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ON $(h-m)$-CONVEXITY AND HADAMARD-TYPE INEQUALITIES

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ABSTRACT. In this paper, a new class of convex functions as a generalization of convexity which is called $(h-m)$-convex functions and some properties of this class is given. We also prove some Hadamard's type inequalities.

1. Introduction

The concept of $m-$convexity has been introduced by Toader in [12], an intermediate between the ordinary convexity and starshaped property, as following:

Definition 1. *The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be $m-$convex, where $m\in[0,1]$, if we have*

$$f(tx+m(1-t)y)\le tf(x)+m(1-t)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$. We say that f is $m-$concave if $-f$ is $m-$convex.

Several papers have been written on $m-$convex functions and we refer the papers [11], [12], [13], [14], [15] and [17]. In [13], Dragomir and Toader proved following inequality for $m-$convex functions.

Theorem 1. *Let $f:[0,\infty)\to\mathbb{R}$ be a $m-$convex function with $m\in(0,1]$. If $0\le a<b<\infty$ and $f\in L_1[a,b]$, then one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx\le\min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2},\frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{1}$$

In [17], Dragomir established following inequalities of Hadamard-type similar to above.

Theorem 2. *Let $f:[0,\infty)\to\mathbb{R}$ be a $m-$convex function with $m\in(0,1]$. If $0\le a<b<\infty$ and $f\in L_1[a,b]$, then one has the inequality:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\int_a^b\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx \\ &\le \frac{m+1}{4}\left[\frac{f(a)+f(b)}{2}+m\frac{f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)}{2}\right]. \end{aligned} \tag{2}$$

Theorem 3. *Let $f:[0,\infty)\to\mathbb{R}$ be a $m-$convex function with $m\in(0,1]$. If $f\in L_1[am,b]$ where $0\le a<b<\infty$, then one has the inequality:*

$$\frac{1}{m+1}\left[\int_a^{mb}f(x)dx+\frac{mb-a}{b-ma}\int_{ma}^b f(x)dx\right]\le(mb-a)\frac{f(a)+f(b)}{2}. \tag{3}$$

In [16], Breckner introduced a new class of convex functions, a generalization of the ordinary convexity, is called $s-$convexity, as following;

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Definition 2. *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be $s-$convex (in the second sense), or that f belongs to the class K_s^2, if*

$$f(\alpha x + (1-\alpha) y) \leq \alpha^s f(x) + (1-\alpha)^s f(y)$$

for all $x, y \in [0,\infty)$ and $\alpha \in [0,1]$.

Some properties of $s-$convexity have been given in [4] and Kırmacı *et al.* proved some inequalities for $s-$convex functions in [10]. In [9], Dragomir and Fitzpatrick established the following Hadamard's type inequalities;

Theorem 4. *Suppose that $f : [0,\infty) \to [0,\infty)$ is an $s-$convex function in the second sense, where $s \in (0,1]$ and let $a, b \in [0,\infty)$, $a < b$. If $f \in L_1[0,1]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{s+1}. \tag{4}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (4). The above inequalities are sharp.

In [3], Godunova and Levin introduced the following class of functions.

Definition 3. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class of $Q(I)$ if it is nonnegative and, for all $x, y \in I$ and $\lambda \in (0,1)$ satisfies the inequality;*

$$f(\lambda x + (1-\lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}$$

In [2], Dragomir *et al.* defined following new class of functions.

Definition 4. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is P function or that f belongs to the class of $P(I)$, if it is nonnegative and for all $x, y \in I$ and $\lambda \in [0,1]$, satisfies the following inequality;*

$$f(\lambda x + (1-\lambda)y) \leq f(x) + f(y)$$

In [2], Dragomir *et al.* proved two inequalities of Hadamard's type for class of Godunova-Levin functions and $P-$ functions.

Theorem 5. *Let $f \in Q(I), a, b \in I$, with $a < b$ and $f \in L_1[a,b]$. Then the following inequality holds.*

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a}\int_a^b f(x)dx \tag{5}$$

Theorem 6. *Let $f \in P(I), a, b \in I$, with $a < b$ and $f \in L_1[a,b]$. Then the following inequality holds.*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)dx \leq 2[f(a)+f(b)] \tag{6}$$

On all of these, in [5], Varošanec defined $h-$convex functions and gave some properties of this class of functions.

Definition 5. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function . We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is $h-$convex function, or that f belongs to the class $SX(h, I)$, if f is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y). \tag{7}$$

If inequality (7) is reversed, then f is said to be $h-$concave, i.e., $f \in SV(h,I)$. Obviously, if $h(t)=t$, then all nonnegative convex functions belong to $SX(h,I)$ and all nonnegative concave functions belong to $SV(h,I)$; if $h(t)=\frac{1}{t}$, then $SX(h,I)=Q(I)$; if $h(t)=1$, $SX(h,I)\supseteq P(I)$; and if $h(t)=t^s$, where $s\in(0,1)$, then $SX(h,I)\supseteq K_s^2$.

Theorem 7 (See [1], Theorem 6)**.** *Let $f\in SX(h,I)$, $a,b\in I$, with $a<b$ and $f\in L_1([a,b])$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\le\frac{1}{(b-a)}\int_a^b f(x)\,dx\le[f(a)+f(b)]\int_0^1 h(\alpha)\,d\alpha \tag{8}$$

For some recent results for $h-$convex functions we refer the interest of reader to the papers [1], [6], [7] and [8].

The main aim of this paper is to give a new class of convex functions and to give some properties of this functions, by using a similar way to proof of properties of $h-$convexity (see [5]). Therefore, some inequalities of Hadamard-type related to this new class of convex functions are given.

2. MAIN RESULTS

We will introduce a new class of convex functions in the following definition.

Definition 6. *Let $h:J\subseteq\mathbb{R}\to\mathbb{R}$ be a non-negative function. We say that $f:[0,b]\to\mathbb{R}$ is a $(h-m)-$convex function, if f is non-negative and for all $x,y\in[0,b]$, $m\in[0,1]$ and $\alpha\in(0,1)$, we have*

$$f(\alpha x+m(1-\alpha)y)\le h(\alpha)f(x)+mh(1-\alpha)f(y). \tag{9}$$

If the inequality (9) is reversed, then f is said to be $(h-m)-$concave function on $[0,b]$.

Obviously, if we choose $m=1$, then we have $h-$convex functions. If we choose $h(\alpha)=\alpha$, then we obtain non-negative $m-$convex functions. If we choose $m=1$ and $h(\alpha)=\left\{\alpha,\ 1,\ \frac{1}{\alpha},\ \alpha^s\right\}$, then we obtain the following classes of functions, non-negative convex functions, $P-$functions, Godunova-Levin functions and $s-$convex functions (in the second sense), respectively.

Remark 1. *Let h be a non-negative function such that*

$$h(\alpha)\ge\alpha$$

for all $\alpha\in(0,1)$. If f is a non-negative $m-$convex function on $[0,b]$, then for all $x,y\in[0,b]$, $m\in[0,1]$ and $\alpha\in(0,1)$, we have

$$f(\alpha x+m(1-\alpha)y)\le\alpha f(x)+m(1-\alpha)f(y)\le h(\alpha)f(x)+mh(1-\alpha)f(y).$$

This shows that f is a $(h-m)-$convex function. By a similar way, one can see that, if

$$h(\alpha)\le\alpha$$

for all $\alpha\in(0,1)$. Then, all non-negative $m-$concave functions are $(h-m)-$concave function on $[0,b]$.

Proposition 1. *Let h_1,h_2 be non-negative functions defined on $J\subseteq\mathbb{R}$ such that*

$$h_2(t)\le h_1(t)$$

for $t\in(0,1)$. If f is $(h_2-m)-$convex, then f is $(h_1-m)-$convex.

Proof. If f is $(h_2-m)-$convex, then for all $x,y\in[0,b]$ and $\alpha\in(0,1)$, we can write

$$\begin{aligned} f(\alpha x+m(1-\alpha)y) &\leq h_2(\alpha)f(x)+mh_2(1-\alpha)f(y) \\ &\leq h_1(\alpha)f(x)+mh_1(1-\alpha)f(y). \end{aligned}$$

Which completes the proof of $(h_1-m)-$convexity of f. □

Proposition 2. *If f,g are $(h-m)-$convex functions and $\lambda>0$, then $f+g$ and λf are $(h-m)-$convex functions.*

Proof. By using definition of $(h-m)-$convex functions, we can write

$$f(\alpha x+m(1-\alpha)y)\leq h(\alpha)f(x)+mh(1-\alpha)f(y) \tag{10}$$

and

$$g(\alpha x+m(1-\alpha)y)\leq h(\alpha)g(x)+mh(1-\alpha)g(y) \tag{11}$$

for all $x,y\in[0,b]$, $m\in[0,1]$ and $\alpha\in(0,1)$. If we add (10) and (11), we get

$$(f+g)(\alpha x+m(1-\alpha)y)\leq h(\alpha)(f+g)(x)+mh(1-\alpha)(f+g)(y).$$

This shows that $f+g$ is $(h-m)-$convex function. Therefore, to prove $(h-m)-$convexity of λf, from the definition, we have

$$\lambda f(\alpha x+m(1-\alpha)y)\leq h(\alpha)\lambda f(x)+mh(1-\alpha)\lambda f(y).$$

This completes the proof. □

The following inequality of Hadamard-type for $(h-m)-$convex functions holds.

Theorem 8. *Let $f:[0,\infty)\to\mathbb{R}$ be a $(h-m)-$convex function with $m\in(0,1]$, $t\in[0,1]$. If $0\leq a<b<\infty$ and $f\in L_1[a,b]$, then the following inequality holds;*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \min\left\{f(a)\int_0^1 h(t)\,dt+mf\left(\frac{b}{m}\right)\int_0^1 h(1-t)\,dt,\right. \tag{12}$$

$$\left. f(b)\int_0^1 h(t)\,dt+mf\left(\frac{a}{m}\right)\int_0^1 h(1-t)\,dt\right\}. \tag{13}$$

Proof. From the definition of $(h-m)-$convex functions, we can write

$$f(tx+m(1-t)y)\leq h(t)f(x)+mh(1-t)f(y)$$

for all $x,y\geq 0$. It follows that; for all $t\in[0,1]$,

$$f(ta+(1-t)b)\leq h(t)f(a)+mh(1-t)f\left(\frac{b}{m}\right)$$

and

$$f(tb+(1-t)a)\leq h(t)f(b)+mh(1-t)f\left(\frac{a}{m}\right).$$

Integrating these inequalities on $[0,1]$, with respect to t, we obtain

$$\int_0^1 f(ta+(1-t)b)\,dt\leq f(a)\int_0^1 h(t)\,dt+mf\left(\frac{b}{m}\right)\int_0^1 h(1-t)\,dt$$

and

$$\int_0^1 f(tb+(1-t)a)\,dt\leq f(b)\int_0^1 h(t)\,dt+mf\left(\frac{a}{m}\right)\int_0^1 h(1-t)\,dt.$$

It is easy to see that;

$$\int_0^1 f(ta+(1-t)b)\,dt=\int_0^1 f(tb+(1-t)a)\,dt=\frac{1}{b-a}\int_a^b f(x)dx.$$

Using this equality, we obtain the required result. □

Corollary 1. *If we choose $h(t)=1$ in (12), we obtain the following inequality;*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{f(a)+mf\left(\frac{b}{m}\right),\ f(b)+mf\left(\frac{a}{m}\right)\right\}.$$

Corollary 2. *If we choose $m=1$ in (12), we obtain the following inequality;*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx \quad &\le \quad \min\left\{f(a)\int_0^1 h(t)\,dt+f(b)\int_0^1 h(1-t)\,dt,\right.\\ &\left. f(b)\int_0^1 h(t)\,dt+f(a)\int_0^1 h(1-t)\,dt\right\}.\end{aligned}$$

Remark 2. *If we choose $h(t)=t$ in (12), we obtain the inequality (1).*

Remark 3. *If we choose $m=1$ and $h(t)=t$ in (12), we obtain the right hand side of the Hadamard's inequality. If we choose $m=1$ and $h(t)=1$ in (12), we obtain the right hand side of the inequality (6). If we choose $m=1$ and $h(t)=t^s$ in (12), we obtain the right hand side of the inequality (4).*

Another result of Hadamard-type for $(h-m)-$convex functions is emboided in the following theorem.

Theorem 9. *Let $f:[0,\infty)\to\mathbb{R}$ be a $(h-m)-$convex function with $m\in(0,1]$, $t\in[0,1]$. If $0\le a<b<\infty$ and $f\in L_1[a,b]$, then the following inequality holds;*

$$\begin{aligned}f\left(\frac{a+b}{2}\right) \quad &\le \quad \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b\left[f(x)+mf\left(\frac{x}{m}\right)\right]dx \qquad (14)\\ &\le \quad h\left(\frac{1}{2}\right)\left[\frac{f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)+m^2f\left(\frac{b}{m^2}\right)}{2}\right]\int_0^1 h(t)dt\end{aligned}$$

Proof. For $x,y\in[0,\infty)$ and $\alpha=\frac{1}{2}$, we can write definition of $(h-m)-$convex function as following;

$$f\left(\frac{x+y}{2}\right)\le h\left(\frac{1}{2}\right)f(x)+mh\left(\frac{1}{2}\right)f\left(\frac{y}{m}\right)$$

If we choose $x=ta+(1-t)b$ and $y=tb+(1-t)a$, we get

$$f\left(\frac{a+b}{2}\right)\le h\left(\frac{1}{2}\right)f(ta+(1-t)b)+mh\left(\frac{1}{2}\right)f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)$$

for all $t\in[0,1]$. By integrating the result on $[0,1]$ with respect to t, we have

$$f\left(\frac{a+b}{2}\right)\le h\left(\frac{1}{2}\right)\int_0^1 f(ta+(1-t)b)\,dt+mh\left(\frac{1}{2}\right)\int_0^1 f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)dt. \qquad (15)$$

By the facts that

$$\int_0^1 f(ta+(1-t)b)\,dt=\frac{1}{b-a}\int_a^b f(x)dx$$

and

$$\int_0^1 f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)dt=\frac{m}{b-a}\int_{\frac{a}{m}}^{\frac{b}{m}} f(x)dx=\frac{1}{b-a}\int_a^b f\left(\frac{x}{m}\right)dx.$$

Using these equalities in (15), we obtain the first inequality of (14). By the $(h-m)$–convexity of f, we can write

$$h\left(\frac{1}{2}\right)\left[f\left(ta+(1-t)b\right)+mf\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)\right] \tag{16}$$
$$\leq h\left(\frac{1}{2}\right)\left[h(t)f(a)+mh(1-t)f\left(\frac{b}{m}\right)+mh(1-t)f\left(\frac{a}{m}\right)+m^2h(t)f\left(\frac{b}{m^2}\right)\right].$$

Integrating the inequality (16) on $[0,1]$ with respect to t, we have

$$\frac{h\left(\frac{1}{2}\right)}{b-a}\left[\int_a^b f(x)dx+m\int_a^b f\left(\frac{x}{m}\right)dx\right]$$
$$\leq h\left(\frac{1}{2}\right)\left[\frac{f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)+m^2f\left(\frac{b}{m^2}\right)}{2}\right]\int_0^1 h(t)dt$$

which completes the proof. □

Corollary 3. *If we choose $h(t)=1$ in (14), we obtain the following inequality;*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \left[f(x)+mf\left(\frac{x}{m}\right)\right]dx$$
$$\leq \left[\frac{f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)+m^2f\left(\frac{b}{m^2}\right)}{2}\right].$$

Corollary 4. *If we choose $m=1$ and $h(t)=t^s$ in (14), we obtain the following inequality;*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int\lim_a^b f(x)dx\leq\frac{f(a)+f(b)}{2(s+1)}$$

which is similar to (4).

Remark 4. *If we choose $m=1$ in* (12)*, we obtain the right hand side of the inequality* (8).

Remark 5. *If we choose $m=1$ and $h(t)=t$ in* (14)*, we obtain the Hadamard's inequality.*

Remark 6. *If we choose $h(t)=t$ in* (14)*, we obtain the inequality* (2).

The following inequality also holds for $(h-m)$ −convex functions.

Theorem 10. *Let $f:[0,\infty)\to\mathbb{R}$ be a $(h-m)$ −convex function with $m\in(0,1]$, $t\in[0,1]$. If $0\leq a<b<\infty$ and $f\in L_1[ma,b]$, then the following inequality holds;*

$$\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb}f(x)dx+\frac{1}{b-ma}\int_{ma}^b f(x)dx\right] \tag{17}$$
$$\leq \frac{f(a)+f(b)}{2}\left[\int_0^1 h(t)\,dt+\int_0^1 h(1-t)\,dt\right].$$

Proof. From definition of $(h-m)$ −convex functions, we can write

$$f(ta+m(1-t)b)\leq h(t)f(a)+mh(1-t)f(b)$$
$$f((1-t)a+mtb)\leq h(1-t)f(a)+mh(t)f(b)$$
$$f(tb+m(1-t)a)\leq h(t)f(b)+mh(1-t)f(a)$$

and

$$f((1-t)b+mta)\leq h(1-t)f(b)+mh(t)f(a)$$

for all $t \in [0,1]$. By summing these inequalities and integrating on $[0,1]$ with respect to t, we obtain

$$\begin{aligned} & \int_0^1 f(ta+m(1-t)b)\,dt + \int_0^1 f((1-t)a+mtb)\,dt \\ & + \int_0^1 f(tb+m(1-t)a)\,dt + \int_0^1 f((1-t)b+mta)\,dt \\ \leq \; & (f(a)+f(b))(m+1)\left[\int_0^1 h(t)\,dt + \int_0^1 h(1-t)\,dt\right]. \end{aligned} \tag{18}$$

It is easy to show that

$$\int_0^1 f(ta+m(1-t)b)\,dt = \int_0^1 f((1-t)a+mtb)\,dt = \frac{1}{mb-a}\int_a^{mb} f(x)dx$$

and

$$\int_0^1 f(tb+m(1-t)a)\,dt = \int_0^1 f((1-t)b+mta)\,dt = \frac{1}{b-ma}\int_{ma}^{b} f(x)dx.$$

By using these equalities in (18), we get the desired result. □

Corollary 5. *If we choose $h(t)=1$ in* (17)*, we obtain the following inequality;*

$$\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb} f(x)dx + \frac{1}{b-ma}\int_{ma}^{b} f(x)dx\right] \leq f(a)+f(b).$$

Remark 7. *If we choose $m=1$ and $h(t)=t$ in* (17)*, we obtain the right hand side of the Hadamard's inequality. If we choose $m=1$ and $h(t)=1$ in* (17)*, we obtain the right hand side of the inequality* (6)*. If we choose $m=1$ and $h(t)=t^s$ in* (17)*, we obtain the right hand side of the inequality* (4)*.*

Remark 8. *If we choose $h(t)=t$ in* (17)*, we obtain the inequality* (3)*.*

References

[1] Sarıkaya, M.Z., Sağlam, A. and Yıldırım, H., *On some Hadamard-type inequalities for h−convex functions*, Journal of Mathematical Inequalities **2** (3) (2008), 335–341.

[2] Dragomir, S.S., Pečarić, J., Persson, L.E., *Some inequalities of Hadamard type*, Soochow J.Math. **21** (1995), 335–341.

[3] Godunova, E.K., Levin, V.I., *Neravenstva dlja funkcii širokogo klassa, soderžašcego vypuklye, monotonnye i nekotorye drugie vidy funkcii*, in: Vycislitel. Mat. i. Mat. Fiz. Mežvuzov. Sb. Nauc. Trudov, MGPI, Moskva, 1985, pp. 138–142.

[4] Hudzik, H. and Maligranda, L., *Some remarks on s−convex functions*, Aequationes Math. **48** (1994) 100–111.

[5] Varošanec, S., *On h-convexity*, J. Math. Anal. Appl. **326** (2007), 303–311.

[6] Bombardelli, M. and Varošanec, S., *Properties of h−convex functions related to the Hermite-Hadamard-Fejér inequalities*, Computers and Mathematics with Applications **58** (2009), 1869–1877.

[7] Sarıkaya, M.Z., Set, E. and Özdemir, M.E., *On some new inequalities of Hadamard type involving h−convex functions*, Acta Math. Univ. Comenianae, Vol. LXXIX, **2** (2010), pp. 265–272.

[8] Burai, P. and Hazy, A., *On approximately h−convex functions*, Journal of Convex Analysis **18** (2), (2011).

[9] Dragomir, S.S. and Fitzpatrick, S., *The Hadamard's inequality for s−convex functions in the second sense*, Demonstratio Math. **32** (4) (1999), 687–696.

[10] Kırmacı, U.S., Bakula, M.K., Özdemir, M.E. and Pečarić, J., *Hadamard-type inequalities for s−convex functions*, Applied Mathematics and Computation **193** (2007), 26–35.

[11] Özdemir, M.E., Avcı, M. and Set, E., *On some inequalities of Hermite-Hadamard type via m−convexity*, Applied Mathematics Letters **23** (2010), 1065–1070.

[12] Toader, G., *Some generalization of the convexity*, Proc. Colloq. Approx. Opt., Cluj-Napoca, (1985), 329–338.

[13] Dragomir, S.S. and Toader, G., *Some inequalities for $m-$convex functions*, Studia Univ. Babeş-Bolyai Math. **38** (1) (1993), 21–28.

[14] Toader, G., *On a generalization of the convexity*, Mathematica **30** (53) (1988), 83–87.

[15] Bakula, M.K., Pečarić, J. and Ribicic, M., *Companion inequalities to Jensen's inequality for $m-$convex and $(\alpha, m)-$convex functions*, J. Ineq. Pure and Appl. Math. **7** (5) (2006), Art. 194.

[16] Breckner, W.W., *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen*, Pupl. Inst. Math. **23** (1978), 13–20.

[17] Dragomir, S.S., *On some new inequalities of Hermite-Hadamard type for $m-$convex functions*, Tamkang Journal of Mathematics **33** (1) (2002).

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