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A NOTE ON NEW REFINEMENTS AND REVERSES OF YOUNG'S INEQUALITY

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Abstract. In this note we obtain two new refinements and reverses of Young's inequality.

1. Introduction

The famous *Young inequality* for scalars says that if a, $b>0$ and $\nu\in[0,1]$, then

$$a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b \tag{1}$$

with equality if and only if $a=b$. The inequality (1) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [8]

$$S(h):=\begin{cases}\frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty)\\ \\ 1 \text{ if } h=1.\end{cases} \tag{2}$$

It is well known that $\lim_{h\to1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{3}$$

where a, $b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$.

The second inequality in (3) is due to Tominaga [9] while the first one is due to Furuichi [2].

Kittaneh and Manasrah [5], [6] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^{2}\leq(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq R\left(\sqrt{a}-\sqrt{b}\right)^{2} \tag{4}$$

where a, $b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

We also consider the *Kantorovich's ratio* defined by

$$K(h):=\frac{(h+1)^{2}}{4h},\ h>0. \tag{5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\geq1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

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The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b\leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{6}$$

where a, $b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (6) was obtained by Zou et al. in [10] while the second by Liao et al. [7].

In [10] the authors also showed that $K^{r}(h)\geq S(h^{r})$ for $h>0$ and $r\in\left[0,\frac{1}{2}\right]$ implying that the lower bound in (6) is better than the lower bound from (3).

In the recent paper [1] we obtained the following reverses of Young's inequality as well:

$$0\leq(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq\nu(1-\nu)(a-b)(\ln a-\ln b) \tag{7}$$

and

$$1\leq\frac{(1-\nu)\,a+\nu b}{a^{1-\nu}b^{\nu}}\leq\exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{8}$$

where a, $b>0$, $\nu\in[0,1]$.

It has been shown in [1] that there is no ordering for the upper bounds of the quantity $(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}$ as provided by the inequalities (4) and (7). The same conclusion is true for the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}$ incorporated in the inequalities (3), (6) and (8).

In this note we obtain two new refinements and reverses of Young's inequality.

2. Results

We have the following result:

Lemma 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$, the interior of I. If there exists the constants d, D such that*

$$d\leq f''(t)\leq D \text{ for any } t\in\mathring{I}, \tag{9}$$

then

$$\frac{1}{2}\nu(1-\nu)\,d\,(b-a)^{2}\leq(1-\nu)\,f(a)+\nu f(b)-f((1-\nu)\,a+\nu b) \leq\frac{1}{2}\nu(1-\nu)\,D\,(b-a)^{2} \tag{10}$$

for any a, $b\in\mathring{I}$ and $\nu\in[0,1]$.

In particular, we have

$$\frac{1}{8}(b-a)^{2}d\leq\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\leq\frac{1}{8}(b-a)^{2}D, \tag{11}$$

for any a, $b\in\mathring{I}$.

The constant $\frac{1}{8}$ is best possible in both inequalities in (11)*.*

Proof. We consider the auxiliary function $f_{D}:I\subset\mathbb{R}\to\mathbb{R}$ defined by $f_{D}(x)=\frac{1}{2}Dx^{2}-f(x)$. The function f_{D} is differentiable on $\mathring{I}$ and $f_{D}''(x)=D-f''(x)\geq0$, showing that f_{D} is a convex function on $\mathring{I}$.

By the convexity of f_D we have for any $a,\, b \in \mathring{I}$ and $\nu \in [0,1]$ that

$$\begin{aligned}
0 &\le (1-\nu) f_D(a) + \nu f_D(b) - f_D((1-\nu)a + \nu b) \\
&= (1-\nu)\left(\frac{1}{2}Da^2 - f(a)\right) + \nu\left(\frac{1}{2}Db^2 - f(b)\right) \\
&- \left(\frac{1}{2}D((1-\nu)a+\nu b)^2 - f_D((1-\nu)a+\nu b)\right) \\
&= \frac{1}{2}D\left[(1-\nu)a^2 + \nu b^2 - ((1-\nu)a+\nu b)^2\right] \\
&- (1-\nu) f(a) - \nu f(b) + f_D((1-\nu)a+\nu b) \\
&= \frac{1}{2}\nu(1-\nu) D (b-a)^2 - (1-\nu) f(a) - \nu f(b) + f_D((1-\nu)a+\nu b),
\end{aligned}$$

which implies the second inequality in (10).

The first inequality follows in a similar way by considering the auxiliary function $f_d : I \subset \mathbb{R} \to \mathbb{R}$ defined by $f_D(x) = f(x) - \frac{1}{2}dx^2$ that is twice differentiable and convex on $\mathring{I}$.

If we take $f(x) = x^2$, then (9) holds with equality for $d = D = 2$ and (11) reduces to an equality as well. □

If $D > 0$, the second inequality in (10) is better than the corresponding inequality obtained by Furuichi and Minculete in [4] by applying Lagrange's theorem two times. They had instead of $\frac{1}{2}$ the constant 1. Our method also allowed to obtain, for $d > 0$, a lower bound that can not be established by Lagrange's theorem method employed in [4].

We have:

Theorem 1. *For any $a,\, b > 0$ and $\nu \in [0,1]$ we have*

$$\begin{aligned}
\frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \min\{a,b\} &\le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \qquad (12)\\
&\le \frac{1}{2}\nu(1-\nu)(\ln a - \ln b)^2 \max\{a,b\}
\end{aligned}$$

and

$$\begin{aligned}
\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}}\right] &\le \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \qquad (13)\\
&\le \exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}\right].
\end{aligned}$$

Proof. If we write the inequality (10) for the convex function $f : \mathbb{R} \to (0,\infty)$, $f(x) = \exp(x)$, then we have

$$\begin{aligned}
&\frac{1}{2}\nu(1-\nu)(x-y)^2 \min\{\exp x, \exp y\} \qquad (14)\\
&\le (1-\nu)\exp(x) + \nu\exp(y) - \exp((1-\nu)x + \nu y) \\
&\le \frac{1}{2}\nu(1-\nu)(x-y)^2 \max\{\exp x, \exp y\}
\end{aligned}$$

for any $x,\, y \in \mathbb{R}$ and $\nu \in [0,1]$.

Let $a,\, b > 0$. If we take $x = \ln a$, $y = \ln b$ in (14), then we get the desired inequality (12).

Now, if we write the inequality (10) for the convex function $f:(0,\infty)\to\mathbb{R}$, $f(x)=-\ln x$, then we get for any a, $b>0$ and $\nu\in[0,1]$ that

$$\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}}\le\ln((1-\nu)a+\nu b)-(1-\nu)\ln a-\nu\ln b \le\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}. \tag{15}$$

□

The second inequalities in (12) and (13) are better than the corresponding results obtained by Furuichi and Minculete in [4] where instead of constant $\frac{1}{2}$ they had the constant 1.

Now, since

$$\frac{(b-a)^2}{\min^2\{a,b\}}=\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2 \text{ and } \frac{(b-a)^2}{\max^2\{a,b\}}=\left(\frac{\min\{a,b\}}{\max\{a,b\}}-1\right)^2,$$

then (13) can also be written as:

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right]\le\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \le\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right] \tag{16}$$

for any a, $b>0$ and $\nu\in[0,1]$.

Remark 1. *For $\nu=\frac{1}{2}$ we get the following inequalities of interest*

$$\frac{1}{8}(\ln a-\ln b)^2\min\{a,b\}\le\frac{a+b}{2}-\sqrt{ab}\le\frac{1}{8}(\ln a-\ln b)^2\max\{a,b\} \tag{17}$$

and

$$\exp\left[\frac{1}{8}\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right]\le\frac{\frac{a+b}{2}}{\sqrt{ab}}\le\exp\left[\frac{1}{8}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right], \tag{18}$$

for any a, $b>0$.

Consider the functions

$$P_1(\nu,x):=\nu(1-\nu)(x-1)\ln x$$

and

$$P_2(\nu,x):=\frac{1}{2}\nu(1-\nu)(\ln x)^2\max\{x,1\}$$

for $\nu\in[0,1]$ and $x>0$. A $3D$ plot for $\nu\in(0,1)$ and $x\in(0,2)$ reveals that the difference $P_2(\nu,x)-P_1(\nu,x)$ takes both positive and negative values showing that there is no ordering between the upper bounds of the quantity $(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}$ provided by (7) and (12) respectively.

Also, we consider the functions

$$Q_1(\nu,x):=\exp\left[\nu(1-\nu)\frac{(x-1)^2}{x}\right]$$

and

$$Q_2(\nu,x):=\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min^2\{x,1\}}\right]$$

for $\nu \in [0,1]$ and $x > 0$. Since the difference,

$$d(x) := \frac{1}{x} - \frac{1}{2\min^2\{x,1\}} = \frac{2x-1}{2x^2}$$

for $x \in (0,1)$, changes the sign in $\frac{1}{2}$, then it reveals that the difference $Q_2(\nu,x) - Q_1(\nu,x)$ takes also both positive and negative values showing that there is no ordering between the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}$ provided by (8) and (13).

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