14}$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, *then*

$$\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \prec q(z), \quad z \in U, \tag{15}$$

and q *is the best dominant.*

Proof. Let the function p be defined by $p(z) := \frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$

We have $p'(z) = \frac{l+1}{\lambda} \frac{\left(IR_{\lambda,l}^{m+1,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} - \frac{l+1}{\lambda} \frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \cdot \frac{\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)}$.

By using the identity (3), we obtain

$$zp'(z) = \left(\frac{l+1}{\lambda}\right)^2 \frac{IR_{\lambda,l}^{m+2,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)} - \left(\frac{l+1}{\lambda}\right)^2 \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^2. \tag{16}$$

By setting $\theta(w) := \alpha w$ and $\phi(w) := \beta$, it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z) = zq'(z)\phi(q(z)) = \beta z q'(z)$, we find that $Q(z)$ is starlike univalent in U.

Let $h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \beta z q'(z)$.

We have $Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(\frac{\alpha+\beta}{\beta} + z\frac{q''(z)}{q'(z)}\right) > 0$.

By using (16), we obtain $\alpha p(z) + \beta z p'(z) = \beta\left(\frac{l+1}{\lambda}\right)^2 \frac{IR_{\lambda,l}^{m+2,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)} -$ $\beta\left(\frac{l+1}{\lambda}\right)^2 \left(\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)}\right)^2 + \alpha\frac{l+1}{\lambda}\frac{IR_{\lambda,l}^{m+1,n} f(z)}{IR_{\lambda,l}^{m,n} f(z)} - \alpha\left(\frac{1+l}{\lambda} - 1\right)$.

By using (14), we have $\alpha p(z) + \beta z p'(z) \prec \alpha q(z) + \beta z q'(z)$.

From Lemma 1, we have $p(z) \prec q(z)$, $z \in U$, i.e. $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \prec q(z)$, $z \in U$, and q is the best dominant. □

Corollary 7. *Let* $q(z) = \frac{1+Az}{1+Bz}$, $z \in U$, $-1 \leq B < A \leq 1$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. *Assume that* (12) *holds. If* $f \in \mathcal{A}$ *and*

$$\psi_{\lambda,l}^{m,n}(\alpha, \beta; z) \prec \alpha\frac{1+Az}{1+Bz} + \frac{\beta(A-B)z}{(1+Bz)^2},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in* (13), *then* $\frac{z\left(IR_{\lambda,l}^{m,n} f(z)\right)'}{IR_{\lambda,l}^{m,n} f(z)} \prec \frac{1+Az}{1+Bz}$, *and* $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 4 we get the corollary. □

Corollary 8. *Let* $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. *Assume that* (12) *holds. If* $f\in\mathcal{A}$ *and*

$$\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)\prec\alpha\left(\frac{1+z}{1-z}\right)^{\gamma}+\frac{2\beta\gamma z}{1-z^{2}}\left(\frac{1+z}{1-z}\right)^{\gamma},$$

for $\alpha,\beta\in\mathbb{C}$, $0<\gamma\leq 1$, $\beta\neq 0$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in* (13), *then* $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\gamma}$, *and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best dominant.*

Proof. Corollary follows by using Theorem 4 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq 1$. $\square$

Theorem 5. *Let* q *be convex and univalent in* U *such that* $q(0)=1$. *Assume that*

$$Re\left(\frac{\alpha}{\beta}q'(z)\right)>0,\ \text{for } \alpha,\beta\in\mathbb{C},\ \beta\neq 0. \tag{17}$$

If $f\in\mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ *is univalent in* U, *where* $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ *is as defined in* (13), *then*

$$\alpha q(z)+\beta zq'(z)\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta;z) \tag{18}$$

implies

$$q(z)\prec\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)},\quad \delta\in\mathbb{C},\ \delta\neq 0,\ z\in U, \tag{19}$$

and q *is the best subordinant.*

Proof. Let the function p be defined by $p(z):=\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$, $z\in U$, $z\neq 0$, $f\in\mathcal{A}$. The function p is analytic in U and $p(0)=1$.

By setting $\nu(w):=\alpha w$ and $\phi(w):=\beta$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))}=\frac{\alpha}{\beta}q'(z)$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right)=Re\left(\frac{\alpha}{\beta}q'(z)\right)>0$, for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$.

Now, by using (18) we obtain $\alpha q(z)+\beta zq'(z)\prec\alpha p(z)+\beta zp'(z)$, $z\in U$. From Lemma 2, we have $q(z)\prec p(z)=\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$, $z\in U$, and q is the best subordinant. $\square$

Corollary 9. *Let* $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$, $z\in U$, $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. *Assume that* (17) *holds. If* $f\in\mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}[q(0),1]\cap Q$, *and*

$$\alpha\frac{1+Az}{1+Bz}+\frac{\beta(A-B)z}{(1+Bz)^{2}}\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta;z),$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\leq B<A\leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in* (13), *then* $\frac{1+Az}{1+Bz}\prec\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$, *and* $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

Proof. For $q(z)=\frac{1+Az}{1+Bz}$, $-1\leq B<A\leq 1$, in Theorem 5 we get the corollary. $\square$

Corollary 10. *Let* $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. *Assume that* (17) *holds. If* $f\in\mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}\left[q(0),1\right]\cap Q$ *and*

$$\alpha\left(\frac{1+z}{1-z}\right)^{\gamma}+\frac{2\beta\gamma z}{1-z^{2}}\left(\frac{1+z}{1-z}\right)^{\gamma}\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta;z),$$

for $\alpha,\beta\in\mathbb{C}$, $0<\gamma\leq 1$, $\beta\neq 0$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in* (13), *then* $\left(\frac{1+z}{1-z}\right)^{\gamma}\prec\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}$, *and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best subordinant.*

Proof. Corollary follows by using Theorem 5 for $q(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$, $0<\gamma\leq 1$. □

Combining Theorem 4 and Theorem 5, we state the following sandwich theorem.

Theorem 6. *Let* q_1 *and* q_2 *be convex and univalent in* U *such that* $q_1(z)\neq 0$ *and* $q_2(z)\neq 0$, *for all* $z\in U$. *Suppose that* q_1 *satisfies* (12) *and* q_2 *satisfies* (17). *If* $f\in\mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}\left[q(0),1\right]\cap Q$, *and* $\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$ *is as defined in* (13) *univalent in* U, *then*

$$\alpha q_1(z)+\beta z q_1'(z)\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)\prec\alpha q_2(z)+\beta z q_2'(z),$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, *implies*

$$q_1(z)\prec\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec q_2(z),\quad z\in U,$$

and q_1 *and* q_2 *are respectively the best subordinant and the best dominant.*

For $q_1(z)=\frac{1+A_1z}{1+B_1z}$, $q_2(z)=\frac{1+A_2z}{1+B_2z}$, where $-1\leq B_2<B_1<A_1<A_2\leq 1$, we have the following corollary.

Corollary 11. *Let* $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. *Assume that* (12) *and* (17) *hold for* $q_1(z)=\frac{1+A_1z}{1+B_1z}$ *and* $q_2(z)=\frac{1+A_2z}{1+B_2z}$, *respectively. If* $f\in\mathcal{A}$, $\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\in\mathcal{H}\left[q(0),1\right]\cap Q$ *and*

$$\alpha\frac{1+A_1z}{1+B_1z}+\frac{\beta(A_1-B_1)z}{(1+B_1z)^{2}}\prec\psi_{\lambda,l}^{m,n}(\alpha,\beta;z)$$

$$\prec\alpha\frac{1+A_2z}{1+B_2z}+\frac{\beta(A_2-B_2)z}{(1+B_2z)^{2}},\quad z\in U,$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\leq B_2\leq B_1<A_1\leq A_2\leq 1$, *where* $\psi_{\lambda,l}^{m,n}$ *is defined in* (5), *then*

$$\frac{1+A_1z}{1+B_1z}\prec\frac{z\left(IR_{\lambda,l}^{m,n}f(z)\right)'}{IR_{\lambda,l}^{m,n}f(z)}\prec\frac{1+A_2z}{1+B_2z},\quad z\in U,$$

hence $\frac{1+A_1z}{1+B_1z}$ *and* $\frac{1+A_2z}{1+B_2z}$ *are the best subordinant and the best dominant, respectively.*

For $q_1(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0<\gamma_1<\gamma_2\leq 1$, we have the following corollary.

Corollary 12. *Let $m,n\in\mathbb{N}$, $\lambda,l\geq 0$. Assume that* (12) *and* (17) *hold for* $q_1(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ *and* $q_2(z)=\left(\frac{1+z}{1-z}\right)^{\gamma_2}$*, respectively. If* $f\in\mathcal{A}$, $\frac{z\left(IR^{m,n}_{\lambda,l}f(z)\right)'}{IR^{m,n}_{\lambda,l}f(z)}\in\mathcal{H}\left[q(0),1\right]\cap Q$ *and*

$$\alpha\left(\frac{1+z}{1-z}\right)^{\gamma_1}+\frac{2\beta\gamma_1 z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma_1}\prec\psi^{m,n}_{\lambda,l}(\alpha,\beta;z)$$
$$\prec\alpha\left(\frac{1+z}{1-z}\right)^{\gamma_2}+\frac{2\beta\gamma_2 z}{1-z^2}\left(\frac{1+z}{1-z}\right)^{\gamma_2},\quad z\in U,$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $0<\gamma_1<\gamma_2\leq 1$, *where* $\psi^{m,n}_{\lambda,l}$ *is defined in* (5)*, then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma_1}\prec\frac{z\left(IR^{m,n}_{\lambda,l}f(z)\right)'}{IR^{m,n}_{\lambda,l}f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\gamma_2},\quad z\in U,$$

hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ *and* $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ *are the best subordinant and the best dominant, respectively.*

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Department of Mathematics and Computer Science
University of Oradea
1 Universitatii Street, 410087 Oradea, Romania
E-mail address: alblupas@gmail.com