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NEW WEIGHTED OSTROWSKI TYPE INEQUALITIES FOR MAPPINGS WITH FIRST DERIVATIVES OF BOUNDED VARIATION

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Abstract. In this paper, some new weighted Ostrowski type integral inequalities for mappings whose first derivatives are of bounded variation are obtained and midpoint quadrature formula is provided.

1. Introduction

Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e. $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then, we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty , \tag{1}$$

for all $x\in[a,b]$[17]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

Definition 1. *Let $P : a = x_0 < x_1 < ... < x_n = b$ be any partition of $[a,b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$. Then $f(x)$ is said to be of bounded variation if the sum*

$$\sum_{i=1}^{m} |\Delta f(x_i)|$$

is bounded for all such partitions.

Let f be of bounded variation on $[a,b]$, and $\sum(P)$ denotes the sum $\sum_{i=1}^{n} |\Delta f(x_i)|$ corresponding to the partition P of $[a,b]$. The number

$$\bigvee_a^b (f) := \sup\left\{ \sum (P) : P \in P([a,b]) \right\},$$

is called the total variation of f on $[a,b]$. Here $P([a,b])$ denotes the family of partitions of $[a,b]$.

In [9], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$. Then*

$$\left| \int_a^b f(t)dt - (b-a) f(x) \right| \leq \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f)$$

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holds for all $x \in [a,b]$. *The constant* $\frac{1}{2}$ *is the best possible.*

In [7], Dragomir obtained following Ostrowski type inequality for functions of bounded variation:

Theorem 2. *Let* $I_k : a = x_0 < x_1 < ... < x_k = b$ *be a division of the interval* $[a,b]$ *and* $\alpha_i\,(i = 0,1,...,k+1)$ *be* $k+2$ *points so that* $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ $(i = 1,...,k)$, $\alpha_{k+1} = b$. *If* $f : [a,b] \to \mathbb{R}$ *is of bounded variation on* $[a,b]$, *then we have the inequality:*

$$\begin{aligned}
&\left| \int_a^b f(x)dx - \sum_{i=0}^{k} (\alpha_{i+1} - \alpha_i) f(x_i) \right| \\
\leq &\left[\frac{1}{2}\upsilon(h) + \max \left| \alpha_{i+1} - \frac{x_i + x_{i+1}}{2} \right|, i = 0,1,...,k-1 \right] \bigvee_a^b (f) \qquad (2) \\
\leq &\upsilon(h) \bigvee_a^b (f)
\end{aligned}$$

where $\upsilon(h) := \max\{ h_i |\, i = 0,...,n-1\}$, $h_i := x_{i+1} - x_i\,(i = 0,1,...,k-1)$ *and* $\bigvee_a^b (f)$ *is the total variation of* f *on the interval* $[a,b]$.

For some recent results connected with functions of bounded variation see [1],[2],[4]-[6],[8],[10],[11]-[14],[18]-[22].

The aim of this paper is to obtain some generalization of weighted Ostrowski type integral inequalities for functions of bounded variation.

2. Main Results

Theorem 3. *Let* $w : [a,b] \to \mathbb{R}$ *be nonnegative and continuous and let* $f : [a,b] \to \mathbb{R}$ *be differentiable mapping on* $[a,b]$. *If* f' *is of bounded variation on* $[a,b]$, *then we have the weighted inequality*

$$\begin{aligned}
&\left| \left(\int_a^b (u-x)\, w(u)du \right) f'(x) + \left(\int_a^b w(u)du \right) f(x) - \int_a^b w(t)f(t)dt \right| \\
&\leq \left(\int_a^x (x-u)\, w(u)du \right) \bigvee_a^x (f') + \left(\int_x^b (u-x)\, w(u)du \right) \bigvee_x^b (f') \qquad (3)
\end{aligned}$$

for any $x \in [a,b]$, *where* $\bigvee_c^d (f')$ *denotes the total variation of* f' *on* $[c,d]$.

Proof. Define the mapping $P_w(x,t)$ by,

$$P_w(x,t) = \begin{cases} \int_a^t (u-t)\, w(u)du, & a \leq t < x \\ \int_b^t (u-t)\, w(u)du & x \leq t \leq b. \end{cases}$$

Integrating by parts, we get

$$\begin{aligned}
&\int_a^b P_w(x,t)df'(t) \qquad (4)\\
=&\int_a^x\left(\int_a^t (u-t)\,w(u)du\right)df'(t)+\int_x^b\left(\int_b^t (u-t)\,w(u)du\right)df'(t)\\
=&\left(\int_a^t (u-t)\,w(u)du\right)f'(t)\Bigg|_a^x+\int_a^x\left(\int_a^t w(u)du\right)f'(t)dt\\
&+\left(\int_b^t (u-t)\,w(u)du\right)f'(t)\Bigg|_x^b+\int_x^b\left(\int_b^t w(u)du\right)f'(t)dt\\
=&\left(\int_a^x (u-x)\,w(u)du\right)f'(x)-\left(\int_b^x (u-x)\,w(u)du\right)f'(x)\\
&+\left(\int_a^t w(u)du\right)f(t)\Bigg|_a^x-\int_a^x w(t)f(t)dt\\
&+\left(\int_b^t w(u)du\right)f(t)\Bigg|_x^b-\int_x^b w(t)f(t)dt\\
=&\left(\int_a^b (u-x)\,w(u)du\right)f'(x)+\left(\int_b^b w(u)du\right)f(t)-\int_a^b w(t)f(t)dt.
\end{aligned}$$

It is well known [3, see p. 159] that if $g,f:[a,b]\to\mathbb{R}$ are such that g is continuous on $[a,b]$ and f is of bounded variation on $[a,b]$, then $\int_a^b g(t)df(t)$ exist and [3, see p. 177]

$$\left|\int_a^b g(t)df(t)\right|\le\sup_{t\in[a,b]}|g(t)|\bigvee_a^b(f). \qquad (5)$$

Taking modulus in (4) and by using (5), we have

$$\begin{aligned}
&\left|\int_a^b P_w(x,t)df'(t)\right|\\
\le&\left|\int_a^x\left(\int_a^t (u-t)\,w(u)du\right)df'(t)\right|+\left|\int_x^b\left(\int_b^t (u-t)\,w(u)du\right)df'(t)\right|
\end{aligned}$$

$$\leq \sup_{t\in[a,x]}\left|\int_a^t (u-t)\,w(u)du\right|\bigvee_a^x(f')+\sup_{t\in[x,b]}\left|\int_b^t (u-t)\,w(u)du\right|\bigvee_x^b(f')$$

$$\leq \sup_{t\in[a,x]}\left\{\int_a^t (t-u)\,w(u)du\right\}\bigvee_a^x(f')+\sup_{t\in[x,b]}\left\{\int_t^b (u-t)\,w(u)du\right\}\bigvee_x^b(f')$$

$$= \left(\int_a^x (x-u)\,w(u)du\right)\bigvee_a^x(f')+\left(\int_x^b (u-x)\,w(u)du\right)\bigvee_x^b(f')$$

which is the desired result. □

Corollary 1. *Under assumption of Theorem 3 with $w \equiv 1$, then we have the inequality*

$$\left|\left(\frac{a+b}{2}-x\right)f'(x)+f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{2}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^2\bigvee_a^b(f') \tag{6}$$

for any $x\in[a,b]$.

Proof. Choosing $w \equiv 1$ in Theorem 3, we have

$$\left|(b-a)\left(\frac{a+b}{2}-x\right)f'(x)+(b-a)\,f(x)-\int_a^b w(t)f(t)dt\right|$$

$$\leq \frac{1}{2}\left[(x-a)^2\bigvee_a^x(f')+(b-x)^2\bigvee_x^b(f')\right]$$

$$\leq \frac{1}{2}\max\left\{(x-a)^2,(b-x)^2\right\}\bigvee_a^b(f')$$

$$= \frac{1}{2}\left(\max\left\{x-a,b-x\right\}\right)^2\bigvee_a^b(f')$$

$$= \frac{(b-a)^2}{2}\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^2\bigvee_a^b(f')$$

which is required result. □

Remark 1. *If we choose $x=\frac{a+b}{2}$ in* (6)*, then we have the following midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{b-a}{8}\bigvee_a^b(f').$$

which was given by Liu in [16].

Under assumption of of Theorem 3, we have the following corollaries:

Corollary 2. *Let $f \in C^{(2)}[a,b]$. Then we have the inequality*

$$\begin{aligned} &\left| \left(\int_a^b (u-x)\, w(u)du \right) f'(x) + \left(\int_a^b w(u)du \right) f(x) - \int_a^b w(t)f(t)dt \right| \\ &\leq \left(\int_a^x (x-u)\, w(u)du \right) \|f''\|_{[a,x],1} + \left(\int_x^b (u-x)\, w(u)du \right) \|f''\|_{[x,b],1} \end{aligned} \tag{7}$$

for all $x \in [a,b]$, where $\|.\|_{[c,d],1}$ is the L_1-norm, namely

$$\|f''\|_{[c,d],1} = \int_c^d |f''(t)|\, dt.$$

Remark 2. *If we choose $w \equiv 1$ and $x = \frac{a+b}{2}$ in (7), then we have the inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{b-a}{8} \|f''\|_{[a,b],1}$$

which was given by Liu in [15].

Corollary 3. *Let $f' : [a,b] \to \mathbb{R}$ be a Lipschitzian mapping with the constants $L_1, L_2 > 0$. Then, we have the inequality*

$$\begin{aligned} &\left| \left(\int_a^b (u-x)\, w(u)du \right) f'(x) + \left(\int_a^b w(u)du \right) f(x) - \int_a^b w(t)f(t)dt \right| \\ &\leq \left(\int_a^x (x-u)\, w(u)du \right) (x-a)\, L_1 + \left(\int_x^b (u-x)\, w(u)du \right) (b-x)\, L_2 \\ &\leq L \left[\left(\int_a^x (x-u)\, w(u)du \right) (x-a) + \left(\int_x^b (u-x)\, w(u)du \right) (b-x) \right] \end{aligned} \tag{8}$$

for all $x \in [a,b]$ and $L = \max\{L_1, L_2\}$.

3. Application to Quadrature Formula

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < ... < x_n = b$ with $h_i := x_{i+1} - x_i$ and $\upsilon(h) := \max\{h_i | \ i = 0, ..., n-1\}$. Then the following Theorem holds:

Theorem 4. *Let $f : Q \to \mathbb{R}$ is of bounded variation on Q and $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, ..., n-1$). Then we have the quadrature formula:*

$$\int_a^b f(t)dt = \sum_{i=0}^{n-1} \left[\left(\frac{x_i + x_{i+1}}{2} - \xi_i \right) f'(\xi_i) + f(\xi_i) \right] h_i + R(I_n, f, \xi).$$

The remainder term $R(I_n, f, \xi)$ satisfies

$$|R(I_n, f, \xi)| \leq \frac{\upsilon^2(h)}{2} \left[\frac{1}{2} + \max_{i \in \{0,...,n-1\}} \left| \frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i} \right| \right]^2 \bigvee_a^b (f').$$

Proof. Applying Corollary 1 for interval $[x_i, x_{i+1}]$, we have

$$\begin{aligned} &\left| \left(\frac{x_i + x_{i+1}}{2} - x \right) f'(\xi_i) h_i + f(\xi_i) h_i - \int_{x_i}^{x_{i+1}} f(t)dt \right| \\ &\leq \frac{h_i^2}{2} \left[\frac{1}{2} + \left| \frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i} \right| \right]^2 \bigvee_{x_i}^{x_{i+1}} (f') . \end{aligned} \tag{9}$$

Summing the inequality (9) over i from 0 to $n-1$, then we have

$$\begin{aligned} & |R(I_n, f, \xi)| \\ \leq \; & \sum_{i=0}^{n-1} \frac{h_i^2}{2} \left[\frac{1}{2} + \left| \frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i} \right| \right]^2 \bigvee_{x_i}^{x_{i+1}} (f') \\ \leq \; & \max_{i \in \{0,\ldots,n-1\}} \frac{h_i^2}{2} \left[\frac{1}{2} + \left| \frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i} \right| \right]^2 \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f') \\ \leq \; & \frac{\upsilon^2(h)}{2} \left[\frac{1}{2} + \max_{i \in \{0,\ldots,n-1\}} \left| \frac{\xi_i - \frac{x_i + x_{i+1}}{2}}{h_i} \right| \right]^2 \bigvee_a^b (f') . \end{aligned}$$

This completes the proof of the Theorem. □

Remark 3. *If we choose* $\xi_i = \frac{x_i + x_{i+1}}{2}$ $(i = 0, ..., n-1)$ *in Theorem 4, we have the midpoint rule*

$$\int_a^b f(t)dt = \sum_{i=0}^{n-1} f(\xi_i) h_i + R_M(I_n, f)$$

and the remainder term satisfies

$$|R_M(I_n, f)| \leq \frac{\upsilon(h)}{8} \bigvee_a^b (f') .$$

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