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HERMITE-HADAMARD TYPE INEQUALITIES FOR THE PRODUCT TWO DIFFERENTIABLE MAPPINGS

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Abstract. In this paper we extend some estimates of the right hand side of a Hermite-Hadamard type inequality for the product two differentiable functions whose derivatives absolute values are convex. Some natural applications to special weighted means of real numbers are given. Finally, an error estimate for the Simpson's formula is also addressed.

1. Introduction

Let $I = [c, d]$ be an interval on the real line $\mathbb{R}$, let $f : I \to \mathbb{R}$ be a convex function and let $a, b \in [c, d], a < b$. We consider the well-known Hadamard's inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality.

The Hermite-Hadamard inequality has several applications in nonlinear analysis and the geometry of Banach spaces, see [6]. In recent years several extensions and generalizations have been considered for classical convexity. We would like to refer the reader to [2, 5, 8] and references therein for more information. A number of papers have been written on this inequality *providing some inequalities analogous to Hadamard's inequality* given in (1) involving two convex functions, see [3, 1, 7].

The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$.

Let X be a vector space, $x, y \in X$, $x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty \;:\; t \in [0, 1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0, 1] \to \mathbb{R},$$
$$g(x, y)(t) := f((1-t)x + ty), t \in [0, 1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$. For any convex function defined on a segment $[x, y] \subseteq X$, we have the Hermite-Hadamard integral inequality

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f((1-t)x + ty)dt \le \frac{f(x)+f(y)}{2}, \tag{2}$$

which can be derived from the classical Hermite-Hadamard inequality (1) for the convex function $g(x, y) : [0; 1] \to \mathbb{R}$.

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For functions $f : [a, b] \to \mathbb{R}$ that are differentiable on (a, b), Dragomir and Agarwal [4] used the formula,

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{3}$$

to prove the following results.

Theorem 1. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f'|$ is convex on $[a, b]$ then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}. \tag{4}$$

Theorem 2. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$ then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{2(p+1)^{1/p}} \times \left[\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{5}$$

Motivated by the above results in the present paper, we extend some estimates of the right hand side of a Hermite-Hadamard type inequality for the product two differentiable functions whose derivatives absolute values are convex. Also we give their natural applications to special weighted means of real numbers. Finally, *an* error estimate for the Simpson's formula is also addressed.

2. THE MAIN RESULTS

In this section, we investigate some estimates of the right hand side of a Hermite-Hadamard type inequality involving the weighted arithmetic mean of $f(a)$ and $f(b)$.

Theorem 3. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b) and $r > 0, s > 0$. If $|f'|$ is convex on $[a, b]$ then the following inequality holds*

$$\begin{aligned}\left|\frac{sf(a)+rf(b)}{s+r} - \frac{1}{b-a}\int_a^b f(x)dx\right| \\ \le (b-a)\Bigg[\left(\frac{r^3+3rs^2+2s^3}{6(r+s)^3}\right)|f'(a)|+ \\ \left(\frac{s^3+3r^2s+2r^3}{6(r+s)^3}\right)|f'(b)|\Bigg].\end{aligned} \tag{6}$$

Proof. Similar to equality (3) we have the following equality for a differentiable function f.

$$\begin{aligned}\int_0^1 \big(r-(r+s)t\big)f'(ta+(1-t)b)dt \\ = \frac{sf(a)+rf(b)}{b-a} - \frac{r+s}{b-a}\int_0^1 f(ta+(1-t)b)dt.\end{aligned} \tag{7}$$

Since

$$\int_0^1 \big|r-(r+s)t\big|tdt = \left(\frac{r^3+3rs^2+2s^3}{6(r+s)^2}\right),$$
$$\int_0^1 \big|r-(r+s)t\big|(1-t)dt = \left(\frac{s^3+3r^2s+2r^3}{6(r+s)^2}\right), \tag{8}$$

therefore we get the desired inequality (6). □

Theorem 4. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$ then the following inequality holds*

$$\left|\frac{sf(a)+rf(b)}{s+r} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)(r^{p+1}+s^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(r+s)^{\frac{p+1}{p}}} \times \left[\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{9}$$

Proof. *Using Holder's inequality*

$$\int_0^1 \Big|\big(r-(r+s)t\big)f'(ta+(1-t)b)\Big|dt \le \left(\int_0^1 \Big|\big(r-(r+s)t\big)\Big|^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \Big|f'(ta+(1-t)b)\Big|^q dt\right)^{\frac{1}{q}}, \tag{10}$$

we obtain inequality (9) from inequalities (8) and (10). □

Theorem 5. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f, g : [a, b] \to \mathbb{R}$ are two differentiable functions on (a, b). Assume $q, t \in \mathbb{R}$ with $q, t > 1$ and $q + t < qt$.*

(a) If $|f'|^q, |g|^r, |f|^r, |g'|^q$ are convex on $[a, b]$ then the following inequality holds

$$\left|\frac{sf(a)g(a)+rf(b)g(b)}{s+r} - \frac{1}{b-a}\int_a^b f(x)g(x)dx\right| \le \frac{(b-a)(r^{p+1}+s^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(r+s)^{\frac{p+1}{p}}} \times \left[\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\left(\frac{|g(a)|^t+|g(b)|^t}{2}\right)^{\frac{1}{t}} + \left(\frac{|f(a)|^t+|f(b)|^t}{2}\right)^{\frac{1}{t}}\left(\frac{|g'(a)|^q+|g'(b)|^q}{2}\right)^{\frac{1}{q}}\right], \tag{11}$$

(b) If $|f'|^q, |g|^t, |f|^q, |g'|^t$ are convex on $[a, b]$ then the following inequality holds

$$\left|\frac{sf(a)g(a)+rf(b)g(b)}{s+r} - \frac{1}{b-a}\int_a^b f(x)g(x)dx\right| \le \frac{(b-a)(r^{p+1}+s^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(r+s)^{\frac{p+1}{p}}} \times \left[\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\left(\frac{|g(a)|^t+|g(b)|^t}{2}\right)^{\frac{1}{t}} + \left(\frac{|f(a)|^q+|f(b)|^q}{2}\right)^{\frac{1}{q}}\left(\frac{|g'(a)|^t+|g'(b)|^t}{2}\right)^{\frac{1}{t}}\right], \tag{12}$$

where $p = \frac{qt}{qt-(q+t)}$.

Proof. By using (7) for two differentiable functions f and g we have

$$\int_0^1 \big(r-(r+s)t\big)(fg)'(ta+(1-t)b)dt$$
$$= \frac{sf(a)g(a)+rf(b)g(b)}{b-a} - \frac{r+s}{b-a}\int_0^1 (fg)(ta+(1-t)b)dt. \quad (13)$$

Inequalities (11) and (12) follow from (13) and (10). □

Theorem 6. *Assume* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $f, g : [a, b] \to \mathbb{R}$ *are differentiable functions on* (a, b). *If* $|f'|, |f|$ *and* $|g'|, |g|$ *are convex on* $[a, b]$ *then the following inequality holds*

$$\left|\frac{sf(a)g(a)+rf(b)g(b)}{s+r} - \frac{1}{b-a}\int_a^b f(x)g(x)dx\right|$$
$$\le (b-a)\Bigg[\left(\frac{r^4+6r^2s^2+8rs^3+3s^4}{12(r+s)^4}\right)M(a)$$
$$+\left(\frac{3r^4+8r^3s+6r^2s^2+s^4)}{12(r+s)^4}\right)M(b)$$
$$+\left(\frac{r^4+2r^3s+2rs^3+s^4}{12(r+s)^4}\right)\Big(N(a,b)+N(b,a)\Big)\Bigg], \quad (14)$$

where $M(a) = |f'(a)g(a)| + |f(a)g'(a)|$ *and* $N(a, b) = |f'(a)g(b)| + |f(b)g'(a)|$.

Proof. Using (13) it follows that

$$\left|\frac{sf(a)g(a)+rf(b)g(b)}{s+r} - \frac{1}{b-a}\int_a^b f(x)g(x)dx\right|$$
$$= \frac{b-a}{r+s}\left|\int_0^1 (r-(r+s)t)\big[f'(ta+(1-t)b)g(ta+(1-t)b)\right.$$
$$\left. + f(ta+(1-t)b)g'(ta+(1-t)b)\big]dt\right|$$
$$\le \frac{b-a}{r+s}\int_0^1 |(r-(r+s)t)|\big[|f'(ta+(1-t)b)||g(ta+(1-t)b)|$$
$$+ |f(ta+(1-t)b)||g'(ta+(1-t)b)|\big]dt. \quad (15)$$

Since $|f'|, |g|, |f|, |g'|$ are convex on $[a, b]$, it follows that

$$\begin{aligned}
|f'(ta+(1-t)b)| &\le t|f'(a)| + (1-t)|f'(b)| \\
|g(ta+(1-t)b)| &\le t|g(a)| + (1-t)|g(b)| \\
|f(ta+(1-t)b)| &\le t|f(a)| + (1-t)|f(b)| \\
|g'(ta+(1-t)b)| &\le t|g'(a)| + (1-t)|g'(b)|.
\end{aligned} \quad (16)$$

By using (16), inequality (15) becomes

$$\begin{aligned}&\left|\frac{sf(a)g(a)+rf(b)g(b)}{s+r}-\frac{1}{b-a}\int_a^b f(x)g(x)dx\right|\\ &\quad\le\frac{b-a}{r+s}\Bigg[\big(|f'(a)g(a)|+|f(a)g'(a)|\big)\int_0^1|r-(r+s)t|t^2dt\\ &\quad+\big(|f'(b)g(b)|+|f(b)g'(b)|\big)\int_0^1|r-(r+s)t|(1-t)^2dt\\ &\quad+\Big(|f'(a)g(b)|+|f'(b)g(a)|+|f(a)g'(b)|+|f(b)g'(a)|\Big)\\ &\qquad\times\int_0^1|r-(r+s)t|t(1-t)dt\Bigg].\qquad(17)\end{aligned}$$

Now we note that

$$\begin{aligned}\int_0^1|r-(r+s)t|t^2dt&=\left(\frac{r^4+6r^2s^2+8rs^3+3s^4}{12(r+s)^3}\right)\\ \int_0^1|r-(r+s)t|t(1-t)dt&=\left(\frac{r^4+2r^3s+2rs^3+s^4}{12(r+s)^3}\right)\qquad(18)\\ \int_0^1|r-(r+s)t|(1-t)^2dt&=\left(\frac{3r^4+8r^3s+6r^2s^2+s^4}{12(r+s)^3}\right).\end{aligned}$$

We conclude inequality (14) from inequalities (17) and (18). □

In the following we give some examples of functions f, g such that $|f'|, |g'|$ are convex functions but $|(fg)'|$ is not convex. Therefore function $h = fg$ doesn't satisfy the assumptions of Theorem 3 and in this case we can't use inequality (6) for h. However, inequality (14) will be available that is an estimate of the right hand side of a Hermite-Hadamard type inequality for function h.

Example 1. (a) We consider real value functions $f(x) = x^{-2}, g(x) = x^{\frac{7}{2}}$ on $(0,\infty)$. The functions $|f'(x)| = 2x^{-3}$ and $|g'(x)| = \frac{7}{2}x^{\frac{5}{2}}$ are convex on $(0,\infty)$ but $|(fg)'(x)| = \frac{3}{2}x^{\frac{1}{2}}$ is not convex.

(b) Real value functions $f(x) = x, g(x) = \frac{1}{3}x^2 - 4$ on $\mathbb{R}$, in this case $|(fg)'(x)| = |x^2 - 4|$.

(c) Real value functions $f(x) = x, g(x) = e^{-x}$ on segment $(1, 4)$, in this case $|(fg)'(x)| = |1 - x|e^{-x}$.

(d) If we put $g = 1$ in inequality (14) then this inequality becomes (6), and if we put $r = s = 1$ in inequality (6), then we obtain inequality (4).

3. Application to special means

We consider some weighted means for arbitrary real numbers a, b $(a \neq b)$ with the weight $w_0 = (s, r),\ s, r \geq 0$. We take

1. *Weighted arithmetic mean:*

$$A_{w_0}(a,b)=\frac{sa+rb}{s+r},\qquad s+r\neq 0,\ a,b\in\mathbb{R},$$

2. *Weighted geometric mean:*

$$G_{w_0}(a,b)=(a^sb^r)^{\frac{1}{s+r}},\qquad s+r\neq 0,\ a,b\geq 0,$$

3. *Weighted harmonic mean:*

$$H_{w_0}(a,b) = \frac{s+r}{\frac{s}{a}+\frac{r}{b}}, \qquad a,b \neq 0,$$

4. *Weighted power mean:*

$$M_{p,w_0}(a,b) = \left(\frac{sa^p + rb^p}{s+r}\right)^{\frac{1}{p}}, \qquad r+s \neq 0,\ a,b \geq 0,\ p \neq 0,$$

5. *Logarithmic mean:*

$$L(a,b) = \frac{b-a}{ln|b| - ln|a|}, \qquad |a| \neq |b|,\ a,b \neq 0,$$

6. *Generalized logarithmic mean:*

$$L_p(a,b) = \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, \qquad p \in \mathbb{R} \setminus \{0,-1\},\ a \neq b.$$

If $r = s$ in the weight function $w_0 = (s,r)$, we take A, G, H, M_p for arithmetic mean, geometric mean, harmonic mean, power mean, respectively.

Proposition 1. *Let $a,b \in \mathbb{R},\ 0 \leq a < b$ and $p \in \mathbb{R},\ p \geq 2$ or $p \leq 1$. Then, the following inequality holds:*

$$|M^p_{p,w_0}(a,b) - L^p_p(a,b)| \leq |p|(b-a)(w_1+w_2)M^{p-1}_{p-1,w}(|a|,|b|),$$

where $w = (w_1,w_2) = \left(\frac{r^3+3rs^2+2s^3}{6(r+s)^3},\ \frac{s^3+3r^2s+2r^3}{6(r+s)^3}\right)$.

Proof. The proof is immediate from Theorem 3 applied for $f(x) = x^p, x \in [a,b]$. □

Proposition 2. *Let $a,b \in \mathbb{R},\ a < b$ and $0 \notin [a,b]$. Then, the following inequality holds:*

$$|H^{-1}_{w_0}(a,b) - L^{-1}(a,b)| \leq (b-a)(w_1+w_2)H^{-1}_w(|a|^2,|b|^2),$$

where $w = (w_1,w_2) = \left(\frac{r^3+3rs^2+2s^3}{6(r+s)^3},\ \frac{s^3+3r^2s+2r^3}{6(r+s)^3}\right)$.

Proof. The proof is immediate from Theorem 3 applied for $f(x) = \frac{1}{x}, x \in [a,b]$. □

Proposition 3. *Let $a,b \in \mathbb{R},\ 0 \leq a < b$ and $p,k \in \mathbb{R},\ p \geq 1,\ k \geq 2,$ or $k \leq 1$. Then, the following inequality holds:*

$$|M^k_{k,w_0}(a,b) - L^k_k(a,b)| \leq \frac{(b-a)(r^{p+1}+s^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(r+s)^{\frac{p+1}{p}}}|k|M^{\frac{p-1}{p}}_{\frac{p-1}{p}}(|a|^{k-1},|b|^{k-1}).$$

Proof. The proof is immediate from Theorem 4 applied for $f(x) = x^k, x \in [a,b]$. □

Proposition 4. *Let $a,b \in \mathbb{R},\ a < b$ and $0 \notin [a,b]$. Then, the following inequality holds:*

$$|H^{-1}_{w_0}(a,b) - L^{-1}(a,b)| \leq \frac{(b-a)(r^{p+1}+s^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(r+s)^{\frac{p+1}{p}}}M^{\frac{p-1}{p}}_{\frac{p-1}{p}}(|a|^{-2},|b|^{-2}).$$

Proof. The proof is immediate from Theorem 4 applied for $f(x) = \frac{1}{x}, x \in [a,b]$. □

Proposition 5. *Let $a,b \in \mathbb{R},\ 0 \leq a < b$ and $q,t \in \mathbb{R},\ q \geq 1,\ t \geq 1,$ with $q+t < qt$. Then, the following inequality holds:*

$$|M^{t+q}_{t+q,w_0}(a,b) - L^{t+q+1}_{t+q+1}(a,b)| \leq \frac{(b-a)(r^{p+1}+s^{p+1})^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}(r+s)^{\frac{p+1}{p}}}$$
$$\times \left[tM_q(|a|^{t-1},|b|^{t-1})M_t(|a|^q,|b|^q) + qM_q(|a|^t,|b|^t)M_t(|a|^{q-1},|b|^{q-1})\right].$$

Proof. The proof is immediate from Theorem 5 applied for $f(x) = x^t,\ g(x) = x^q,\ x \in [a, b]$ and $p = \frac{qt}{qt-(q+t)}$. □

4. An application to Simpson's formula

Theorem 7. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b) and $r > 0, s > 0$. If $|f'|$ is convex on $[a, b]$ then the following inequality holds*

$$\left|\frac{f(a) + 4f(\frac{a+b}{2}) + f(b)}{6} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{648}\left(16|f'(a)| + 58\left|f'\left(\frac{a+b}{2}\right)\right| + 16|f'(b)|\right). \quad (19)$$

Proof.

$$\left|\frac{f(a) + 4f(\frac{a+b}{2}) + f(b)}{6} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2}\left|\frac{f(a) + 2f(\frac{a+b}{2})}{3} - \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(x)dx\right| + \frac{1}{2}\left|\frac{2f(\frac{a+b}{2}) + f(b)}{3} - \frac{2}{b-a}\int_{\frac{a+b}{2}}^b f(x)dx\right|. \quad (20)$$

Using the inequality (6) for $s = 1,\ r = 2$ and for $s = 2,\ r = 1$, we get

$$\left|\frac{f(a) + 4f(\frac{a+b}{2}) + f(b)}{6} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{16}{162}|f'(a)| + \frac{29}{162}\left|f'\left(\frac{a+b}{2}\right)\right|\right) + \frac{b-a}{4}\left(\frac{29}{162}\left|f'\left(\frac{a+b}{2}\right)\right| + \frac{16}{162}|f'(b)|\right). \quad (21)$$

Hence (19) follows from (21). □

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