TJMM
7 (2015), No. 2, 131-136

APPROXIMATE SOLUTION OF A NONLINEAR FRACTIONAL ORDINARY DIFFERENTIAL EQUATION BY DGJ METHOD

YIYING FENG AND YUE HU

Abstract. Applying the new iterative method (DGJM), which have been used to handle the nonlinear models, we investigate the approximate analytical solutions for a nonlinear fractional ordinary differential equation (FODE), where the fractional derivatives are considered in Caputo sense. On the process of dealing with nonlinear terms, we particularly employ Taylor series expansion to obtain the analytical solutions.

1. Introduction

Fractional differential equations have caused increasing attention for decades since it plays a vital role in various research fields as diverse as physics, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electrodynamics of complex medium, viscoelasticity, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, etc[1, 2, 3, 4, 5, 6]. Compared to integer order differential equations, fractional differential equations have the advantage that the definition of the fractional one involves all the values of the function. In recent years, many authors have paid attention to studying the solutions of nonlinear fractional differential equations by using various methods. Among these are the Adomian decomposition method (ADM)[7, 8, 9, 10, 11, 12, 13], the homotopy perturbation method (HPM)[14, 15, 16, 17, 18, 19], the variational iteration method(VIM)[20, 21, 22, 23], the Homotopy Analysis Method(HAM)[24, 25, 26].

Daftardar-Gejji and Jafari [27] have devised a New Iterative Method(DGJM) in 2006 to solve nonlinear differential equations. This method is free from rounding off errors since it does not involve discretization and has fairly simple algorithm. Also it does not require prior knowledge of the concepts such as Lagrange multipliers(VIM) or homotopy(HAM). In this article, DGJM is applied to solve a nonlinear fractional ordinary differential equation.

2. Preliminaries and notations

We give some basic definitions and properties of the fractional calculus theory.

Definition 1. A real function $s(x)$, $x > 0$, is said to be in the space $\mathcal{C}_\mu$, $\mu \in \mathfrak{R}$ if there exists a real number $p(> \mu)$, such that $s(x) = x^p s_1(x)$, where $s_1(x) \in \mathcal{C}[0, \infty)$, and it is said to be in the space $\mathcal{C}_\mu^m$ if $s^{(m)} \in \mathcal{C}_\mu$, $m \in N$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha \geqslant 0$, of a

2010 *Mathematics Subject Classification.* 34A08.
Key words and phrases. fractional differential equations, Taylor series; Caputo's fractional derivative, DGJM..

function $s \in \mathcal{C}_\mu$, $\mu \geqslant -1$, is defined as

$$\begin{aligned} J^\alpha s(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} s(t)dt, \quad \alpha > 0, x > 0, \\ J^0 s(x) &= s(x). \end{aligned} \tag{1}$$

where $\Gamma(z)$ is the well-known Gamma function.

Properties of the operator J^α can be found in [28, 29, 30, 31], we mention only the following.

For $s \in \mathcal{C}_\mu$, $\mu \geqslant -1$, $\alpha, \beta \geqslant 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta s(x) = J^{\alpha+\beta} s(x)$,
2. $J^\alpha J^\beta s(x) = J^\beta J^\alpha s(x)$,
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1) x^{\alpha+\gamma}}$.

The Riemann-Liouville derivative has certain disadvantages when trying to model real world phenomena with fractional differential eqautions. Therefore, we shall introduce a modified fractional differential operator D^α proposed by Caputo in his work on the theory of viscoelasticity[32].

Definition 3. The fractional derivative $s(x)$ in the Caputo sense is defined as

$$D^\alpha s(x) = J^{m-\alpha} D^m s(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} s^{(m)}(t)dt, \tag{2}$$

for $m-1 < \alpha \leqslant m$, $m \in N$, $x > 0$, $s \in \mathcal{C}_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 1. If $m-1 < \alpha \leqslant m$, $m \in N$, and $s \in \mathcal{C}_\mu^m$, $\mu \geqslant -1$, then

$$D^\alpha J^\alpha s(x) = s(x)$$

and

$$J^\alpha D^\alpha s(x) = s(x) - \sum_{k=0}^{m-1} s^k(0^+) \frac{x^k}{k!}, \quad x > 0. \tag{3}$$

The Caputo fractional derivatives are considered here because it allows traditional initial conditions to be included in the formulation of the problem.

Definition 4. For m to be the smallest integer that exceeds α, the Caputo fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m u(\tau)}{dt^m} d\tau & for \quad m-1 < \alpha < m, \\ \frac{d^m u(x,y,t)}{dt^m} & for \quad \alpha = m \in N. \end{cases} \tag{4}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

3. New iterative method-DGJM

Consider the following general functional equation[27]:

$$u(\bar{x}) = f(\bar{x}) + N(u(\bar{x})), \tag{5}$$

where N is a nonlinear operator from a Banach space $B \to B$ and f is a known function. $\bar{x} = (x_1, x_2, \ldots, x_n)$. We are looking for a solution u of Eq.(5) having the series form

$$u(\bar{x}) = \sum_{i=0}^{\infty} u_i(\bar{x}). \tag{6}$$

The nonlinear operator N can be decomposed as

$$\begin{aligned} N\left(\sum_{i=0}^{\infty} u_i\right) &= N(u_0) + \{N(u_0+u_1) - N(u_0)\} + \{N(u_0+u_1+u_2) - N(u_0+u_1)\} \\ &\quad + \{N(u_0+u_1+u_2+u_3) - N(u_0+u_1+u_2)\} + \cdots \\ &= N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \end{aligned} \tag{7}$$

From Eqs.(6) and (7), Eq.(5) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \tag{8}$$

We define the recurrence relation

$$\begin{cases} u_0 = f, \\ u_1 = N(u_0), \\ u_{m+1} = \{N(u_0 + \cdots + u_m) - N(u_0 + \cdots + u_{m-1}), \\ m = 1, 2, \ldots \end{cases} \tag{9}$$

Then

$$(u_1 + \cdots + u_{m+1}) = N(u_0 + \cdots + u_m), \quad m = 1, 2, \ldots \tag{10}$$

and

$$\sum_{i=0}^{\infty} u_i = f + N\left(\sum_{i=0}^{\infty} u_i\right). \tag{11}$$

The k-term approximate solution of (5) and (6) is given by $u = u_0 + u_1 + \cdots + u_{k-1}$.

4. Solution of a nonlinear fractional ordinary differential equation

In this section, let us consider the DGJ method for solving the following nonlinear fractional ordinary differential equation:

$$D^{\alpha} u(t) = u^{1/3}(t), \quad 0 < t \leqslant 0.5, \tag{12}$$

where $0 < \alpha \leqslant 1$, subject to the initial condition

$$u(0) = (2/3)^{3/2}. \tag{13}$$

The exact solution of the FODE (12) for $\alpha = 1$, is

$$u(t) = \frac{2\sqrt{6}}{9}(t+1)^{3/2}. \tag{14}$$

This type of FODE has been discussed in [33] by Odibat with VIM. Here we solve the equation with DGJM. Firstly, expanding the nonlinear term $u^{1/3}$ in (12) by using the Taylor series, we get

$$u^{1/3} \approx \frac{40\sqrt{6}}{243} + \frac{10}{9}u - \frac{\sqrt{6}}{3}u^2 + \frac{5}{16}u^3. \tag{15}$$

Then, the FDE (12) can be approximated by

$$D^{\alpha} u = \frac{40\sqrt{6}}{243} + \frac{10}{9}u - \frac{\sqrt{6}}{3}u^2 + \frac{5}{16}u^3. \tag{16}$$

According to the DGJM, in view of the algorithm(11), we construct the following recurrence relation:

$$
\begin{aligned}
u_0(t) =& \frac{2\sqrt{6}}{9},\\
u_1(t) =& \frac{\sqrt{6}t^{\alpha}}{3\Gamma(\alpha+1)},\\
u_2(t) =& \frac{\sqrt{6}t^{2\alpha}}{12\Gamma(\alpha+1)^2} - \frac{\sqrt{6}t^{3\alpha}}{36\Gamma(\alpha+1)^3} + \frac{5\sqrt{6}t^{4\alpha}}{288\Gamma(\alpha+1)^4},\\
u_3(t) =& \frac{\sqrt{6}t^{3\alpha}}{72\Gamma(\alpha+1)^3} - \frac{\sqrt{6}t^{4\alpha}}{72\Gamma(\alpha+1)^4} + \frac{\sqrt{6}t^{5\alpha}}{72\Gamma(\alpha+1)^5} - \frac{11\sqrt{6}t^{6\alpha}}{6912\Gamma(\alpha+1)^6}\\
&+ \frac{\sqrt{6}t^{7\alpha}}{13824\Gamma(\alpha+1)^7} + \frac{5\sqrt{6}t^{8\alpha}}{6144\Gamma(\alpha+1)^8} - \frac{55\sqrt{6}t^{9\alpha}}{497664\Gamma(\alpha+1)^9} + \frac{29\sqrt{6}t^{10\alpha}}{3981312\Gamma(\alpha+1)^{10}}\\
&+ \frac{575\sqrt{6}t^{11\alpha}}{29196288\Gamma(\alpha+1)^{11}} - \frac{125\sqrt{6}t^{12\alpha}}{31850496\Gamma(\alpha+1)^{12}} + \frac{625\sqrt{6}t^{13\alpha}}{828112896\Gamma(\alpha+1)^{13}},\\
&\vdots
\end{aligned}
$$

Then, the DGJM series solution of the FDE (12) can be approximated as

$$u(t) = \frac{\sqrt{6}t^{\alpha}}{3\Gamma(\alpha+1)} + \frac{\sqrt{6}t^{2\alpha}}{12\Gamma(\alpha+1)^2} - \frac{\sqrt{6}t^{3\alpha}}{72\Gamma(\alpha+1)^3} + \frac{\sqrt{6}t^{4\alpha}}{192\Gamma(\alpha+1)^4} + \cdots. \tag{17}$$

For the particular case $\alpha = 1$,

$$u(t) = \frac{2\sqrt{6}}{9} + \frac{\sqrt{6}}{3}t + \frac{\sqrt{6}}{12}t^2 - \frac{\sqrt{6}}{72}t^3 + \frac{\sqrt{6}}{192}t^4 + \cdots. \tag{18}$$

TABLE 1. Approximate solution of (12) for some values of α using the 4-term DGJM approximation

t	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$	$\alpha = 1$(DGJM)	$\alpha = 1$(Exact)
0.0	0.54433105	0.54433105	0.54433105	0.54433105	0.54433105
0.1	1.12594977	0.86049370	0.70973466	0.62798949	0.62798915
0.2	1.25457320	1.00574494	0.83071446	0.71555255	0.71554175
0.3	1.34538453	1.12342547	0.94220811	0.80690495	0.80682276
0.4	1.41863747	1.22749320	1.04931472	0.90203303	0.90168567
0.5	1.48149133	1.32352807	1.15440910	1.00106445	1.00000000

Numerical results is given in Table 1 on the $[0, 0.5]$. We can see that the numerical solution is agree well with the exact solution when $\alpha = 1$. Therefore, we hold that the solution for $\alpha = 0.25$, $\alpha = 0.5$ and $\alpha = 0.75$ is also credible.

5. CONCLUSIONS

In the present study, the new iterative method(DGJM) has been applied to solve a fractional differential equation. This method is shown as a straightforward and popular tool for handling many types of fractional differential equations. Combined with Taylor series, the solutions agree well with the exact solutions.

References

[1] Guo, S., Mei, L., Li Y, et al., *The improved fractional sub-equation method and its applications to the space–time fractional differential equations in fluid mechanics*, Physics Letters A, 2012, **376** (4), 407–411.

[2] Fec, M., Zhou, Y., Wang J R., *On the concept and existence of solution for impulsive fractional differential equations*, Communications in Nonlinear Science and Numerical Simulation, 2012, **17** (7), 3050–3060.

[3] Salahshour, S., Allahviranloo, T., Abbasbandy, S., *Solving fuzzy fractional differential equations by fuzzy Laplace transforms*, Communications in Nonlinear Science and Numerical Simulation, 2012, **17** (3), 1372–1381.

[4] Lu, B., *Backlund transformation of fractional Riccati equation and its applications to nonlinear fractional partial differential equations*, Physics Letters A, 2012, **376** (28), 2045–2048.

[5] Zhang, L., Ahmad, B., Wang, G., et al., *Nonlinear fractional integro-differential equations on unbounded domains in a Banach space*, Journal of Computational and Applied Mathematics, 2013, 249, 51–56.

[6] He, J H., *Exp-function method for fractional differential equations*, International Journal of Nonlinear Sciences and Numerical Simulation, 2013, **14** (6), 363–366.

[7] Adomian, G., *A review of the decomposition method in applied mathematics*, J. Math. Anal. Appl. **135** (1988), 501–544.

[8] Momani, S., Odibat, Z., *Analytical solution of a time-fractional Navier–Stokes equation by Adomian decomposition method*, Applied Mathematics and Computation, 2006, **177** (2), 488–494.

[9] Rach, R., Wazwaz, A.M., Duan, J.S., *A reliable modification of the Adomian decomposition method for higher-order nonlinear differential equations*, Kybernetes, 2013, **42** (2), 282–308.

[10] Rach, R., Duan, J.S., Wazwaz, A.M., *Solving coupled Lane–Emden boundary value problems in catalytic diffusion reactions by the Adomian decomposition method*, Journal of Mathematical Chemistry, 2014, **52** (1), 255–267.

[11] Mekheimer, K.S., Hemada, K.A., Raslan, K.R., et al., *Numerical study of a non-linear peristaltic transport: Application of Adomian decomposition method (ADM)*, Gen, 2014, **20** (2), 22–49.

[12] Wang, H.H., Sun, K.H., He, S.B., *Dynamic analysis and implementation of a digital signal processor of a fractional-order Lorenz-Stenflo system based on the Adomian decomposition method*, Physica Scripta, 2015, **90** (1), 015206.

[13] Al-Hayani, W., *An Efficient Advanced Adomian Decomposition Method to Solve Second-Order Boundary Value Problems with Neumann Conditions*, International Mathematical Forum. 2015, **10** (1), 13–23.

[14] Biazar, J., Ghazvini., H., *Convergence of the homotopy perturbation method for partial differential equations*, Nonlinear Anal. Real World Appl. **10** (2009), 2633–2640.

[15] Gepreel, K.A., *The homotopy perturbation method applied to the nonlinear fractional Kolmogorov–Petrovskii–Piskunov equations*, Applied Mathematics Letters, 2011, **24** (8), 1428–1434.

[16] Jabbar Hussein, A., *Study of error and convergence of homotopy perturbation method for two and three dimension linear Schrödinger equation*, J. Coll. Edu. **1** (2) (2011), 21–43.

[17] Turkyilmazogluglu, M., *Convergence of the homotopy perturbation method*, Int. J. Nonlinear Sci. Numer. Simul. **12** (2011), 9–14.

[18] Zhang, X., Zhao, J., Liu, J., et al., *Homotopy perturbation method for two dimensional time-fractional wave equation*, Applied Mathematical Modelling, 2014, **38** (23), 5545–5552.

[19] Biazar, J., Asadi, M.A., Salehi, F., *Rational Homotopy Perturbation Method for solving stiff systems of ordinary differential equations*, Applied Mathematical Modelling, 2015, **39** (3), 1291–1299.

[20] Momani, S., Abusaad, S., *Application of He's variational-iteration method to Helmholtz equation*, Chaos, Solitons Fractals **27** (5) (2005), 1119–1123.

[21] Abulwafa, E.M., Abdou, M.A., Mahmoud, A.A., *The solution of nonlinear coagulation problem with mass loss*, Chaos, Solitons Fractals **29** (2006), 313–330.

[22] Momani, S., Odibat, Z., *Analytical approach to linear fractional partial differential equations arising in fluid mechanics*, Phys. Lett. A 1 (53) (2006), 1–9.

[23] Sakar, M.G., Ergoren, H., *Alternative variational iteration method to solve the time-fractional Fornberg-Whitham equation*, Applied Mathematical Modelling, 2014.

[24] Behzadia, S.S., Abbasbandy, S., Yildirim, A., *Application of homotopy analysis method for solving a class of nonlinear Volterra-Fredholm integro-differential equations*, Journal of Applied Analysis and Computation, 2012, **2** (2), 127–136.

[25] Mahmoud, Y., Kazemian, E.M., *The Homotopy Analysis Method for Solving the Kuramoto-Tsuzuki Equation*, World Applied Sciences Journal, 2013, **21** (12), 1776–1781.

[26] Odibat, Z., Sami Bataineh, A., *An adaptation of homotopy analysis method for reliable treatment of strongly nonlinear problems: construction of homotopy polynomials*, Mathematical Methods in the Applied Sciences, 2014.

[27] Daftardar-Gejji, V., Jafari, H., *An iterative method for solving nonlinear functional equations*, J. Math. Anal. Appl. **316** (2006), 753–763.

[28] Miller, K.S., Ross, B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.

[29] Samko, S.G., Kilbas, A.A., Marichev, O.I., *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[30] Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[31] West, B.J., Bolognab, M., Grigolini, P., *Physics of Fractal Operators*, Springer, New York, 2003.

[32] Luchko, Y., Gorneflo, R., *The initial value problem for some fractional differential equations with the Caputo derivative*, Preprint, Ser. A-98-08, Fachbreich Mathematik und Informatik, Freic Universitat Berlin, 1998.

[33] Odibat, Z., *On Legendre polynomial approximation with the VIM or HAM for numerical treatment of nonlinear fractional differential equations*, Journal of Computational and Applied Mathematics, 2011, **235** (9), 2956–2968.

Henan Polytechnic University
College of Mathematics and Information Science
Jiaozuo 454000, China

Henan Polytechnic University
College of Mathematics and Information Science
Jiaozuo 454000, China
E-mail address: huu3y2@163.com