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ON THE DETERMINATION OF THE EIGENVALUES FOR AIRY FRACTIONAL DIFFERENTIAL EQUATION WITH TURNING POINT

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Abstract. The present paper reports the result of a study on eigenvalue approximation of the Airy fractional differential equation, using a new definition of fractional derivative called conformable fractional derivative.

We have tried to present the approximate solution of the eigenvalue of Airy fractional differential equation (AFDE) on the right half-line and the left half-line with a turning point through applying the Adomian decomposition method. All numerical calculations in this manuscript were performed on a PC applying some programs written in *Mathematica*.

1. Introduction

The Airy differential equation was named after by George Biddell Airy (1801-1892), who was particularly involved in optics; for this reason, he was also interested in the calculation of light intensity in the neighborhood of a caustic surface (see [1, 2]).

A number of scientist have acknowledged the significance of the Airy equation over the world as it constitutes a classical equation of mathematical physics. Airy equation has various applications in this particular field. Its applications includes modeling the diffraction of light and optic problems, though it is not restricted to this area [3, 4].

The underlying basis of the Airy differential equation was originally the form of the intensity near a directional caustic, such as a rainbow [5]. In fact, this was the problem that resulted in the development of the Airy function [6].

It is also interesting to note that the Airy function is also the solution to Schrödinger's equation for a particle confined within a triangular potential well and for a particle in a one-dimensional constant force field [7]. Moreover, Airy function can be used to find the solutions to a large number of other problems.

Some more applications can be associated with the differential equation

$$y''(x) - \lambda\, r(x)\, y(x) = 0, \quad \lambda \in \mathbb{R},$$

with turning points (a finite number of zeros of $r(x)$ or the points that $r(x)$ changes sign are called turning point) [8]. Turning point idea is a member of the asymptotic theory of ordinary differential equations that rely upon a singular way on a parameter. These points are certain exceptional points in that idea. Differential equations with turning points are extensively used to describe many significant phenomena and dynamic processes in physics, engineering, geophysics, mechanics [9, 10].

Fractional calculus has been one of the most fascinating issues that has attracted the attention of a large group of scholars, particularly in the fields of mathematics and engineering. This is due to the fact that boundary value problems of fractional differential equation can be employed to explain various natural phenomena. Many authors in

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different fields such as physics, fluid flows, electrical networks, and viscoelasticity have attempted to introduce a model for these phenomena through using fractional differential equation [11, 12, 13, 14, 15]. Interested readers can check other books and papers in the related literature to get further information about fractional calculus [13, 14].

The well-known Airy differential equation with integer derivatives has gradually developed over two centuries as an interesting and important field of research due to its significance in many areas of science, mathematics, and engineering [16].

In this work, for the first time, we introduced AFDE and found eigenvalues with the turning point

$$\begin{gathered} D^{\alpha} y(x) - \lambda\, x\, y(x) = 0, \quad 1 < \alpha \leq 2 \\ y(0) = A, \quad y'(0) = B, \ -\infty < x, y < +\infty \end{gathered} \tag{1}$$

in which $x = 0$ is turning point, $\lambda \in \mathbb{R}$ and D^{α} signifies conformable fractional derivative operator of order α.

2. Preliminaries

In this part of the paper, we present a definition of the left and right (conformable) fractional derivatives and fractional integrals of higher orders and then the action of fractional derivatives and integrals to each other are presented [17].

Definition 1. The (left) fractional derivative starting from a of a function $f : [a, \infty)$ of order $0 < \alpha \leq 1$, is defined by

$$D_a^{\alpha} f(x) = \lim_{\epsilon \longrightarrow 0} \frac{f\left(x + \epsilon (x-a)^{1-\alpha}\right) - f(x)}{\epsilon}. \tag{2}$$

The (right) fractional derivative of order $0 < \alpha \leq 1$ terminating at b of f is defined by

$$_bD^{\alpha} f(x) = -\lim_{\epsilon \longrightarrow 0} \frac{f\left(x + \epsilon (b-x)^{1-\alpha}\right) - f(x)}{\epsilon}. \tag{3}$$

Notation 1. If f is differentiable, then $D_a^{\alpha} f(x) = (x-a)^{1-\alpha} f'(x)$ and $_bD^{\alpha} f(x) = -(b-x)^{1-\alpha} f'(x)$.

Notation 2. We have $I_a^{\alpha} f(x) = \int_a^x f(t)\, d^{\alpha}(t,a) = \int_a^x (t-a)^{\alpha-1} f(t)\, dt$.

Likewise, in the right case we have $_bI^{\alpha} f(x) = \int_x^b f(t)\, d^{\alpha}(b,t) = \int_x^b (t-a)^{\alpha-1} f(t)\, dt$.

For $0 < \alpha \leq 1$, the operators I_a^{α} and $_bI^{\alpha}$ are called conformable left and right fractional integrals, respectively.

Definition 2. Suppose $\alpha \in (n\ ,\ n+1]$, $\forall n \geq 1$ and set $\beta = \alpha - n$. Then, the (left) fractional derivative starting from a of a function $f : [\ a,\ \infty\ )$ of order α, where $f^{(n)}(x)$ exists, is defined by

$$D_a^{\alpha} f(x) = D_a^{\beta}\left(f^{(n)}(x)\right). \tag{4}$$

When $a = 0$, we write D^{α}.

The (right) fractional derivative of order α terminating at b of f is defined by

$$_bD^{\alpha} f(x) = (-1)^{n+1}\ _bD^{\beta}\left(f^{(n)}(x)\right). \tag{5}$$

Note that if $\alpha = n+1$, then $\beta = 1$ and the fractional derivative of f converts to $f^{(n+1)}(x)$. when $n = 0$ (or $\alpha \in (0,1)$), then $\beta = \alpha$ and the definition equals with those in Definition (2).

Definition 3. Assume $\alpha \in (n\ ,\ n+1]$, then the left fractional integral starting at a if order α is defined as

$$I_a^{\alpha} f(x) = I_{n+1}^{\alpha}\left((x-a)^{\beta-1} f\right) = \frac{1}{n!}\int_a^x (x-t)^n\, (t-a)^{\beta-1}\, f(t)\, dt. \tag{6}$$

Definition 4. Suppose $\alpha \in (n\ ,\ n+1]$, then the left fractional integral starting at a if order α is defined as

$$ {}_bI^{\alpha} f(x) = I^{\alpha}_{n+1}\left((b-x)^{\beta-1} f\right) = \frac{1}{n!}\int_x^b (t-x)^n\,(b-t)^{\beta-1}\, f(t)\,dt. \tag{7}$$

Proposition 1. [18] Let $\alpha \in (n\ ,\ n+1]$ and $f:[\,a,\infty) \longrightarrow \mathbb{R}$ be $(n+1)$ times differentiable for $x>a$. Then, for all $x>a$, we have

$$ I^{\alpha}_a\, D^{\alpha}_a f(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\,(x-a)^k\,. \tag{8}$$

Proposition 2. [18] Consider $\alpha \in (n\ ,\ n+1]$ and $f:(-\infty,\,b\,] \longrightarrow \mathbb{R}$ be $(n+1)$ times differentiable for $b<x$. Then, for all $x>a$, we have

$$ {}_bI^{\alpha}\, {}_bD^{\alpha} f(x) = f(x) - \sum_{k=0}^{n} \frac{(-1)^k f^{(k)}(b)}{k!}\,(b-x)^k\,. \tag{9}$$

3. Numerical method of solution

In this section, for the purpose of convenience, consider AFDE again

$$ \begin{gathered} D^{\alpha}y(x) - \lambda\, x\, y(x) = 0, \qquad 1<\alpha\le 2 \\ y(0)=A, \quad y'(0)=B,\ -\infty < x, y < +\infty, \end{gathered} \tag{10}$$

in which $x=0$ is the turning point, $\lambda \in \mathbb{R}$ and D^{α} signifies conformable fractional derivative operator of order α. In order to obtain a solution for this equation, we firstly convert the AFDE in (10) into the following form

$$ \begin{gathered} I^{\alpha}\, D^{\alpha}y(x) - \lambda I^{\alpha}\, x\, y(x) = 0, \qquad 1<\alpha\le 2 \\ y(0)=A, \quad y'(0)=B,\ -\infty < x, y < +\infty. \end{gathered} \tag{11}$$

If $x>0$, we consider

$$ \begin{gathered} I^{\alpha}_0\, D^{\alpha}_0 y(x) + \lambda\ I^{\alpha}_0\,[x\, y(x)] = 0, \qquad 1<\alpha\le 2 \\ y(0)=A, \quad y'(0)=B,\ y:(0,\,+\infty)\to\mathbb{R}. \end{gathered} \tag{12}$$

If $x<0$, we consider

$$ \begin{gathered} {}_0I^{\alpha}\, {}_0D^{\alpha} y(x) - \lambda\ {}_0I^{\alpha}\,[x\, y(x)] = 0, \qquad 1<\alpha\le 2 \\ y(0)=A, \quad y'(0)=B,\ y:(-\infty,\,0)\to\mathbb{R}. \end{gathered} \tag{13}$$

According to equations (6), (8) and (12), we have the following integral differential equation

$$ y(x) - y(0) - x\, y'(0) = \lambda \int_0^x (x-t)\, t^{\alpha-1}\, y(t)\, dt. \tag{14}$$

Thus, according to equations (7), (9) and (13), we have the following Volterra Integro-Differential equations

$$ y(x) - y(0) + x\, y'(0) = (-1)^{\alpha-1}\,\lambda \int_x^0 (t-x)\, t^{\alpha-1}\, y(t)\, dt. \tag{15}$$

The approximate solution for Volterra Integro-Differential equations was obtained by applying Adomian decomposition method(ADM) implemented in *Mathematica* software.

4. Numerical results

In this section, we focus on the approximate computation of the eigenvalues with the turning point (10) using the ADM. The general solution of equation (10) on right half-line is given by

$$y(x) = y(0) + x\, y'(0) + \lambda \int_0^x (x-t)\, t^{\alpha-1}\, y(t)\, dt, \qquad x \in (0,\ 2) \tag{16}$$
$$y(0) = y'(0) = 1.$$

Then, the purpose of the first 5 sentences as an estimation of the solution for Eq.(16) with $\alpha = 3/2$ is

$$y(x) = 1 + x + \frac{4}{105} x^{5/2}(7+3x)\lambda + \frac{1}{525} x^5 (7+2x)\lambda^2 + \frac{4x^{15/2}(119+26x)\lambda^3}{1740375} + \frac{2x^{10}(1309+234x)\lambda^4}{861485625} + \frac{8x^{25/2}(3927+598x)\lambda^5}{1486062703125}. \tag{17}$$

Fig.1 displays the eigenfunction $y(x)$ corresponding to the third eigenvalue (λ_4) as α

Table 1. The numerical values of the eigenvalues of 17 for different values of α

λ_k	$\alpha = 1.85$	$\alpha = 1.9$	$\alpha = 1.95$	$\alpha = 2$
1	-40.767	-42.3165	-43.8936	-45.4981
2	$-35.936 - 54.829i$	$-37.237 - 56.900i$	$-38.5613 - 59.008i$	$-39.9088 - 61.152i$
3	$-35.936 + 54.829i$	$-37.237 + 56.900i$	$-38.5613 + 59.008i$	$-39.9088 + 61.152i$
4	-10.086	-10.5024	-10.926	-11.3569

varies from 1.85 to 2.

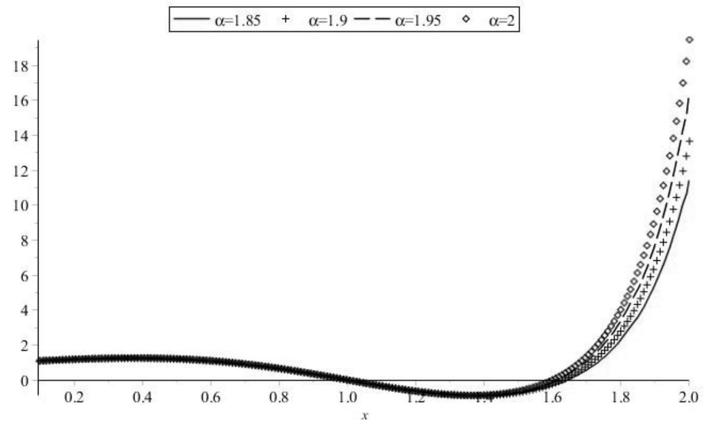

Figure 1. The approximate eigenfunction corresponding to the forth eigenvalue (λ_4) in Eq.(17)

5. Conclusion

In this paper, we have presented a simple and efficient numerically computable eigenvalues of Airy fractional differential equation. We have shown that the proposed method is simple and computational. The Adomian decomposition method proved to be very competent in calculating the eigenvalues of the equation suggested in this paper.

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