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HANKEL TRANSFORM IN BICOMPLEX SPACE AND APPLICATIONS

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Abstract. Motivated by the work of Zemanian A.H. in this paper we generalize the complex Hankel transform to bicomplex Hankel transform and derive some of it's basic properties. A table of bicomplex Hankel transform is given for some functions of importance. It has found applications in solving partial differential equation of bicomplex-valued functions, signal processing, optics, electromagnetic field theory and other related problems. The application of Hankel transform has been illustrated by solving bicomplex Cauchy problem.

1. Introduction

In this paper, we investigate Hankel transform in bicomplex space which is generalization of complex Hankel transform which is given by Koh E.L. and Zemanian A.H. [9] and it's properties. In 1892, Segre Corrado [14] defined bicomplex numbers as

$$C_2 = \{\xi : \xi = x_0 + i_1 x_1 + i_2 x_2 + j x_3 |\ x_0, x_1, x_2, x_3 \in C_0\},$$

or

$$C_2 = \{\xi : \xi = z_1 + i_2 z_2 |\ z_1, z_2 \in C_1\}.$$

where i_1 and i_2 are imaginary units such that $i_1^2 = i_2^2 = -1,\ \ i_1 i_2 = i_2 i_1 = j,\ j^2 = 1$ and C_0, C_1 and C_2 are sets of real numbers, complex numbers and bicomplex numbers respectively. The set of bicomplex numbers is a commutative ring with unit and zero divisors. Hence, contrary to quaternions, bicomplex numbers are commutative with some non-invertible elements situated on the null cone.

In 1928 and 1932, Futagawa Michiji originated the concept of holomorphic functions of a bicomplex variable in a series of papers [5], [6]. In 1934, Dragoni [4] gave some basic results in the theory of bicomplex holomorphic functions while Price G.B. [12] and Rönn S. [15] have developed the bicomplex algebra and function theory.

In recent developments, authors have done efforts to extend Polygamma function [7], inverse Laplace transform, it's convolution theorem [2], Stieltjes transform [1], Tauberian Theorem of Laplace-Stieltjes transform [3] and Bochner Theorem of Fourier-Stieltjes transform in the bicomplex variable from their complex counterpart. In their procedure the idempotent representation of bicomplex plays a vital role.

In 1966, Zemanian A.H. [18] extended the classical Hankel transformation to generalized functions of slow growth and in 1968, Koh and Zemanian [9] generalized the Hankel transform in complex variable in new space. In 1997, Tuan V.K. [16] extended the range of the Hankel transform. Motivated by the work of Koh and Zemanian [9], we derive the bicomplex Hankel transform with convergence conditions that are capable of transferring signals from real-valued (t) domain to bicomplexified frequency (ξ) domain. Bicomplex

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Hankel transform is highly applicable in solving partial differential equation of bicomplex-valued function, signal processing, optics and other related problems.
Idempotent Representation: Every bicomplex number can be uniquely expressed as a complex combination of e_1 and e_2, viz.

$$\xi = (z_1 + i_2 z_2) = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2,$$

(where $e_1 = \frac{1+j}{2}, e_2 = \frac{1-j}{2}; e_1 + e_2 = 1$ and $e_1 e_2 = e_2 e_1 = 0$).

This representation of a bicomplex number is known as Idempotent Representation of ξ. The coefficients $(z_1 - i_1 z_2)$ and $(z_1 + i_1 z_2)$ are called the Idempotent Components of the bicomplex number $\xi = z_1 + i_2 z_2$ and $\{e_1, e_2\}$ is called idempotent basis.
Norm: The norm $\| \cdot \| : C_2 \to C_0^+$ of a bicomplex number is defined as

$$\|\xi\| = \left(x_0^2 + x_1^2 + x_2^2 + x_3^2\right)^{\frac{1}{2}}$$

where C_0^+ denote the set of all non-negative real numbers and $\xi = x_0 + i_1 x_1 + i_2 x_2 + i_1 i_2 x_3$.
Cartesian Set: The Auxiliary complex spaces A_1 and A_2 are defined as follows:

$$A_1 = \{z_1 - i_1 z_2,\ \forall\ z_1, z_2 \in C_1\},\ A_2 = \{z_1 + i_1 z_2,\ \forall\ z_1, z_2 \in C_1\}.$$

A Cartesian set determined by X_1 and X_2 in A_1 and A_2 respectively is denoted as $X_1 \times_e X_2$ and is defined as:

$$X_1 \times_e X_2 = \{z_1 + i_2 z_2 \in C_2 : z_1 + i_2 z_2 = w_1 e_1 + w_2 e_2,\ w_1 \in X_1, w_2 \in X_2\}.$$

With the help of idempotent representation, we define functions $P_1 : C_2 \to A_1 \subseteq C_1$, $P_2 : C_2 \to A_2 \subseteq C_1$ as follows:

$$P_1(z_1 + i_2 z_2) = P_1[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 - i_1 z_2) \in A_1,\ \forall\ z_1 + i_2 z_2 \in C_2,$$

$$P_2(z_1 + i_2 z_2) = P_2[(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2] = (z_1 + i_1 z_2) \in A_2,\ \forall\ z_1 + i_2 z_2 \in C_2.$$

In the following theorem, Price G.B. had discussed the convergence of bicomplex function with respect to it's idempotent complex component functions. This theorem is useful in proving our results.

Theorem 1. *(Price G.B. [12]): $F(\xi) = F_{e_1}(\xi_1)e_1 + F_{e_2}(\xi_2)e_2$ is convergent in domain $D \subseteq C_2$ iff $F_{e_1}(\xi_1)$ and $F_{e_2}(\xi_2)$, the projections under the functions $P_1 : D \to D_1 \subseteq C_1$ and $P_2 : D \to D_2 \subseteq C_1$, are convergent in domains D_1 and D_2, respectively.*

The organization of this paper is as follows:
In section 2, we define testing spaces for bicomplex functions and their duals. In Section 3, we establish Hankel transform in bicomplex space and its convergence conditions. In Section 4, we present basic and useful properties of bicomlex Hankel transform. In Section 5 , the inversion theorem for bicomplex Hankel transform has been established and bicomplex Hankel transform has been illustrated for functions of importance. In section 6, operational calculus for Hankel transform operator has been discussed. In section 7, application of bicomplex Hankel transform is illustrated by solving bicomplex Cauchy problem. The last Section 8 contains the conclusion.

2. The testing spaces $\mathcal{J}_{\mu,a}$ and $\mathcal{J}_{\mu}(\sigma)$ and their duals

In this section, we define the space of bicomplex-valued testing functions extending the space defined by Koh and Zemanian [9]. Let a denote a positive real number and μ any bicomplex number. Then for each pair of a and μ we define $\mathcal{J}_{\mu,a}$ as the space of testing functions ϕ which are bicomplex-valued, which are defined and smooth on $0 < x < \infty$ and for which

$$\tau_k^{\mu,a}(\phi) = \sup_{0<x<\infty} \left\| e^{-ax} x^{-\mu-1/2} \left(x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2} \right)^k \phi(x) \right\| < \infty,$$

$$k = 0, 1, 2, \cdots, \; D \equiv \frac{d}{dx}.$$

Theorem 2. *Let $a > b > 0$, then $\mathcal{J}_{\mu,b} \subset \mathcal{J}_{\mu,a}$.*

Proof. Let $\phi \in \mathcal{J}_{\mu,b}$, then

$$\tau_k^{\mu,b}(\phi) = \sup_{0<x<\infty} \left\| e^{-bx} x^{-\mu-1/2} \left(x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2} \right)^k \phi(x) \right\| < \infty.$$

Since $a > b > 0$, therefore

$$\begin{aligned} &\sup_{0<x<\infty} \left\| e^{-ax} x^{-\mu-1/2} \left(x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2} \right)^k \phi(x) \right\| \\ &\leq \sup_{0<x<\infty} \left\| e^{-bx} x^{-\mu-1/2} \left(x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2} \right)^k \phi(x) \right\| < \infty \\ &\Rightarrow \quad \tau_k^{\mu,a}(\phi) \leq \tau_k^{\mu,b}(\phi) < \infty. \end{aligned}$$

Therefore,

$$\begin{aligned} &\tau_k^{\mu,a}(\phi) = \sup_{0<x<\infty} \left\| e^{-ax} x^{-\mu-1/2} \left(x^{-\mu-1/2} D x^{2\mu+1} D x^{-\mu-1/2} \right)^k \phi(x) \right\| < \infty \\ &\Rightarrow \phi \in \mathcal{J}_{\mu,a} \\ &\therefore \quad \mathcal{J}_{\mu,b} \subset \mathcal{J}_{\mu,a}. \end{aligned}$$

□

$\mathcal{J}_{\mu,a}$ is the linear space over the field of complex numbers as c_1, $c_2 \in C_1$ and ϕ, $\psi \in \mathcal{J}_{\mu,a} \Rightarrow c_1\phi + c_2\psi \in \mathcal{J}_{\mu,a}$. Let $\{a_\nu\}_{\nu=1}^{\infty}$ be a monotonically increasing sequence of positive numbers tending to σ. By the Theorem 2 if $a_1 > b_1 > 0$, then $\mathcal{J}_{\mu_1,b_1} \subset \mathcal{J}_{\mu_1,a_1}$.

This follows that $\{\mathcal{J}_{\mu,a_\nu}\}_{\nu=1}^{\infty}$ is a sequence such that $\mathcal{J}_{\mu,a_1} \subset \mathcal{J}_{\mu,a_2} \subset \mathcal{J}_{\mu,a_3} \cdots$. Let $\mathcal{J}_{\mu}(\sigma) = \bigcup_{\nu=1}^{\infty} \mathcal{J}_{\mu,a_\nu}$ denote the countable-union space generated by the above sequence of spaces. The dual of $\mathcal{J}_{\mu,a}$ and $\mathcal{J}_{\mu}(\sigma)$ are denoted by $\mathcal{J}'_{\mu,a}$ and $\mathcal{J}'_{\mu}(\sigma)$ respectively.

Let bicomplex-valued function $f(x)$ be locally integrable on $0 < x < \infty$ and such that $\int_0^\infty \left\| f(x) e^{ax} x^{\mu+\frac{1}{2}} \right\| dx < \infty$. Then $f(x)$ generates a regular generalized function in $\mathcal{J}'_{\mu,a}$ defined by

$$\langle f, \phi \rangle = \int_0^\infty f(x)\phi(x)dx, \qquad \phi \in \mathcal{J}_{\mu,a}.$$

3. Bicomplex Hankel Transform

Let $\mu_1 \in C_1$ be restricted to $\mathrm{Re}(\mu_1) \geq -\frac{1}{2}$. If $a_1 > b_1 > 0$, then $\mathcal{J}_{\mu_1,b_1} \subset \mathcal{J}_{\mu_1,a_1}$. This follows immediately from the inequality $\tau_k^{\mu_1,a_1}(\phi) \leq \tau_k^{\mu_1,b_1}(\phi)$ for $\phi \in \mathcal{J}_{\mu_1,a_1}$. Hence, the restriction of $f_1 \in \mathcal{J}'_{\mu_1,a_1}$ to $\mathcal{J}_{\mu_1,b_1}$ is in $\mathcal{J}'_{\mu_1,b_1}$, and convergence in $\mathcal{J}'_{\mu_1,a_1}$ implies convergence in $\mathcal{J}'_{\mu_1,b_1}$. For every $f_1 \in \mathcal{J}'_{\mu_1,a_1}$, $\exists$ a unique real number σ_1 such that

$$\begin{aligned} &f_1 \in \mathcal{J}'_{\mu_1,b_1} \text{ if } b_1 < \sigma_1 \\ &f_1 \notin \mathcal{J}'_{\mu_1,b_1} \text{ if } b_1 > \sigma_1. \end{aligned}$$

Therefore, $f_1 \in \mathcal{J}'_{\mu}(\sigma_1)$. The Hankel transform $F(s_1)$ f_1 of order μ_1 defined as

$$F(s_1) = H_{\mu_1}\{f_1(x)\} = \langle f_1(x), \sqrt{xs_1}J_{\mu_1}(xs_1)\rangle \,, \tag{1}$$

where

$$s_1 \in \Omega_1 = \{s_1 : |\mathrm{Im}(s_1)| < \sigma_1,\ s_1 \notin (-\infty, 0]\} \,. \tag{2}$$

Similarly, for every $f_2 \in \mathcal{J}'_{\mu_2}(\sigma_2)$, another Hankel transform $F(s_2)$ of f_2 of order μ_2 defined as

$$F(s_2) = H_{\mu_2}\{f_2(x)\} = \langle f_2(x), \sqrt{xs_2}J_{\mu_2}(xs_2)\rangle \,, \tag{3}$$

where

$$s_2 \in \Omega_2 = \{s_2 : |\mathrm{Im}(s_2)| < \sigma_2,\ s_2 \notin (-\infty, 0]\} \,. \tag{4}$$

Since $F(s_1)$ and $F(s_2)$ are analytic and convergent in Ω_1 and Ω_2 respectively for $\sigma = \min(\sigma_1, \sigma_2)$, taking the linear combination with idempotent component e_1 and e_2 such as:

$$\begin{aligned}
&F(s_1)e_1 + F(s_2)e_2 \\
&= \langle f_1(x), \sqrt{xs_1}J_{\mu_1}(xs_1)\rangle\, e_1 + \langle f_2(x), \sqrt{xs_2}J_{\mu_2}(xs_2)\rangle\, e_2 \\
&= \left\langle f_1(x)e_1 + f_2(x)e_2, \sqrt{x(s_1e_1 + s_2e_2)}J_{\mu_1 e_1+\mu_2 e_2}(x(s_1e_1 + s_2e_2))\right\rangle \\
&= \left\langle f(x), \sqrt{x\xi}J_{\mu}(x\xi)\right\rangle = F(\xi)
\end{aligned} \tag{5}$$

(where $f(x) = f_1(x)e_1 + f_2(x)e_2,\ \mu = \mu_1e_1 + \mu_2e_2$ and $\xi = s_1e_1 + s_2e_2)$.

Since $F(s_1)$ and $F(s_2)$ are complex functions which are convergent and analytic in Ω_1 and Ω_2 respectively, so a bicomplex function $F(\xi) = F(s_1)e_1 + F(s_2)e_2$ will be convergent and analytic in the region Ω defined as:

$$\Omega = \{\xi : \xi = s_1e_1 + s_2e_2;\ |\mathrm{Im}(s_1)| < \sigma,\ |\mathrm{Im}(s_2)| < \sigma \text{ and } s_1, s_2 \notin (-\infty, 0]\} \tag{6}$$

For better geometrical understanding of the region of convergence of bicomplex Hankel transform it will be advantageous to use the general four dimensional representation of bicomplex numbers. For this we take conventional representation of $s_1, s_2 \in C_1$ as

$$s_1 = x_1 + i_1y_1, \quad s_2 = x_2 + i_1y_2; \qquad x_1, x_2, y_1, y_2 \in C_0.$$

Then by (6) $|y_1| < \sigma,\ |y_2| < \sigma$ and if $y_1 = y_2 = 0$ then $x_1, x_2 \notin (-\infty, 0]$. Now,

$$\begin{aligned}
\xi &= s_1e_1 + s_2e_2 = (x_1 + i_1y_1)e_1 + (x_2 + i_1y_2)e_2 \\
&= (x_1 + i_1y_1)\left(\frac{1 + i_1i_2}{2}\right) + (x_2 + i_1y_2)\left(\frac{1 + i_1i_2}{2}\right) \\
&= \frac{x_1 + x_2}{2} + \left(\frac{y_1 + y_2}{2}\right)i_1 + \left(\frac{y_2 - y_1}{2}\right)i_2 + \left(\frac{x_1 - x_2}{2}\right)i_1i_2. \\
&= a_0 + i_1a_1 + i_2a_2 + i_1i_2a_3 \qquad \text{(say)}
\end{aligned}$$

On the basis of restriction on y_1 and y_2, three possible cases occur:

(1) If $y_1 = y_2$ then $\frac{y_2 - y_1}{2} = 0$ and $\frac{y_1 + y_2}{2} = y_1 = y_2$. Hence $a_2 = 0$ and $|a_1| < \sigma$. In particular, if $y_1 = y_2 = 0$ and $x_1, x_2 \notin (-\infty, 0]$. Clearly if $a_1, a_2 = 0$ then $|a_3| < a_0$.

(2) If $y_1 > y_2$ then $-\sigma < \frac{y_2 - y_1}{2} < 0$, $\frac{y_1+y_2}{2} < \frac{y_2+\sigma}{2} < \frac{y_2+\sigma}{2} + \frac{\sigma-y_1}{2} = \sigma + \frac{y_2-y_1}{2}$ and $\frac{y_1+y_2}{2} > \frac{y_1-\sigma}{2} > \frac{y_1-\sigma}{2} - \frac{\sigma+y_2}{2} = -\sigma - \frac{y_2-y_1}{2}$. Hence $-\sigma < a_2 < 0$ and $-\sigma - a_2 < a_1 < \sigma + a_2$.

(3) If $y_1 < y_2$ then $0 < \frac{y_2-y_1}{2} < \sigma$, $\frac{y_1+y_2}{2} < \frac{y_1+\sigma}{2} < \frac{y_1+\sigma}{2} + \frac{\sigma-y_1}{2} = \sigma - \frac{y_2-y_1}{2}$ and $\frac{y_1+y_2}{2} > \frac{y_2-\sigma}{2} > \frac{y_2-\sigma}{2} + \frac{-\sigma-y_1}{2} = -\sigma + \frac{y_2-y_1}{2}$. Hence $0 < a_2 < \sigma$ and $-\sigma + a_2 < a_1 < \sigma - a_2$.

Considering all of these results we conclude that the region of convergence of $F(\xi)$ as

$$\Omega = \{\xi : \xi = a_0 + i_1a_1 + i_2a_2 + i_1i_2a_3 \in C_2,\ -\sigma + |a_2| < a_1 < \sigma - |a_2|,$$
$$0 < |a_2| < \sigma \text{ and if } a_1 = a_2 = 0 \text{ then } |a_3| < a_0\} \tag{7}$$

Similarly, for $\mathrm{Re}(\mu_1) \geq -\frac{1}{2}$ and $\mathrm{Re}(\mu_2) \geq -\frac{1}{2}$, we have

$$\mathrm{Re}(\mu) \geq -\frac{1}{2} + |\mathrm{Im}_j(\mu)|\,, \quad (\text{where } \mu \in C_2 \text{ and } \mathrm{Im}_j \text{ denotes imaginary part w.r.t. } j.)$$

Let bicomplex-valued function $f(x)$ be locally integrable on $0 < x < \infty$ and such that

$$\int_0^\infty \left\| f(x)e^{ax}x^{\mu+\frac{1}{2}} \right\| dx < \infty, \qquad \forall\, a < \sigma. \tag{8}$$

Therefore, (5) can be written as

$$F(\xi) = \left\langle f(x), \sqrt{x\xi}J_\mu(x\xi) \right\rangle = \int_0^\infty f(x)\sqrt{x\xi}J_\mu(x\xi)dx$$
$$\therefore\ F(\xi) = \int_0^\infty f(x)\sqrt{x\xi}J_\mu(x\xi)dx, \qquad \forall\, \xi \in \Omega. \tag{9}$$

Now, we can define bicomplex Hankel transform as follows:

Definition 1. *Let $\mu \in C_2$ be restricted to $Re(\mu) \geq -\frac{1}{2} + |Im_j(\mu)|$. If $a > b > 0$, then $\mathcal{J}_{\mu,b} \subset \mathcal{J}_{\mu,a}$. This follows immediately from the inequality $\tau_k^{\mu,a}(\phi) \leq \tau_k^{\mu,b}(\phi)$ for $\phi \in \mathcal{J}_{\mu,a}$. Hence, the restriction of $f \in \mathcal{J}'_{\mu,a}$ to $\mathcal{J}_{\mu,b}$ is in $\mathcal{J}'_{\mu,b}$, and convergence in $\mathcal{J}'_{\mu,a}$ implies convergence in $\mathcal{J}'_{\mu,b}$. For every bicomplex-valued function $f \in \mathcal{J}'_{\mu,a}$, $\exists$ a unique real number σ such that $f \in \mathcal{J}'_{\mu,b}$ if $b < \sigma$ and $f \notin \mathcal{J}'_{\mu,b}$ if $b > \sigma$. Therefore, $f \in \mathcal{J}'_\mu(\sigma)$. The μ^{th} order bicomplex Hankel transform $F(\xi)$ of f is defined as*

$$F(\xi) = H_\mu\{f(x)\} = \left\langle f(x), \sqrt{x\xi}J_\mu(x\xi) \right\rangle, \qquad \forall\, \xi \in \Omega$$

where Ω is defined in (6) and (7). If bicomplex-valued function $f(x)$ is locally integrable on $0 < x < \infty$ and satisfy the condition (8). Then bicomplex Hankel transform is defined as

$$F(\xi) = \int_0^\infty f(x)\sqrt{x\xi}J_\mu(x\xi)dx, \qquad \forall\, \xi \in \Omega.$$

4. Properties of Bicomplex Hankel Transform

In this section, some properties of bicomplex Hankel transform viz. linearity property, change of scale property, analyticity of $F(\xi)$, relationship with bicomplex Laplace transform and others have been discussed.

Theorem 3. *(Linearity Property). Let $F(\xi)$ and $G(\xi)$ be the bicomplex Hankel transform of order μ of bicomplex-valued functions $f(x)$ and $g(x)$ respectively, then*

$$H_\mu\{f(x) + g(x)\} = F(\xi) + G(\xi), \qquad \xi \in \Omega$$

where Ω defined in (6).

Proof. By applying the definition of bicomplex Hankel transform,

$$\begin{aligned}
&H_\mu\{f(x)+g(x)\}\\
&=\left\langle f(x)+g(x), \sqrt{x\xi}J_\mu(x\xi)\right\rangle\\
&=\left\langle f_1(x)e_1+f_2(x)e_2+g_1(x)e_1+g_2(x)e_2, \sqrt{x(s_1e_1+s_2e_2)}J_\mu(x(s_1e_1+s_2e_2))\right\rangle\\
&=\langle f_1(x)+g_1(x), \sqrt{xs_1}J_\mu(xs_1)\rangle e_1+\langle f_2(x)+g_2(x), \sqrt{xs_2}J_\mu(xs_2)\rangle e_2\\
&=\langle f_1(x), \sqrt{xs_1}J_\mu(xs_1)\rangle e_1+\langle g_1(x), \sqrt{xs_1}J_\mu(xs_1)\rangle e_1\\
&\quad+\langle f_2(x), \sqrt{xs_2}J_\mu(xs_2)\rangle e_2+\langle g_2(x), \sqrt{xs_2}J_\mu(xs_2)\rangle e_2\\
&=\left\langle f_1(x)e_1+f_2(x)e_2, \sqrt{x(s_1e_1+s_2e_2)}J_\mu(x(s_1e_1+s_2e_2))\right\rangle\\
&\quad+\left\langle g_1(x)e_1+g_2(x)e_2, \sqrt{x(s_1e_1+s_2e_2)}J_\mu(x(s_1e_1+s_2e_2))\right\rangle\\
&=\left\langle f(x), \sqrt{x\xi}J_\mu(x\xi)\right\rangle+\left\langle g(x), \sqrt{x\xi}J_\mu(x\xi)\right\rangle\\
&=F(\xi)+G(\xi).
\end{aligned}$$

□

Theorem 4 (Change of Scale Property)**.** *Let $F(\xi)$ be the bicomplex Hankel transform of order μ of bicomplex-valued function $f(x)$ and satisfy the condition (8). Then*

$$H_\mu\{f(ax)\}=\frac{1}{a}F\left(\frac{\xi}{a}\right), \quad a\neq 0\in C_0,\ \xi\in\Omega$$

where Ω defined in (6).

Proof. By applying the definition of bicomplex Hankel transform

$$\begin{aligned}
H_\mu\{f(ax)\}&=\int_0^\infty f(ax)\sqrt{x\xi}J_\mu(x\xi)dx\\
&\text{Put } ax=t\\
&=\frac{1}{a}\int_0^\infty f(t)\sqrt{t\frac{\xi}{a}}J_\mu\left(t\frac{\xi}{a}\right)dt\\
&=\frac{1}{a}F\left(\frac{\xi}{a}\right).
\end{aligned}$$

□

Theorem 5. *Let $H_\mu\{f(x)\}$ be the bicomplex Hankel transform of order μ of bicomplex-valued locally integrable function $f(x)$ and satisfy the condition (8). Then*

$$H_\mu\left\{\frac{df}{dx}\right\}=\frac{\xi}{2}\left(H_{\mu+1}\{f(x)\}-H_{\mu-1}\{f(x)\}\right)-\frac{1}{2}H_\mu\left\{\frac{f}{x}\right\}, \qquad \xi\in\Omega$$

where Ω defined in (6).

Proof. If $F(\xi)$ be the bicomplex Hankel transform of order μ of $f(x)$ i.e.

$$H_\mu\{f(x)\}=\int_0^\infty f(x)\sqrt{x\xi}J_\mu(x\xi)dx,$$

then the bicomplex Hankel transform of $\frac{df}{dx}$ is

$$H_\mu\left\{\frac{df}{dx}\right\}=\int_0^\infty\frac{df}{dx}\sqrt{x\xi}J_\mu(x\xi)dx$$

on integrating by parts and assuming that $\sqrt{x}f(x) \to 0$ as $x \to 0,\ x \to \infty$, we get

$$\begin{aligned}
H_\mu\left\{\frac{df}{dx}\right\} &= -\int_0^\infty f(x)\frac{d}{dx}\left(\sqrt{x\xi}J_\mu(x\xi)\right)dx \\
&= -\int_0^\infty f(x)\left(\frac{\sqrt{\xi}}{2\sqrt{x}}J_\mu(x\xi) + \frac{\xi}{2}\sqrt{x\xi}J_{\mu-1}(x\xi) - \frac{\xi}{2}\sqrt{x\xi}J_{\mu+1}(x\xi)\right)dx \\
&= -\frac{1}{2}\int_0^\infty \frac{f(x)}{x}\sqrt{x\xi}J_\mu(x\xi)dx \\
&\quad -\frac{\xi}{2}\int_0^\infty f(x)\sqrt{x\xi}J_{\mu-1}(x\xi)dx + \frac{\xi}{2}\int_0^\infty f(x)\sqrt{x\xi}J_{\mu+1}(x\xi)dx \\
&= -\frac{1}{2}H_\mu\left\{\frac{f}{x}\right\} - \frac{\xi}{2}H_{\mu-1}(\xi) + \frac{\xi}{2}H_{\mu+1}(\xi) \\
\therefore\ H_\mu\left\{\frac{df}{dx}\right\} &= \frac{\xi}{2}\left(H_{\mu+1}\{f(x)\} - H_{\mu-1}\{f(x)\}\right) - \frac{1}{2}H_\mu\left\{\frac{f}{x}\right\}.
\end{aligned}$$

□

Theorem 6. *Let $H_\mu\{f(x)\}$ be the bicomplex Hankel transform of order μ of bicomplex-valued locally integrable function $f(x)$ and satisfy the condition* (8)*. Then*

$$\begin{aligned}
H_\mu\left\{\frac{d^2f}{dx^2}\right\} =& \frac{\xi^2}{4}\left(H_{\mu+2}\{f(x)\} - 2H_\mu\{f(x)\}\right) \\
&-\frac{\xi}{4}\left(H_{\mu-1}\left\{\frac{f}{x}\right\} - 2H_{\mu+1}\left\{\frac{f}{x}\right\}\right) - \frac{1}{4}H_\mu\left\{\frac{f}{x^2}\right\}, \qquad \xi \in \Omega
\end{aligned}$$

where Ω defined in (6)*.*

Proof. By Theorem 5 we have,

$$H_\mu\left\{\frac{df}{dx}\right\} = \frac{\xi}{2}\left(H_{\mu+1}\{f(x)\} - H_{\mu-1}\{f(x)\}\right) - \frac{1}{2}H_\mu\left\{\frac{f}{x}\right\}. \tag{10}$$

By inserting $\frac{df}{dx}$ in place of f in (10) we have

$$\begin{aligned}
H_\mu\left\{\frac{d^2f}{dx^2}\right\} =& \frac{\xi}{2}\left(H_{\mu+1}\left\{\frac{df}{dx}\right\} - H_{\mu-1}\left\{\frac{df}{dx}\right\}\right) - \frac{1}{2}H_\mu\left\{\frac{1}{x}\frac{df}{dx}\right\} \\
=& \frac{\xi}{2}\left(\xi H_{\mu+2}\{f(x)\} - 2\xi H_\mu\{f(x)\} + \xi H_{\mu-2}\{f(x)\} - H_{\mu+1}\left\{\frac{f}{x}\right\}\right. \\
&\left. + H_{\mu-1}\left\{\frac{f}{x}\right\}\right) - \frac{1}{2}H_\mu\left\{\frac{1}{x}\frac{df}{dx}\right\}.
\end{aligned} \tag{11}$$

Now,

$$H_\mu\left\{\frac{1}{x}\frac{df}{dx}\right\} = \int_0^\infty f(x)\frac{1}{x}\sqrt{x\xi}J_\mu(x\xi)dx.$$

On integrating by parts and assuming that $\frac{f(x)}{\sqrt{x}} \to 0$ as $x \to 0,\ x \to \infty$ we have

$$\begin{aligned}
&H_\mu\left\{\frac{1}{x}\frac{df}{dx}\right\} \\
&= -\int_0^\infty f(x)\frac{d}{dx}\left(\sqrt{\frac{\xi}{x}}J_\mu(x\xi)\right)dx \\
&= -\int_0^\infty \sqrt{\xi}f(x)\left(-\frac{1}{2x^{3/2}}J_\mu(x\xi)+\frac{\xi}{\sqrt{x}}\left(\frac{1}{2}\left(J_{\mu-1}(x\xi)-J_{\mu+1}(x\xi)\right)\right)\right)dx \\
&= \frac{1}{2}\int_0^\infty \frac{f(x)}{x^2}\sqrt{x\xi}J_\mu(x\xi)dx - \frac{\xi}{2}\int_0^\infty \frac{f(x)}{x}\sqrt{x\xi}J_{\mu-1}(x\xi)dx \\
&\quad + \frac{\xi}{2}\int_0^\infty \frac{f(x)}{x}\sqrt{x\xi}J_{\mu+1}(x\xi)dx \\
&= \frac{1}{2}H_\mu\left\{\frac{f}{x^2}\right\} - \frac{\xi}{2}H_{\mu-1}\left\{\frac{f}{x}\right\} + \frac{\xi}{2}H_{\mu+1}\left\{\frac{f}{x}\right\}.
\end{aligned}$$

By putting the value in (11) and after simplification we have

$$\begin{aligned}
H_\mu\left\{\frac{d^2 f}{dx^2}\right\} =& \frac{\xi^2}{4}\left(H_{\mu+2}\{f(x)\} - 2H_\mu\{f(x)\}\right) \\
&- \frac{\xi}{4}\left(H_{\mu-1}\left\{\frac{f}{x}\right\} - 2H_{\mu+1}\left\{\frac{f}{x}\right\}\right) - \frac{1}{4}H_\mu\left\{\frac{f}{x^2}\right\}.
\end{aligned}$$

□

Theorem 7 (Relationship between bicomplex Hankel transform and bicomplex Laplace transform)**.** *Let $H_\mu\{f(x);\ \xi\}$ and $L\{f(x);\ \eta\}$ be bicomplex Hankel transform and bicomplex Laplace transform respectively. Then*

$$H_\mu\left\{e^{-\eta x}f(x);\ \xi\right\} = L\left\{\sqrt{x\xi}J_\mu(x\xi)f(x);\ \eta\right\}, \qquad \xi \in \Omega$$

where $Re(P_1 : \eta) > 0$, $Re(P_2 : \eta) > 0$ and Ω defined in (6)*.*

Proof. By the definition of bicomplex Hankel transform, we have

$$\begin{aligned}
H_\mu\left\{e^{-\eta x}f(x);\ \xi\right\} &= \int_0^\infty e^{-\eta x}f(x)\sqrt{x\xi}J_\mu(x\xi)dx \\
&= \int_0^\infty e^{-\eta x}\left(f(x)\sqrt{x\xi}J_\mu(x\xi)\right)dx \\
&= L\left\{\sqrt{x\xi}J_\mu(x\xi)f(x);\ \eta\right\}.
\end{aligned}$$

□

Theorem 8 (Analyticity of $F(\xi)$)**.** *$F(\xi)$, as defined in* (5)*, is an analytic function of ξ in the region Ω defined in* (6)*, and*

$$D_\xi F(\xi) = \left\langle f(x), \frac{\partial}{\partial\xi}\sqrt{x\xi}J_\mu(x\xi)\right\rangle, \qquad \xi = s_1e_1 + s_2e_2 \in \Omega.$$

Proof. Clearly, the bicomplex function $F(\xi)$ is analytic in Ω. By Theorem 1 in [9], we have

$$D_{s_1}F(s_1) = \left\langle f_1(x), \frac{\partial}{\partial s_1}\sqrt{xs_1}J_{\mu_1}(xs_1)\right\rangle, \qquad s_1 \in \Omega_1. \tag{12}$$

Similarly,

$$D_{s_2}F(s_2) = \left\langle f_2(x), \frac{\partial}{\partial s_2}\sqrt{xs_2}J_{\mu_2}(xs_2)\right\rangle, \qquad s_2 \in \Omega_2. \tag{13}$$

Since (12) and (13) are analytic in Ω_1 and Ω_2 respectively. Therefore, taking linear combination of (12) and (13) with e_1 and e_2 respectively.

$$\begin{aligned} D_{s_1}F(s_1)e_1 + D_{s_2}F(s_2)e_2 = &\left\langle f_1(x), \frac{\partial}{\partial s_1}\sqrt{xs_1}J_{\mu_1}(xs_1)\right\rangle e_1 \\ &+ \left\langle f_2(x), \frac{\partial}{\partial s_2}\sqrt{xs_2}J_{\mu_2}(xs_2)\right\rangle e_2 \end{aligned}$$

$$\begin{aligned} &D_{(s_1e_1+s_2e_2)}F(s_1e_1+s_2e_2) \\ &= \left\langle f_1(x)e_1 + f_2(x)e_2, \frac{\partial}{\partial(s_1e_1+s_2e_2)}\sqrt{x(s_1e_1+s_2e_2)}J_{\mu_1e_1+\mu_2e_2}(x(s_1e_1+s_2e_2))\right\rangle \\ &\therefore\ D_\xi F(\xi) = \left\langle f(x), \frac{\partial}{\partial \xi}\sqrt{x\xi}J_\mu(x\xi)\right\rangle \end{aligned}$$

(where $\xi = s_1e_1 + s_2e_2$, $f(x) = f_1(x)e_1 + f_2(x)e_2$ and $\mu = \mu_1e_1 + \mu_2e_2$). □

5. Inversion of Bicomplex Hankel Transform

In this section, we discuss inversion formula for bicomplex Hankel transform. We require the following theorem by Koh and Zemanian [9, Theorem 4] for inverse Hankel transform to define it's bicomplex form.

Theorem 9. *Let $F(s) = H_\mu\{f(x)\}$, $f \in \mathcal{J}_\mu^{'}(\sigma)$ be complex Hankel transform of $f(x)$ where s is restricted to the real positive axis. Let $\mu \geq -\frac{1}{2}$. Then, in the sense of convergence in $\mathcal{D}^{'}(I)$,*

$$f(x) = \lim_{r\to\infty}\int_0^r H(s)\sqrt{xs}J_\mu(xs)ds.$$

$\mathcal{D}(I)$ denotes the space of smooth functions that have compact support on I and $\mathcal{D}^{'}(I)$ is dual of space $\mathcal{D}(I)$.

Theorem 10. *Let $F(\xi) = H_\mu\{f(x)\}$, $f \in \mathcal{J}_\mu^{'}(\sigma)$ as in (5) where ξ restricted to the real positive axis and $f(x)$ and $F(\xi)$ are bicomplex-valued functions. Then for $Re(\mu) \geq -\frac{1}{2} + |Im_j(\mu)|$, in the sense of convergence in $\mathcal{D}^{'}(I)$,*

$$f(x) = \lim_{r\to\infty}\int_0^r F(\xi)\sqrt{x\xi}J_\mu(x\xi)d\xi.$$

Proof. Let $F_1(\xi) = H_{\mu_1}\{f_1(x)\}$, $f_1 \in J_{\mu_1}^{'}(\sigma)$ be complex-valued Hankel transform of complex-valued function $f(x)$, where ξ restricted to the real positive axis. Then for $\mathrm{Re}(\mu_1) \geq -\frac{1}{2}$ by Theorem 9 we have

$$f_1(x) = \lim_{r\to\infty}\int_0^r F_1(\xi)\sqrt{x\xi}J_{\mu_1}(x\xi)d\xi. \tag{14}$$

Similarly, let $F_2(\xi) = H_{\mu_2}\{f_2(x)\}$, $f_2 \in J_{\mu_2}^{'}(\sigma)$. Then for $\mathrm{Re}(\mu_2) \geq -\frac{1}{2}$,

$$f_2(x) = \lim_{r\to\infty}\int_0^r F_2(\xi)\sqrt{x\xi}J_{\mu_2}(x\xi)d\xi. \tag{15}$$

Taking the linear combination of (14) and (15) with e_1 and e_2 respectively, we have

$$f_1(x)e_1 + f_2(x)e_2 = \lim_{r\to\infty}\left[\left(\int_0^r F_1(\xi)\sqrt{x\xi}J_{\mu_1}(x\xi)d\xi\right)e_1 + \left(\int_0^r F_2(\xi)\sqrt{x\xi}J_{\mu_2}(x\xi)d\xi\right)e_2\right]$$

$$f(x) = \lim_{r\to\infty}\int_0^r \{F_1(\xi)e_1 + F_2(\xi)e_2\}\sqrt{x\xi}J_{(\mu_1e_1+\mu_2e_2)}(x\xi)d\xi$$

$$f(x) = \lim_{r\to\infty}\int_0^r F(\xi)\sqrt{x\xi}J_\mu(x\xi)d\xi$$

(where $f(x) = f_1(x)e_1 + f_2(x)e_2$, $\mu = \mu_1e_1 + \mu_2e_2$ and $F(\xi) = F_1(\xi)e_1 + F_2(\xi)e_2$) which complete our proof. □

Following is the illustration to find bicomplex Hankel transform of a bicomplex-valued function.

Example 1. *If $F(\xi) = H_\mu\{f(x);\, \xi\}$, $\xi \in \Omega$ be the bicomplex Hankel transform, then show that*

$$H_\mu\left\{x^{\mu-\frac{1}{2}}e^{-\eta x};\, \xi\right\} = \frac{2^\mu\Gamma\left(\mu+\frac{1}{2}\right)}{\sqrt{\pi}(\xi^2+\eta^2)^{\mu+\frac{1}{2}}},$$

where $Re(P_1:\eta) > 0$, $Re(P_2:\eta) > 0$, $Re(\mu) > -\frac{1}{2} + |Im_j(\mu)|$ and Ω defined in (6).

Solution. By the definition of bicomplex Hankel transform, we have

$$H_\mu\left\{x^{\mu-\frac{1}{2}}e^{-\eta x};\, \xi\right\} = \int_0^\infty x^{\mu-\frac{1}{2}}e^{-\eta x}\sqrt{x\xi}J_\mu(x\xi)dx$$

$$= \sqrt{\xi}\int_0^\infty e^{-\eta x}x^\mu J_\mu(x\xi)dx$$

$$= \sqrt{\xi}\int_0^\infty e^{-\eta x}x^\mu\sum_{r=0}^\infty \frac{(-1)^r}{r!\Gamma(\mu+r+1)}\left(\frac{x\xi}{2}\right)^{\mu+2r}dx$$

$$= \sqrt{\xi}\sum_{r=0}^\infty \frac{(-1)^r}{r!\Gamma(\mu+r+1)}\left(\frac{\xi}{2}\right)^{\mu+2r}\int_0^\infty e^{-\eta x}x^{2\mu+2r}dx$$

$$= \frac{\xi^{\mu+\frac{1}{2}}}{2^\mu}\sum_{r=0}^\infty \frac{(-1)^r}{r!\Gamma(\mu+r+1)}\left(\frac{\xi}{2}\right)^{2r}\frac{\Gamma(2\mu+2r+1)}{\eta^{2\mu+2r+1}}$$

Applying Duplication formula for Gamma function [13, p. 24]

$$= \frac{\xi^{\mu+\frac{1}{2}}}{2^\mu\eta^{2\mu+1}}\sum_{r=0}^\infty \frac{(-1)^r}{r!}\left(\frac{\xi}{2\eta}\right)^{2r}\frac{\Gamma(\mu+r+\frac{1}{2})}{\sqrt{\pi}}2^{2\mu+2r}$$

$$= \frac{2^\mu\Gamma(\mu+\frac{1}{2})\xi^{\mu+\frac{1}{2}}}{\sqrt{\pi}\eta^{2\mu+1}}\sum_{r=0}^\infty \frac{(-1)^r}{r!}\left(\mu+\frac{1}{2}\right)_r\left(\frac{\xi^2}{a^2}\right)^r$$

$$= \frac{2^\mu\Gamma(\mu+\frac{1}{2})\xi^{\mu+\frac{1}{2}}}{\sqrt{\pi}\eta^{2\mu+1}}\left(1+\frac{\xi^2}{\eta^2}\right)^{-\mu-\frac{1}{2}}$$

$$= \frac{2^\mu\Gamma(\mu+\frac{1}{2})\xi^{\mu+\frac{1}{2}}}{\sqrt{\pi}(\xi^2+\eta^2)^{\mu+\frac{1}{2}}}.$$

Table 1. Bicomplex Hankel transform of some functions

S.No.	$f(x)$	Order μ	Bicomplex Hankel Transform $F(\xi)$	Region of Convergence
1.	$x^{-\frac{1}{2}}$	$\mu = 0$	$\xi^{-\frac{1}{2}}$	$\xi \in \Omega$
2.	$\begin{cases} x^{\mu+\frac{1}{2}}, & 0 < x < a \\ 0, & x > a \end{cases}$	$\mathrm{Re}(\mu) \geq -\frac{1}{2} + \lvert\mathrm{Im}_j(\mu)\rvert$	$\frac{a^{\mu+1}}{\sqrt{\xi}} J_{\mu+1}(a\xi)$	$\xi \in \Omega$
3.	$x^{\frac{1}{2}}(a^2+x^2)^{-\frac{1}{2}}$	$\mu = 0$	$\xi^{-\frac{1}{2}} e^{-a\xi}$	$\xi \in \Omega$, $\mathrm{Re}(P_1 : a) > 0$, $\mathrm{Re}(P_2 : a) > 0$
4.	$x^{\frac{1}{2}}(x^2+a^2)$	$\mu = 0$	$\frac{\xi^{\frac{1}{2}}}{a} e^{-a\xi}$	$\xi \in \Omega$, $\mathrm{Re}(P_1 : a) > 0$, $\mathrm{Re}(P_2 : a) > 0$
5.	$x^{-\frac{1}{2}} e^{-ax}$	$\mu = 0$	$\xi^{\frac{1}{2}}(\xi^2+a^2)^{-\frac{1}{2}}$	$\xi \in \Omega$, $\mathrm{Re}(P_1 : a) > 0$, $\mathrm{Re}(P_2 : a) > 0$

6. An Operational Calculus

$y = x^{\frac{1}{2}} J_\mu(x)$ satisfies the following differential equation [17, p. 158]

$$x^2 \frac{d^2y}{dx^2} + \left[x^2 - \left(\mu^2 - \frac{1}{4}\right)\right] y = 0. \tag{16}$$

Let $\Delta_\mu \equiv x^{-\mu-\frac{1}{2}} D x^{2\mu+1} D x^{-\frac{1}{2}-\mu}$, Then

$$\begin{aligned}
\Delta_\mu(f) =& x^{-\mu-\frac{1}{2}} D x^{2\mu+1} D x^{-\frac{1}{2}-\mu} f(x) \\
=& x^{-\mu-\frac{1}{2}} \left\{(2\mu+1)x^{2\mu+1} D x^{-\mu-\frac{1}{2}} f(x) + x^{2\mu+1} D^2 \left[x^{-\mu-\frac{1}{2}} f(x)\right]\right\} \\
=& x^{-\mu-\frac{1}{2}} \left\{(2\mu+1)x^{2\mu+1}\left(-\mu-\frac{1}{2}\right) x^{-\mu-\frac{3}{2}} f(x) + (2\mu+1)x^{2\mu+1} x^{-\mu-\frac{1}{2}} f^{'}(x) \right. \\
&+ x^{2\mu+1} \left[\left(-\mu-\frac{1}{2}\right)\left(-\mu-\frac{3}{2}\right) x^{-\mu-\frac{5}{2}} f(x) + 2\left(-\mu-\frac{1}{2}\right) x^{-\mu-\frac{3}{2}} f^{'}(x) \right. \\
&\left.\left. + x^{-\mu-\frac{1}{2}} f^{''}(x)\right]\right\} \\
=& f^{''}(x) + x^{-2}\left(\frac{1}{4} - \mu^2\right) f(x).
\end{aligned}$$

Therefore,

$$\Delta_\mu \equiv x^{-\mu-\frac{1}{2}} D x^{2\mu+1} D x^{-\frac{1}{2}-\mu} \equiv D_x^2 + x^{-2}\left(\frac{1}{4} - \mu^2\right). \tag{17}$$

The operator satisfies (see Koh and Zemanian [9, p. 951])

$$\Delta_\mu^k \left[\sqrt{xs_1} J_\mu(xs_1)\right] = (-1)^k s_1^{2k} J_\mu(xs_1), \quad s_1 \in C_1 \tag{18}$$

Similarly,

$$\Delta_\mu^k \left[\sqrt{xs_2} J_\mu(xs_2)\right] = (-1)^k s_2^{2k} J_\mu(xs_2), \quad s_2 \in C_1 \tag{19}$$

By taking linear combination of (18) and (19) w.r.t. e_1 and e_2 respectively, we have

$$\begin{aligned}\Delta_\mu^k\left[\sqrt{xs_1}J_\mu(xs_1)\right]e_1+\Delta_\mu^k\left[\sqrt{xs_2}J_\mu(xs_2)\right]e_2 &= \left((-1)^k s_1^{2k}\sqrt{xs_1}J_\mu(xs_1)\right)e_1\\ &\quad+\left((-1)^k s_2^{2k}\sqrt{xs_2}J_\mu(xs_2)\right)e_2\\ \Delta_\mu^k\left[\sqrt{x\xi}J_\mu(x\xi)\right] &= (-1)^k\xi^{2k}\sqrt{x\xi}J_\mu(x\xi), \quad (\text{where } \xi=s_1e_1+s_2e_2\in C_2). \end{aligned} \tag{20}$$

Now we define the operator

$$\Delta_\mu : \mathcal{J}_\mu(\sigma)\to\mathcal{J}_\mu(\sigma)$$

by the relation

$$\langle \Delta_\mu f(x),\phi(x)\rangle = \langle f(x),\Delta_\mu\phi(x)\rangle\,, \quad \forall\ f\in\mathcal{J}_\mu^{'}(\sigma),\ \phi\in\mathcal{J}_\mu(\sigma) \tag{21}$$

From (20) and (21), we get

$$\left\langle \Delta_\mu^k f(x),\sqrt{x\xi}J_\mu(x\xi)\right\rangle = (-1)^k\xi^{2k}\left\langle f(x),\sqrt{x\xi}J_\mu(x\xi)\right\rangle. \tag{22}$$

Therefore, bicomplex Hankel transform of $\Delta_\mu^k f(x)$ is

$$\begin{aligned}H_\mu\left\{\Delta_\mu^k f(x)\right\} &= \left\langle \Delta_\mu^k f(x),\sqrt{x\xi}J_\mu(x\xi)\right\rangle\\ &= (-1)^k\xi^{2k}\left\langle f(x),\sqrt{x\xi}J_\mu(x\xi)\right\rangle, \qquad [\text{Using (22)}]\\ &= (-1)^k\xi^{2k}H_\mu\left\{f(x)\right\}\\ &= (-1)^k\xi^{2k}F(\xi). \end{aligned} \tag{23}$$

7. Application

In [11], Malgonde S.P. et al. used generalized Hankel transform to solve the Cauchy problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{2\mu+1}{x}\frac{\partial u}{\partial x}-\frac{\nu^2-\mu^2}{x^2}u=\lambda\frac{\partial u}{\partial t}, \tag{24}$$

with initial condition

$$u(x,t)\to f(x) \text{ in } D^{'}(I), \text{ where } f\in\mathbb{H}_{\mu,\nu}^{'}(\sigma) \text{ for some } \sigma>0 \text{ as } t\to 0^+.$$

In the above equation notations and terminologies are as defined in Zemanian A.H. [19] and I denotes the open interval $(0,\infty)$.

In the similar manner, if Cauchy problem of bicomplex-valued function $u: C_0\to C_2$ is in the form

$$\frac{\partial^2 u}{\partial x^2}-\frac{\mu^2-\frac{1}{4}}{x^2}u=\lambda\frac{\partial u}{\partial t}, \tag{25}$$

with initial condition

$$u(x,t)\to f(x) \text{ in } \mathcal{D}^{'}(I), \text{ where } f\in\mathcal{J}_{\mu,\nu}^{'}(\sigma) \text{ for some } \sigma>0 \text{ as } t\to 0^+.$$

By taking bicomplex Hankel transform of (25) and by using (23) we have

$$\begin{aligned}&-\xi^2U(\xi,t)=\lambda\frac{\partial}{\partial t}U(\xi,t), \quad [\text{where } U(\xi,t)=H_\mu\left\{u(x,t)\right\}]\\ &\frac{\partial}{\partial t}U(\xi,t)+\frac{\xi^2}{\lambda}U(\xi,t)=0.\end{aligned}$$

Therefore, by using initial condition we get

$$U(\xi,t)=F(\xi)e^{-\frac{\xi^2}{\lambda}t}, \quad [\text{where } F(\xi)=H_\mu\left\{f(x)\right\}]\,. \tag{26}$$

By using the bicomplex inverse Hankel transform of (26), we get

$$u(x,t) = \lim_{r\to\infty} \int_0^r F(\xi) e^{-\frac{\xi^2}{\lambda} t} \sqrt{x\xi} J_\mu(x\xi) d\xi \tag{27}$$

which is the solution of equation (25).

In [10], Kong F.N. discussed the application of Hankel transform in the dipole antenna radiation in conductive medium. In [8], Gupta et al. discussed that computation of electromagnetic fields, for one-dimensional layered earth model, requires evaluation of Hankel transform of the electromagnetic kernel function. Motivated by the work of Kong F.N. and Gupta et al. bicomplex Hankel transform can be used in large class of frequency domain and it is advantageous than complex Hankel transform due to the large class of frequency domain.

8. Conclusion

In this paper, we derive bicomplex Hankel transform and it's properties which is a natural extension of the complex Hankel transform [9]. It is applicable in signal processing, solving partial differential equation of bicomplex-valued functions, optics, electromagnetic field theory and other related problems due to large class. Bicomplex numbers being basically four dimensional hypercomplex numbers, provide large class of frequency domain.

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