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THE GENERALIZED (s,t)-FIBONACCI AND FIBONACCI MATRIX SEQUENCES

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Abstract. In this paper, we study the generalizations of the (s,t)-Fibonacci and Lucas sequences and the (s,t)-Fibonacci and Lucas matrix sequences. We present relationship between the (s,t)-Fibonacci matrix and generalized Fibonacci matrix sequences. Binet's formula for the generalized (s,t)-Fibonacci matrix sequence is derived. We establish several identites for the generalized (s,t)-Fibonacci and Fibonacci matrix sequence. We give some partial sum formulas for the generalized (s,t)-Fibonacci and Fibonacci matrix sequence. Also, we find out relationship between the (s,t)-Fibonacci matrix sequence and the famous Bernoulli numbers.

1. Introduction

The literature includes many papers dealing with the special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Mersenne, Fermat, Padovan, Perrin (see, e.g., [1-17] and the references cited therein). The books written by Hoggat [6], Koshy [10] and Vajda [13] collects and classifies many results dealing with the these number sequences , most of them are obtained quite recently. These numbers have been generalized in many ways [1-5, 7, 9, 11, 14, 15].

Falcón and Plaza [4] defined the following one-parameter generalization of the Fibonacci sequence in which it generalizes both the classic Fibonacci sequence and the Pell sequence, and deduced many properties of these numbers from elementary matrix algebra.

Definition 1 ([4]). *For any integer number $k \geq 1$, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1$$

with initial conditions

$$F_{k,0} = 0; \quad F_{k,1} = 1.$$

However, two-parameters generalizations of the Fibonacci and Lucas sequences, respectively, are given in [1] and [2] by

Definition 2 ([1]). *For any integer numbers $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$; the nth (s,t)-Fibonacci sequence, say $\{F_n(s,t)\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$F_{n+1}(s,t) = sF_n(s,t) + tF_{n-1}(s,t) \quad \text{for } n \geq 1, \tag{1}$$

with $F_0(s,t) = 0$, $F_1(s,t) = 1$.

Definition 3 ([2]). *For any integer numbers $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$; the nth (s,t)-Lucas sequence, say $\{L_n(s,t)\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$L_{n+1}(s,t) = sL_n(s,t) + tL_{n-1}(s,t) \quad \text{for } n \geq 1,$$

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with $L_0(s,t) = 2, \; L_1(s,t) = s.$

The following table summarizes special cases of $F_n(s,t)$ and $L_n(s,t)$:

(s,t)	F_n	L_n
$(1,1)$	Fibonacci numbers	Lucas numbers
$(2,1)$	Pell numbers	Pell-Lucas numbers
$(1,2)$	Jacobsthal numbers	Jacobsthal-Lucas numbers
$(3,-2)$	Mersenne numbers	Fermat numbers

On the other hand, the matrix sequences that concern this special number sequences have taken so much interest [1, 2, 15, 16,]. In [1] and [2], it is defined new matrix generalizations of the Fibonacci and Lucas sequences such that:

Definition 4 ([1])**.** *For any integer numbers $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$; the nth (s,t)-Fibonacci matrix sequence, say $\{\mathfrak{F}_n(s,t)\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$\mathfrak{F}_{n+1}(s,t) = s\mathfrak{F}_n(s,t) + t\mathfrak{F}_{n-1}(s,t) \quad \textit{for } n \geq 1, \tag{2}$$

with $\mathfrak{F}_0(s,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\mathfrak{F}_1(s,t) = \begin{pmatrix} s & 1 \\ t & 0 \end{pmatrix}$.

Definition 5 ([2])**.** *For any integer numbers $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$; the nth (s,t)-Lucas matrix sequence, say $\{\mathfrak{L}_n(s,t)\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$\mathfrak{L}_{n+1}(s,t) = s\mathfrak{L}_n(s,t) + t\mathfrak{L}_{n-1}(s,t) \quad \textit{for } n \geq 1,$$

with $\mathfrak{L}_0(s,t) = \begin{pmatrix} s & 2 \\ 2t & -s \end{pmatrix}, \; \mathfrak{L}_1(s,t) = \begin{pmatrix} s^2+2t & s \\ st & 2t \end{pmatrix}$.

They showed some properties of these matrix sequences using essentially a matrix approach in [1] and [2]. Moreover, in that papers, various identities based on the matrix representations has been derived for the nth (s,t)-Fibonacci and Lucas sequences.

Our goal in this paper is to generalize several results about the (s,t)-Fibonacci and Lucas matrix sequences and (s,t)-number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Mersenne and Fermat numbers. The purpose of this paper is to obtain some propertis of the generalized (s,t)-Fibonacci matrix sequence and to demonstrate that some properties of the generalized (s,t)-Fibonacci numbers and the matrices defined in [1] and [2] are valid for a more general sequence of matrices, introduced in Section 2.

In the rest of the paper:

- **i-):** the matrix sequence in which it generalizes the (s,t)-Fibonacci and Lucas matrix sequences will be defined.
- **ii-):** the relationship between these matrix sequences will be presented.
- **iii-):** by giving the the Binet formulas and summation formulas over these new matrix sequence, some fundamental properties on the generalized (s,t)-Fibonacci numbers will be obtained.
- **iv-):** using Binet's formula for the generalized (s,t)-Fibonacci matrix sequence, the relationship between the (s,t)-generalized Fibonacci matrix sequence and the famous Bernoulli numbers will be investigated.

Therefore, by the generalized (s,t)-Fibonacci matrix sequence defined in Section 2, we have a great opportunity to obtain some new properties over the (s,t)-generalized Fibonacci numbers defined in Section 2.

2. The generalized (s,t)-Fibonacci matrix sequence

In this section, we will mainly focus on the (s,t)-generalized Fibonacci sequence and the generalized (s,t)-Fibonacci matrix sequence to get some important results.

Definition 6. *For any integer numbers $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$; the nth (s,t)-generalized Fibonacci sequence, say $\{G_n(s,t)\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$G_{n+1}(s,t) = sG_n(s,t) + tG_{n-1}(s,t) \quad \text{for } n \geq 1, \tag{3}$$

where $G_0(s,t) = a_0$ and $G_1(s,t) = a_1$, with $a_0, a_1 \in \mathbb{R}$.

In this paper, we will denote simply G_n, $\mathcal{F}_n$ and $\mathcal{L}_n$ instead of $G_n(s,t), \mathcal{F}_n(s,t)$ and $\mathcal{L}_n(s,t)$, respectively.

Definition 7. *For any integer numbers $s > 0$ and $t \neq 0$ with $s^2 + 4t > 0$; the nth generalized (s,t)-Fibonacci matrix sequence, say $\{\Re_n(s,t)\}_{n\in\mathbb{N}}$ is defined recurrently by*

$$\Re_{n+1}(s,t) = s\Re_n(s,t) + t\Re_{n-1}(s,t), \ n \geq 1, \tag{4}$$

with $\Re_0(s,t) = \begin{pmatrix} a_1 & a_0 \\ ta_0 & a_1 - sa_0 \end{pmatrix}$ *and* $\Re_1(s,t) = \begin{pmatrix} sa_1 + ta_0 & a_1 \\ ta_1 & ta_0 \end{pmatrix}$.

Again, in this paper we denote $\Re_n$ instead of $\Re_n(s,t)$.

From the recurrence relations in (1) and (2), Civciv and Turkmen obtained

$$\mathcal{F}_n = \begin{pmatrix} F_{n+1} & F_n \\ tF_n & tF_{n-1} \end{pmatrix} \text{ [1]}. \tag{5}$$

Similarly, from the recurrence relations in (3) and (4), we obtain, for $n \geq 0$,

$$\Re_n = \begin{pmatrix} G_{n+1} & G_n \\ tG_n & tG_{n-1} \end{pmatrix}. \tag{6}$$

Theorem 1 ([1])**.** *$\mathfrak{F}_{m+n} = \mathfrak{F}_m\mathfrak{F}_n$ for any integers $m, n \geq 0$.*

Lemma 1. *For $n \geq 0$ holds:*

$$\Re_{n+1} = \Re_1\mathfrak{F}_n. \tag{7}$$

Proof. We use the second principle of finite induction on n to prove this lemma. When $n = 0$, since $\mathcal{F}_0 = I$, the result is true. Let $n = 1$. Then the lemma yields

$$\Re_2 = \Re_1\mathcal{F}_1,$$

which defines the matrix $\Re_2$. Now assume that $\Re_{n+1} = \Re_1\mathcal{F}_n$ for $n \leq N$. Then

$$\begin{aligned} \Re_1\mathcal{F}_{N+1} &= \Re_1\mathcal{F}_N\mathcal{F}_1 \text{ by Theorem 1} \\ &= \Re_{N+1}\mathcal{F}_1 \text{ by our assumption} \\ &= \begin{pmatrix} G_{n+2} & G_{n+1} \\ tG_{n+1} & tG_n \end{pmatrix}\begin{pmatrix} s & 1 \\ t & 0 \end{pmatrix} \\ &= \Re_{N+2}. \end{aligned}$$

Thus it is true for every nonnegative integer n. □

2.1. **Binet' formula for the generalized (s,t)-Fibonacci matrix sequence.** Binet's formulas are well known in the Fibonacci and Lucas numbers theory [10]. Binet's formulas for the recurrence relation (1) is widely used for simulation of various physical and biological phenomena.

In our case, Binet's formula allows us to express the (s,t)-generalized Fibonacci matrix sequences in function of the roots α and β of the characteristic equation $x^2 = sx + t$, associated to the recurrence relation (1), or (3).

Theorem 2 ([1])**.**

$$\mathfrak{F}_n = \left(\frac{\mathfrak{F}_1 - \beta\mathfrak{F}_0}{\alpha-\beta}\right)\alpha^n - \left(\frac{\mathfrak{F}_1 - \alpha\mathfrak{F}_0}{\alpha-\beta}\right)\beta^n,\ n \geq 0, \tag{8}$$

Corollary 1 ([1])**.** *The nth (s,t)-Fibonacci number is given by*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha-\beta}. \tag{9}$$

Corollary 2. *The nth generalized (s,t)-Fibonacci matrix sequence is given by*

$$\Re_{n+1} = \left(\frac{\Re_2 - \beta\Re_1}{\alpha-\beta}\right)\alpha^n - \left(\frac{\Re_2 - \alpha\Re_1}{\alpha-\beta}\right)\beta^n,\ n \geq 0. \tag{10}$$

Proof. From the formula (8) and Lemma 1, the proof is completed. □

Corollary 3. *The nth (s,t)-generalized Fibonacci number is given by*

$$G_n = \frac{a_1 - a_0\beta}{\alpha-\beta}\alpha^n - \frac{a_1 - a_0\alpha}{\alpha-\beta}\beta^n. \tag{11}$$

Proof. The proof of this corollary is trial from Corollary 2. □

2.2. **Identities for the generalized (s,t)-Fibonacci matrix sequence using Binet's formula.** In this section, we derive several identites for the generalized (s,t)-Fibonacci and Fibonacci matrix sequences by simple matrix algebra.

Corollary 4. $\Re_{m+n+1} = \Re_{m+1}\mathfrak{F}_n$ *for any integers* $m, n \geq 0$.

Proof. The proof is obvious from Theorem 1 and Lemma 1. □

Corollary 5.

$$G_{m+n+2} = G_{m+2}F_{n+1} + tG_{m+1}F_n.$$

Proof. From (5), (6) and Corollary 4, we have

$$\begin{pmatrix} G_{m+n+2} & G_{m+n+1} \\ tG_{m+n+1} & tG_{m+n} \end{pmatrix} = \begin{pmatrix} G_{m+2} & G_{m+1} \\ tG_{m+1} & tG_m \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ tF_n & tF_{n-1} \end{pmatrix}.$$

From where, we obtain the result. □

For $s = t = 1$, we obtain the convolution formula of Honsberger.

Theorem 3.

$$\mathfrak{F}_{n-r}\mathfrak{F}_{n+r} = \mathfrak{F}_n^2 \tag{12}$$

Proof. Let $X = \mathcal{F}_1 - \beta\mathcal{F}_0$ and $Y = \mathcal{F}_1 - \alpha\mathcal{F}_0$. By using Eq. (8) and taking into account that $\alpha\beta = -t$ and $YX = XY$ it is obtained

$$\mathcal{F}_{n-r}\mathcal{F}_{n+r} - \mathcal{F}_n^2 = \frac{X\alpha^{n-r} - Y\beta^{n-r}}{\alpha-\beta}\frac{X\alpha^{n+r} - Y\beta^{n+r}}{\alpha-\beta} - \left(\frac{X\alpha^n - Y\beta^n}{\alpha-\beta}\right)^2$$
$$= \frac{X^2\alpha^{2n} - XY\alpha^{n-r}\beta^{n+r} - XY\alpha^{n+r}\beta^{n-r} + Y^2\beta^{2n} - X^2\alpha^{2n} + 2XY\alpha^n\beta^n - Y^2\beta^{2n}}{(\alpha-\beta)^2}$$
$$= \frac{1}{(\alpha-\beta)^2}\left[-(\alpha\beta)^n\left(\frac{\beta}{\alpha}\right)^r XY - (\alpha\beta)^n\left(\frac{\alpha}{\beta}\right)^r XY + 2(\alpha\beta)^n XY\right]$$
$$= \frac{(\alpha\beta)^{n+1}}{(\alpha-\beta)^2}\left[\frac{\alpha^{2r}+\beta^{2r}}{(\alpha\beta)^r} - 2\right]XY$$
$$= (-t)^{n+1-r} F_n^2 XY \quad \text{by Eq. (9)}.$$

Thus, since $XY = 0$, Eq. (12) is proven. □

Corollary 6.

$$F_{2n+1} = F_{n-r+1}F_{n+r+1} + tF_{n-r}F_{n+r}, \tag{13}$$

$$F_{2n} = F_{n-r+1}F_{n+r} + tF_{n-r}F_{n+r-1}. \tag{14}$$

Proof. From Eq. (5) and Theorem 1, we have

$$\mathcal{F}_{n-r}\mathcal{F}_{n+r} = \begin{pmatrix} F_{n-r+1} & F_{n-r} \\ tF_{n-r} & tF_{n-r-1} \end{pmatrix}\begin{pmatrix} F_{n+r+1} & F_{n+r} \\ tF_{n+r} & tF_{n+r-1} \end{pmatrix}$$

and

$$\mathcal{F}_n^2 = \begin{pmatrix} F_{2n+1} & F_{2n} \\ tF_{2n} & tF_{2n-1} \end{pmatrix}.$$

From Theorem 3, since the terms a_{11} and a_{12} of both sides of $\mathcal{F}_{n-r}\mathcal{F}_{n+r} = \mathcal{F}_n^2$ are equal, the Eq. (13) and (14) are obtained. □

Corollary 7.

$$\mathfrak{R}_{n-r+1}\mathfrak{F}_{n+r} = \mathfrak{R}_{n+1}\mathfrak{F}_n.$$

Proof. The proof is trial from Lemma 1 and Theorem 3. □

Corollary 8.

$$G_{n-r+2}F_{n+r+1} + tG_{n-r+1}F_{n+r} = G_{n+2}F_{n+1} + tG_{n+1}F_n, \tag{15}$$

$$G_{n-r+2}F_{n+r} + tG_{n-r+1}F_{n+r-1} = G_{n+2}F_n + tG_{n+1}F_{n-1}. \tag{16}$$

Proof. From Eq. (5) and (6), we get

$$\mathfrak{R}_{n-r+1}\mathcal{F}_{n+r} = \begin{pmatrix} G_{n-r+2} & G_{n-r+1} \\ tG_{n-r+1} & tG_{n-r} \end{pmatrix}\begin{pmatrix} F_{n+r+1} & F_{n+r} \\ tF_{n+r} & tF_{n+r-1} \end{pmatrix}$$

and

$$\mathfrak{R}_{n+1}\mathcal{F}_n = \begin{pmatrix} G_{n+2} & G_{n+1} \\ tG_{n+1} & tG_n \end{pmatrix}\begin{pmatrix} F_{n+1} & F_n \\ tF_n & tF_{n-1} \end{pmatrix}.$$

From Corollary 7, since the terms a_{11} and a_{12} of both sides of $\mathfrak{R}_{n-r+1}\mathcal{F}_{n+r} = \mathfrak{R}_{n+1}\mathcal{F}_n$ are equal, the Eq. □

It is known that the limit of the ratio of two adjacent Fibonacci numbers (as well as the adjacent Lucas numbers and the adjacent numbers of any numerical sequence that is given by the recurrence relation (1)) tends to the positive characteristic root, i.e.,

$$\lim_{n\to\infty}\frac{f_n}{f_{n-1}}=\frac{1+\sqrt{5}}{2},$$

where f_n is the nth classical Fibonacci number. The Golden propertion $\frac{1+\sqrt{5}}{2}$ is the positive root of the characteristic equation $x^2 = x + 1$, that is also called the *Golden section equation.*

The Hadamard product, or the Schur product, of two matrices A and B of the same size is defined to be the entrywise product $A \circ B = (a_{ij}b_{ij})$. The Hadamard unit matrix U is such a matrix whose all entries are 1(the size of U being understood) [8]. The matrix A called *Hadamard invertible* if all its entries are non-zero. Then $A^{\circ -1} = \left(a_{ij}^{-1}\right)$ is called as the Hadamard inverse of A [12].

An useful property in the (s,t)-generalized Fibonacci matrix sequences is that the limit of the Hadamard quotient of two consecutive terms is equal to αU, where U is the Hadamard unit matrix.

Theorem 4.

$$\lim_{n\to\infty}\Re_{n+1}\circ\Re_n^{\circ -1}=\alpha U_2, \tag{17}$$

where U_2 is the 2×2 Hadamard unit matrix.

Proof. Let $X = \Re_2 - \beta\Re_1$ and $Y = \Re_2 - \alpha\Re_1$. By using Eq. (10),

$$\begin{aligned}\lim_{n\to\infty}\Re_{n+1}\circ\Re_n^{\circ -1} &= \lim_{n\to\infty}\left(X\alpha^n - Y\beta^n\right)\circ\left(X\alpha^{n-1}-Y\beta^{n-1}\right)^{\circ -1}\\ &= \lim_{n\to\infty}\left(X-\left(\frac{\beta}{\alpha}\right)^nY\right)\circ\left(\frac{1}{\alpha}X-\frac{1}{\beta}\left(\frac{\beta}{\alpha}\right)^nY\right)^{\circ -1},\end{aligned}$$

and taking into account that $\lim_{n\to\infty}\left(\frac{\beta}{\alpha}\right)^n = 0$ since $|\beta| < \alpha$, Eq. (17) is obtained. □

Corollary 9. *For the generalized (s,t)-Fibonacci sequence, $\lim_{n\to\infty}\frac{G_n}{G_{n-1}} = \alpha$ where α is the positive root of the characterist equation $x^2 = sx + t$.*

2.3. **Partial Sums for the generalized (s,t)-Fibonacci matrix sequence using Binet's formula.** In this section, we present partial sum formulas for the generalized (s,t)-Fibonacci and Fibonacci matrix sequences using Binet's formula given for $\mathcal{F}_n$.

Theorem 5. *Let $s + t \neq 1$ and S_n be the sum of the first $(n + 1)$ terms of the (s,t)-Fibonacci matrix sequence, that is $S_n = \sum_{i=0}^{n}\mathfrak{F}_i$. Then,*

$$S_n=\frac{1}{s+t-1}\left[\mathfrak{F}_{n+1}+t\mathfrak{F}_n-\mathfrak{F}_1+(s-1)\,\mathfrak{F}_0\right]. \tag{18}$$

Proof. Let $s + t \neq 1$ and $X = \mathcal{F}_1 - \beta\mathcal{F}_0$ and $Y = \mathcal{F}_1 - \alpha\mathcal{F}_0$. Considering Eq. (8), S_n may be written as

$$S_n=\frac{1}{\alpha-\beta}\sum_{i=0}^{n}\left(X\alpha^i-Y\beta^i\right).$$

Now, by summing up to geometric partial sums $\sum_{i=0}^{n}\alpha^i$, or $\sum_{i=0}^{n}\beta^i$, we obtain

$$S_n=\frac{1}{\alpha-\beta}\left(\frac{\alpha^{n+1}-1}{\alpha-1}X-\frac{\beta^{n+1}-1}{\beta-1}Y\right). \tag{19}$$

From where, after some algebra, we get

$$\begin{aligned}
S_n &= \frac{1}{(\alpha-\beta)(\alpha-1)(\beta-1)}\left[\left(-\alpha^n-\alpha^{n+1}-\beta+1\right)X+\left(\beta^n+\alpha+\beta^{n+1}-1\right)Y\right] \\
&= \frac{1}{s+t-1}\left[\frac{X\alpha^{n+1}-Y\beta^{n+1}}{\alpha-\beta}+t\frac{X\alpha^n-Y\beta^n}{\alpha-\beta}-\frac{X-Y}{\alpha-\beta}+\frac{-\alpha Y+\beta X}{\alpha-\beta}\right] \\
&= \frac{1}{s+t-1}\left[\mathcal{F}_{n+1}+t\mathcal{F}_n-\mathcal{F}_0+\frac{-\alpha\left(\mathcal{F}_1-\alpha\mathcal{F}_0\right)+\beta\left(\mathcal{F}_1-\beta\mathcal{F}_0\right)}{\alpha-\beta}\right] \text{ by } (8) \\
&= \frac{1}{s+t-1}\left[\mathcal{F}_{n+1}+t\mathcal{F}_n-\mathcal{F}_1+(s-1)\mathcal{F}_0\right] \text{ with } \alpha+\beta=s,
\end{aligned}$$

which completed the poof. □

Corollary 10. *Let $s+t\neq 1$. Then,*

$$\sum_{i=0}^{n}\Re_{i+1}=\frac{1}{s+t-1}\left[\Re_{n+2}+t\Re_{n+1}-\Re_2+(s-1)\Re_1\right]. \tag{20}$$

Proof. The proof is trial from Lemma 1 and Theorem 5. □

Corollary 11. *Let $s+t\neq 1$. Then,*

$$\sum_{i=0}^{n}G_i=\frac{1}{(s+t-1)}\left[G_{n+1}+tG_n-a_1+(s-1)a_0\right].$$

Proof. From Eq. (20), by obtaining the term a_{22} of the matrix

$$\frac{1}{s+t-1}\left[\Re_{n+2}+t\Re_{n+1}-\Re_2+(s-1)\Re_1\right]$$

since this term is at the same time $\sum_{i=0}^{n}G_i$ from Eq. (6), what we wanted is obtained. □

By summing up the first $(n+1)$ even terms of the (s,t)-Fibonacci matrix sequence we obtain:

Theorem 6. *Let $(s-t+1)(s+t-1)\neq 0$ and T_n be the sum of the first $(n+1)$ even terms of the (s,t)-Fibonacci matrix sequence, that is $T_n=\sum_{i=0}^{n}\mathfrak{F}_{2i}$. Then,*

$$T_n=\frac{1}{(s-t+1)(s+t-1)}\left[\mathfrak{F}_{2n+2}-t^2\mathfrak{F}_{2n}-s\mathfrak{F}_1+\left(s^2+t-1\right)\mathfrak{F}_0\right].$$

Proof. The proof is similar to the proof of Theorem 5, and we only show an outline of it. Let $(s-t+1)(s+t-1)\neq 0$, $X=\mathcal{F}_1-\beta\mathcal{F}_0$ and $Y=\mathcal{F}_1-\alpha\mathcal{F}_0$ and $T_n=\sum_{i=0}^{n}\mathcal{F}_{2i}$. Replacing $\alpha\leftrightarrow\alpha^2$ and $\beta\leftrightarrow\beta^2$ in Eq. (19), we have

$$T_n=\frac{1}{\alpha-\beta}\left(\frac{\left(\alpha^2\right)^{n+1}-1}{\alpha^2-1}X-\frac{\left(\beta^2\right)^{n+1}-1}{\beta^2-1}Y\right). \tag{21}$$

Since,

$$\left(\alpha^2-1\right)\left(\beta^2-1\right)=(t-1)^2-s^2,$$

from (21) we obtain

$$T_n=\frac{1}{(s-t+1)(s+t-1)}\left[\mathcal{F}_{2n+2}-t^2\mathcal{F}_{2n}-\mathcal{F}_0+\frac{\beta^2X-\alpha^2Y}{\alpha-\beta}\right].$$

From where, we get

$$T_n = \frac{1}{(s-t+1)(s+t-1)}\left[\mathcal{F}_{2n+2} - t^2\mathcal{F}_{2n} - s\mathcal{F}_1 + \left(s^2+t-1\right)\mathcal{F}_0\right],$$

since

$$\frac{\beta^2 X - \alpha^2 Y}{\alpha - \beta} = -s\mathcal{F}_1 + \left(s^2+t\right)\mathcal{F}_0.$$

Thus, the proof is completed. □

Corollary 12. *Let* $(s-t+1)(s+t-1) \neq 0$*. Then,*

$$\sum_{i=0}^{n} \Re_{2i+1} = \frac{1}{(s-t+1)(s+t-1)}\left[\Re_{2n+3} - t^2\Re_{2n+1} - s\Re_2 + \left(s^2+t-1\right)\Re_1\right]. \quad (22)$$

Proof. The proof of corollary is obvious from Lemma 1 and 6. □

Corollary 13. *Let* $(s-t+1)(s+t-1) \neq 0$*. Then,*

$$\sum_{i=0}^{n} G_{2i} = \frac{1}{(s-t+1)(s+t-1)}\left[G_{2n+2} - t^2 G_{2n} - sa_1 + \left(s^2+t-1\right)a_0\right].$$

Proof. From Eq. (22), by obtaining the term a_{22} of the matrix

$$\frac{1}{(s-t+1)(s+t-1)}\left[\Re_{2n+3} + \Re_{2n+1} - s\Re_2 + \left(s^2+t-1\right)\Re_1\right]$$

since this term is at the same time $\sum_{i=0}^{n} G_{2i}$ from Eq. (6), what we wanted is obtained. □

Now, considering Theorem 5 and 6 it is rightly obtained the sum of the first odd terms of the (s,t)-Fibonacci matrix sequence:

Theorem 7. *Let* $(s-t+1)(s+t-1) \neq 0$*. Then,*

$$\sum_{i=0}^{n} \mathfrak{F}_{2i+1} = \frac{1}{(s-t+1)(s+t-1)}\left[\mathfrak{F}_{2n+3} - t^2\mathfrak{F}_{2n+1} + (t-1)\mathfrak{F}_1 - st\mathfrak{F}_0\right].$$

Corollary 14. *Let* $(s-t+1)(s+t-1) \neq 0$*. Then,*

$$\sum_{i=0}^{n} \Re_{2i+2} = \frac{1}{(s-t+1)(s+t-1)}\left[\Re_{2n+4} - t^2\Re_{2n+2} + (t-1)\Re_2 - st\Re_1\right].$$

Corollary 15. *Let* $(s-t+1)(s+t-1) \neq 0$*. Then,*

$$\sum_{i=0}^{n} G_{2i+1} = \frac{1}{(s-t+1)(s+t-1)}\left[G_{2n+3} - t^2 G_{2n+1} + (t-1)a_1 - sta_0\right].$$

In a similar way, many formulas for partial sums of the term of the (s,t)-Fibonacci matrix sequence may be obtained and particularized for different values of s and t.

Theorem 8.

$$\sum_{i=0}^{n} \binom{n}{i} s^i t^{n-i} \mathfrak{F}_i = \mathfrak{F}_{2n}$$

Proof. Let $X=\mathcal{F}_1-\beta\mathcal{F}_0$ and $Y=\mathcal{F}_1-\alpha\mathcal{F}_0$. By Eq. (8), we have

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}s^{i}t^{n-i}\mathcal{F}_i&=\frac{1}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}\left(Xs^{i}t^{n-i}\alpha^{i}-Ys^{i}t^{n-i}\beta^{i}\right)\\&=\frac{1}{\alpha-\beta}\left[X\sum_{i=0}^{n}\binom{n}{i}t^{n-i}\left(s\alpha\right)^{i}-Y\sum_{i=0}^{n}\binom{n}{i}t^{n-i}\left(s\beta\right)^{i}\right]\\&=\frac{1}{\alpha-\beta}\left(X\left(t+\alpha s\right)^{n}-Y\left(t+\beta s\right)^{n}\right)\\&=\frac{X\alpha^{2n}-Y\beta^{2n}}{\alpha-\beta}\text{ since }\alpha^{2}=\alpha s+t\text{ and }\beta^{2}=\beta s+t\\&=\mathcal{F}_{2n}.\end{aligned}$$

Thus, the result is obtained. □

Corollary 16.

$$\sum_{i=0}^{n}\binom{n}{i}s^{i}t^{n-i}\Re_{i+1}=\Re_{2n+1}. \tag{23}$$

Corollary 17.

$$\sum_{i=0}^{n}\binom{n}{i}s^{i}t^{n-i}G_i=G_{2n}.$$

Proof. From Eq. (23), by obtaining the term a_{22} of the matrix $\Re_{2n+1}$ since this term is at the same time $\sum\limits_{i=0}^{n}\binom{n}{i}s^{i}t^{n-i}G_i$ from Eq. (6), we get the result. □

Theorem 9.

$$\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^{i}s^{n-i}\mathfrak{F}_i=F_{n+1}\mathfrak{F}_0-F_n\mathfrak{F}_1. \tag{24}$$

Proof. Let $X=\mathcal{F}_1-\beta\mathcal{F}_0$ and $Y=\mathcal{F}_1-\alpha\mathcal{F}_0$. Then, we have,

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^{i}s^{n-i}\mathcal{F}_i&=\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^{i}s^{n-i}\left(\frac{X\alpha^{i}-Y\beta^{i}}{\alpha-\beta}\right)\text{ by (8)}\\&=\frac{X}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}s^{n-i}\left(-\alpha\right)^{i}-\frac{Y}{\alpha-\beta}\sum_{i=0}^{n}\binom{n}{i}s^{n-i}\left(-\beta\right)^{i}\\&=\frac{X}{\alpha-\beta}\left(s-\alpha\right)^{n}-\frac{Y}{\alpha-\beta}\left(s-\beta\right)^{n}\\&=\frac{X\beta^{n}-Y\alpha^{n}}{\alpha-\beta}\\&=\frac{\mathcal{F}_0\left(\alpha^{n+1}-\beta^{n+1}\right)-\mathcal{F}_1\left(\alpha^{n}-\beta^{n}\right)}{\alpha-\beta}\\&=F_{n+1}\mathcal{F}_0-F_n\mathcal{F}_1,\end{aligned}$$

which completes the proof. □

Corollary 18.

$$\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^{i}s^{n-i}\Re_{i+1}=F_{n+1}\Re_1-F_n\Re_2. \tag{25}$$

Proof. The proof is obvious from Lemma 1 and Theorem 9. □

Corollary 19.

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}s^{n-i}G_i = a_0F_{n+1} - a_1F_n.$$

Proof. From Eq. (25), by obtaining the term a_{22} of the matrix $F_{n+1}\Re_1 - F_n\Re_2$ since this term is at the same time $\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}s^{n-i}G_i$, we have the result. □

2.4. **More on identity for the (s,t)-Fibonacci matrix sequence using Binet's formula.** In this section, we use Binet's formula to study the relationship between the (s,t)-generalized Fibonacci matrix sequence and the famous Bernoulli numbers.

Now, let $X = \mathcal{F}_1 - \beta\mathcal{F}_0$ and $Y = \mathcal{F}_1 - \alpha\mathcal{F}_0$. From Eq. (8) we easily deduce that the exponential generating function $g(x)$ is

$$g(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{F}_n x^n = \sum_{n=0}^{\infty}\frac{\alpha^n X - \beta^n Y}{(\alpha-\beta)\,n!}x^n. \tag{26}$$

That is,

$$g(x) = \frac{e^{\alpha x}X - e^{\beta x}Y}{\alpha-\beta} = \frac{e^{\beta x}\left(e^{x\left(\sqrt{s^2+4t}\right)}X - Y\right)}{\sqrt{s^2+4t}},\ \text{since } \alpha-\beta = \sqrt{s^2+4t}.$$

Therefore, we get

$$\frac{e^{\beta x}\left(e^{x\left(\sqrt{s^2+4t}\right)}X - Y\right)}{\left(e^{x\left(\sqrt{s^2+4t}\right)} - 1\right)} = \frac{g(x)}{x}.\frac{x\sqrt{s^2+4t}}{\left(e^{x\left(\sqrt{s^2+4t}\right)} - 1\right)},$$

or

$$e^{\beta x}\left(Y - e^{x\left(\sqrt{s^2+4t}\right)}X\right)\sum_{k=0}^{\infty}e^{kx\left(\sqrt{s^2+4t}\right)} = \frac{g(x)}{x}.\frac{x\sqrt{s^2+4t}}{\left(e^{x\left(\sqrt{s^2+4t}\right)} - 1\right)}. \tag{27}$$

On the other hand, the famous Bernoulli numbers are defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty}B_n\frac{x^n}{n!},\ |x| < 2\pi. \tag{28}$$

A recursion formula involving the Bernoulli numbers are

$$B_n = \sum_{k=0}^{n}\binom{n}{k}B_k,$$

for $n \geq 2$ and $B_0 = 1$, $B_1 = -\frac{1}{2}$. Then, from Eq. (26) and (28) we have

$$e^{\beta x}\left(Y - e^{x\left(\sqrt{s^2+4t}\right)}X\right)\sum_{l=0}^{\infty}e^{lx\left(\sqrt{s^2+4t}\right)} = \left(\sum_{m=0}^{\infty}\frac{\mathcal{F}_m}{m!}x^{m-1}\right)\sum_{n=0}^{\infty}\frac{B_n}{n!}\left(x\sqrt{s^2+4t}\right)^n. \tag{29}$$

From theory of infinite series it is well known that for two absolutely convergent power series $\sum_{n=0}^{\infty}a_nt^n$ and $\sum_{n=0}^{\infty}b_nt^n$, we write

$$\left(\sum_{n=0}^{\infty}a_nt^n\right)\left(\sum_{n=0}^{\infty}b_nt^n\right) = \sum_{n=0}^{\infty}\left(\sum_{u+v=n}a_ub_v\right)t^n,$$

so k times on both sides of formula (29), we obtain

$$\begin{aligned} LHS &= \left[e^{\beta x} \left(Y - e^{x\left(\sqrt{s^2+4t}\right)} X \right) \sum_{l=0}^{\infty} e^{lx\left(\sqrt{s^2+4t}\right)} \right]^k \\ &= e^{k\beta x} \left(Y - e^{x\left(\sqrt{s^2+4t}\right)} X \right)^k \left(\sum_{l=0}^{\infty} e^{lx\left(\sqrt{s^2+4t}\right)} \right)^k \\ &= \sum_{n=0}^{\infty} \frac{(k\beta)^n}{n!} \left(Y - e^{x\left(\sqrt{s^2+4t}\right)} X \right)^k \left(\sum_{l=0}^{\infty} e^{lx\left(\sqrt{s^2+4t}\right)} \right)^k x^n, \end{aligned}$$

and

$$RHS = \sum_{n=0}^{\infty} \sum_{(u_1+u_2+...+u_k+v_1+v_2+...+v_k=n)} \frac{\mathcal{F}_{u_1}}{u_1!} ... \frac{\mathcal{F}_{u_k}}{u_k!} \frac{B_{v_1}}{v_1!} ... \frac{B_{v_k}}{v_k!} \left(\sqrt{s^2+4} \right)^{v_1+v_2+...+v_k} x^{n-k}.$$

Comparing the coefficients of x^{n-k} on the above, we immediately obtain the following identity

$$\begin{aligned} &\frac{(k\beta)^{n-k}}{(n-k)!} \left(Y - e^{x\left(\sqrt{s^2+4t}\right)} X \right)^k \left(\sum_{l=0}^{\infty} e^{lx\left(\sqrt{s^2+4t}\right)} \right)^k \\ &= \sum_{u_1+u_2+...+u_k+v_1+v_2+...+v_k=n} \frac{B_{v_1}}{v_1!} ... \frac{B_{v_k}}{v_k!} \frac{\mathcal{F}_{u_1}}{u_1!} ... \frac{\mathcal{F}_{u_k}}{u_k!} \left(\sqrt{s^2+4} \right)^{v_1+v_2+...+v_k}. \end{aligned}$$

3. Conclusions

In this note, a new matrix generalization of the Fibonacci sequence, which we call the generalized (s,t)-Fibonacci matrix sequence, have been introduced and studied. Using the generalized (s,t)-Fibonacci matrix sequence, many mathematical formulas, which allows us to express in a compact form the generalized (s,t)-Fibonacci, have been given.

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