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AN INTEGRAL EQUATION FROM PHYSICS - A SYNTHESIS SURVEY - PART II

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Abstract. This part of the synthesis survey on the study of the integral equation from physics:

$$x(t) = \int_a^b K(t, s, x(s), x(a), x(b))ds + f(t),\ t \in [a, b].$$

contains the results concerning the continuous data dependence and the differentiability of its solution. This part ends with some examples.

1. Introduction

We consider again, the integral equation:

$$x(t) = \int_a^b K(t, s, x(s), x(a), x(b))ds + f(t),\ t \in [a, b] \tag{1}$$

where $K : [a, b] \times [a, b] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ or $K : [a, b] \times [a, b] \times J^3 \longrightarrow \mathbb{R}$, $J \subset \mathbb{R}$ closed interval, and $f : [a, b] \longrightarrow \mathbb{R}$.

In the second part of the survey, we present the obtained results regarding the continuous data dependence, the differentiability with respect to a and b, and the differentiability with respect to a parameter of the solution of this integral equation. Some of these results were published in the papers [7], [8], [9] and [11].

In the section 2 we present the basic notations and results that were used in order to obtaining the results presented in the next sections.

The section 3 contains the results regarding the continuous data dependence of the solution of the integral equation (1). It also gives a theorem of data dependence of the solution of an integral equations system. For establishing of these results was useful *the General Data Dependence Theorem.*

In the section 4 we present some results of differentiability of the solution of the integral equation (1), that were obtained by applying *the Fiber Generalized Contractions Theorem* and some of the results of I.A. Rus, published in the papers [25], [26], [28] and [29].

Finally, in the section 5 we present some examples.

2. Basic notations and results

Let X be a nonempty set, d a metric on X and $A : X \to X$ an operator. In this part of survey we shall use the following notations:
$F_A := \{x \in X | A(x) = x\}$ – the fixed points set of A
$A^0 := 1_X,\ A^1 := A,\ A^{n+1} := A \circ A^n,\ n \in N$ – the iterate operators of A.

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In order to study the continuous data dependence of the solution of integral equation (1), we consider, as in part I, the Banach space $X = C([a,b], \mathbf{B})$,

$$C([a,b], \mathbf{B}) = \{x : [a,b] \longrightarrow \mathbf{B} | x \; continuous \; function\},$$

endowed with the Chebyshev norm

$$\|x\|_C := \max_{t \in [a,b]} |x(t)|, \quad for \; all \; x \in C([a,b], \mathbf{B}), \tag{2}$$

where $(\mathbf{B}, +, \mathbb{R}, |\cdot|)$ is a Banach space.

Also, in order to study the continuous data dependence of the solution of the integral equations system (1'), it was used the Banach space $C([a,b], \mathbb{R}^m)$,

$$C([a,b], \mathbb{R}^m) = \{x : [a,b] \longrightarrow \mathbb{R}^m | x \; continuous \; function\},$$

endowed with the generalized Chebyshev norm norm defined by the relation:

$$\|x\| := \begin{pmatrix} \|x_1\|_C \\ \cdots \\ \|x_m\|_C \end{pmatrix}, \text{ for all } x := \begin{pmatrix} x_1 \\ \cdots \\ x_m \end{pmatrix} \in C([a,b], \mathbb{R}^m) \tag{3}$$

where $\|x_k\|_C := \max_{t \in [a,b]} |x_k(t)|$, $k = \overline{1,m}$.

This space is a complete generalized Banach space, where for an element $w \in \mathbb{R}^m$, we denoted $\|w\| = (|w_1|, ..., |w_m|)$.

The following definitions and theorems were used to study the data dependence and the differentiability of the solution of the integral equation (1) (see [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]). In order to obtain the results presented in this survey, also, were used some results from [17], [18], [19], [31] and [32].

Definition 1. *(I.A. Rus,* [22] *or* [24]*) Let (X,d) be a metric space. An operator $A : X \longrightarrow X$ is Picard operator (PO) if there exists $x^* \in X$ such that:*

(a) $F_A = \{x^*\}$;
(b) the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges to x^*, for all $x_0 \in X$.

Definition 2. *(I.A. Rus,* [22] *or* [24]*) Let (X,d) be a metric space. An operator $A : X \longrightarrow X$ is weakly Picard operator (WPO) if the sequence $(A^n(x_0))_{n\in\mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.*

If A is a *WPO*, then we consider the following operator $A^\infty : X \longrightarrow X$, defined by $A^\infty(x) = \lim_{n\to\infty} A^n(x)$ and we observe that $A^\infty(X) = F_A$ (see part I, [16]).

In order to obtain a result of continuous data dependence of the solution of the integral equation (1), it was used the following theorem.

Theorem 1. *(General Data Dependence Theorem) Let (X,d) be a complete metric space and A, $B : X \longrightarrow X$ two operators. We suppose that:*

(i) A is an α-contraction ($\alpha < 1$) and $F_A = \{x_A^*\}$;
(ii) $x_B^* \in F_B$;
(iii) there exists $\eta > 0$ such that $d(A(x), B(x)) < \eta$ for all $x \in X$.

In these conditions we have

$$d(x_A^*, x_B^*) \le \frac{\eta}{1-\alpha}.$$

Also, to obtain a result of continuous data dependence of the solution of an integral equations system, the following definition and theorems were applied.

Definition 3. *(see* [20], [27]*) A matrix $Q \in M_{mm}(\mathbb{R})$ converges to zero if Q^k converges to the zero matrix as $k \longrightarrow 1$.*

The following theorem has two conditions which are equivalents with the convergence to zero of a matrix $Q \in M_{mm}(\mathbb{R}_+)$ and it was useful in two examples presented in the last section.

Theorem 2. *(see* [20], [27]*) Let $Q \in M_{mm}(\mathbb{R}_+)$ be a matrix. The following conditions are equivalents:*

(i) $Q^k \to \infty$ as $k \to \infty$;

(ii) The eigenvalues λ_k , $k = \overline{1,m}$ of the matrix Q, satisfies the condition $|\lambda_k| < 1$, $k = \overline{1,m}$;

(iii) The matrix $I - Q$ is non-singular and $(I - Q)^{-1} = I + Q + ... + Q^n + ...$.

Theorem 3. *(A.I. Perov) Let (X, d) be a complete generalized metric space, with the metric $d(x, y) \in \mathbb{R}^m$ and $A : X \longrightarrow X$ an operator. Suppose that there exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$, such that*

(i) $d(A(x), A(y)) \leq Qd(x, y)$ for all $x, y \in X$;

(ii) $Q^n \to \infty$ as $n \to \infty$.

Then

(a) A has a unique fixed point x^*, i.e. $F_A = \{x^*\}$;

(b) the successive approximations sequence $x_n = A^n(x_0)$, converges to x^* for all $x_0 \in X$, i.e.

$$x^* = \lim_{n\to\infty} A^n(x_0), \text{ for all } x_0 \in X.$$

In addition, the following estimate

$$d(A^n(x), x^*) \leq (I_m - Q)^{-1} Q^n d(x_0, A(x_0)), \, n \in \mathbb{N}^*$$

is accomplished.

In order to study the data dependence of the solution of the integral equation (1), the following theorem was useful.

Theorem 4. *(see* [30]*) Let $(X.d)$ be a complete generalized metric space and $A, B : X \longrightarrow X$ two operators. We suppose that:*

(i) A is a Q-contraction (Q converges to zero matrix) and $F_A = \{x_A^*\}$;

(ii) $x_B^* \in F_B$;

(iii) there exists $\eta \in \mathbb{R}_+^m$ such that $d(A(x), B(x)) < \eta$ for all $x \in X$.

In these conditions we have: $d(x_A^*, x_B^*) \leq (I - Q)^{-1} \eta$.

The results presented in the section 4 were obtained by applying *the Fiber Generalized Contractions Theorem*, that we present below.

Theorem 5. *(I.A. Rus,* [25]*) (Fiber Generalized Contractions Theorem) Let (X, d) be a metric space (generalized or not) and (Y, ρ) a complete generalized metric space ($\rho(x, y) \in \mathbb{R}^m$).*

Let $B : X \longrightarrow X$ and $C : X \times Y \longrightarrow Y$ be two operators and $A : X \times Y \longrightarrow X \times Y$ a continuous operator. Suppose that:

(i) $A(x, y) = (B(x), C(x, y))$, for all $x \in X$, $y \in Y$;

(ii) B is a WPO;

(iii) there exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$, $Q^n \to \infty$ as $n \to \infty$, such that

$$\rho\left(C(x, y_1), C(x, y_2)\right) \leq Q\rho\left(y_1, y_2\right), \text{ for all } x \in X, \ y_1, y_2 \in Y.$$

Under these conditions A is a WPO. In addition, if B is a PO, then A is a PO too.

3. Continuous data dependence

In what follows we present two results of continuous data dependence of the solution of the integral equation (1) and of the integral equations system (1'). These results were published in the papers [12], [13] and [14].

3.1. **Data dependence of the solution of integral equation (1).**

In order to obtain the first result of continuous data dependence of the solution, we consider the integral equation with modified argument (1)

$$x(t) = \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t),\ t \in [a,b]$$

and the perturbed integral equation

$$y(t) = \int_a^b H(t,s,y(s),y(a),y(b))ds + h(t),\ t \in [a,b] \tag{4}$$

where $K, H : [a,b] \times [a,b] \times \mathbf{B}^3 \longrightarrow \mathbf{B}$, $f, h : [a,b] \longrightarrow \mathbf{B}$ and $(\mathbf{B}, +, \mathbb{R}, |\cdot|)$ is a Banach space.

We applied the following two theorems (see [13], [16], [23] and [27]).

Theorem 6. *(Dobriţoiu M.,* [12], [16]*). Suppose that*

(i) $K \in C\left([a,b] \times [a,b] \times \mathbf{B}^3, \mathbf{B}\right)$;

(ii) $f \in C\left([a,b], \mathbf{B}\right)$;

(iii) there exists $M_K > 0$ *such that* $|K(t,s,u,v,w)| \le M_K$, *for all* $t,s \in [a,b]$, $u,v,w \in \mathbf{B}$.

Then the integral equation (1) has at least one solution $x^* \in C\left([a,b], \mathbf{B}\right)$.

Theorem 7. *(Dobriţoiu M.,* [12], [16]*). Suppose that*

(i) $K \in C\left([a,b] \times [a,b] \times \mathbf{B}^3, \mathbf{B}\right)$;

(ii) $f \in C\left([a,b], \mathbf{B}\right)$;

(iii) there exists $L_K > 0$ *such that*

$$|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)| \le L_K\left(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|\right)$$

for all $t,s \in [a,b]$, $u_i, v_i \in \mathbf{B}$, $i = 1,2,3$;

(iv) $3L_K(b-a) < 1$.

Then the integral equation (1) has a unique solution $x^* \in C\left([a,b], \mathbf{B}\right)$.

Now, for the integral equation (1), suppose that the conditions of Theorem 7 are satisfied and then, it results that this equation has a unique solution $x^* \in C\left([a,b], \mathbf{B}\right)$.

Also, for the perturbed integral equation (4), suppose that the following conditions are satisfied:

(d_1) $H \in C\left([a,b] \times [a,b] \times \mathbf{B}^3, \mathbf{B}\right)$;

(d_2) $h \in C\left([a,b], \mathbf{B}\right)$;

(d_3) there exists $M_H > 0$ such that $|H(t,s,u,v,w)| \le M_H$, for all $t,s \in [a,b]$, $u,v,w \in \mathbf{B}$.

Then, from the Theorem 6 it results that the integral equation (4) has at least one solution $y^* \in C\left([a,b], \mathbf{B}\right)$.

Finally, using *the General Data Dependence Theorem* we obtained the following result of continuous data dependence of the solution of integral equation (1).

Theorem 8. *(Dobriţoiu M.,* [12]*) Suppose that the conditions of Theorem 7 are fullfiled, and denote with* $x^* \in C\left([a,b], \mathbf{B}\right)$ *the unique solution of the integral equation (1). Moreover, suppose that the conditions* $(d_1) - (d_3)$ *are satisfied. In addition, suppose that:*

(i) there exists η_1, $\eta_2 > 0$ *such that*
$|K(t,s,u_1,u_2,u_3) - H(t,s,u_1,u_2,u_3)| \le \eta_1$, *for all* $t,s \in [a,b]$, $u_1,u_2,u_3 \in \mathbf{B}$, *and*
$|f(t) - h(t)| \le \eta_2$, *for all* $t \in [a,b]$.

Under these conditions, if $y^* \in C([a,b],\mathbf{B})$ *is a solution of the perturbed integral equation (4), then the following estimate is true:*

$$\|x^* - y^*\|_{(C[a,b],B)} \le \frac{\eta_1(b-a)+\eta_2}{1-3L_K(b-a)}. \tag{5}$$

In order to obtain the second result of continuous data dependence of the solution, we consider the integral equation with modified argument (1) and the perturbed integral equation

$$y(t) = \int_a^b H(t,s,y(s),y(a),y(b))ds + f(t),\ t \in [a,b \tag{4'}$$

where $K,H : [a,b] \times [a,b] \times J^3 \to \mathbf{B}$, $J \subset \mathbf{B}$ is a compact subset, $f : [a,b] \to \mathbf{B}$ and $(\mathbf{B},+,\mathbb{R},|\cdot|)$ is a Banach space.

We applied the following two theorems (see [13], [16], [23] and [27]).

Theorem 9. *(Dobriţoiu M.,* [12], [16]*). Suppose that:*

(i) $K \in C([a,b] \times [a,b] \times J^3, \mathbf{B}), J \subset \mathbf{B}$ *is a compact subset;*
(ii) $f \in C([a,b],\mathbf{B})$*;*
(iii) $M_K(b-a) \le r$, *where* $M_K > 0$, *such that,* $|K(t,s,u,v,w)| \le M_K$, *for all* $t,s \in [a,b]$, $u,v,w \in J$.

Then the integral equation (1) has at least one solution $x^* \in \overline{B}(f;r) \subset C([a,b],\mathbf{B})$.

Theorem 10. *(Dobriţoiu M.,* [12], [16]*). Suppose that the following conditions are satisfied:*

(i) $K \in C([a,b] \times [a,b] \times J^3, \mathbf{B}), J \subset \mathbf{B}$ *is a compact subset;*
(ii) $f \in C([a,b],\mathbf{B})$*;*
(iii) $M_K(b-a) \le r$, *where* $M_K > 0$, *such that,* $|K(t,s,u,v,w)| \le M_K$, *for all* $t,s \in [a,b]$, $u,v,w \in J$*;*
(iv) there exists $L_K > 0$ *such that*
$|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)| \le L_K(|u_1-v_1| + |u_2-v_2| + |u_3-v_3|)$,
for all $t,s \in [a,b]$, $u_i,v_i \in J$, $i = 1,2,3$*;*
(v) $3L_K(b-a) < 1$.

Then the integral equation (1) has a unique solution $x^* \in \overline{B}(f;r) \subset C([a,b],\mathbf{B})$.

Now, for the integral equation (1), suppose that the conditions of Theorem 10 are satisfied and then, it results that this integral equation has a unique solution $x^* \in \overline{B}(f;r) \subset C([a,b],\mathbf{B})$.

Also, for the perturbed integral equation (4') suppose that the following conditions are satisfied:

$(d_1^{'})$ $H \in C([a,b] \times [a,b] \times J^3, \mathbf{B})$, $J \subset \mathbf{B}$ is a compact subset;

$(d_2^{'})$ $M_H(b-a) \le r$, where $M_H > 0$, such that, $|H(t,s,u,v,w)| \le M_H$, for all $t,s \in [a,b]$, $u,v,w \in J \subset \mathbf{B}$ compact subset.

Then by the Theorem 9 it results that the perturbed integral equation (4') has at least one solution $y^* \in \overline{B}(h;r) \subset C([a,b],\mathbf{B})$.

Now, using *the General Data Dependence Theorem* it was obtained the following result of continuous data dependence of the solution of integral equation (1).

Theorem 11. *(Dobriţoiu M.,* [12]*). Suppose that the conditions of Theorem 10 are fullfiled, and denote with* $x^* \in \overline{B}(f;r) \subset C([a,b], \mathbf{B})$ *the unique solution of the integral equation (1). Moreover, suppose that the conditions* $(d_1^{'})$ *and* $(d_2^{'})$ *are satisfied. In addition, suppose that*

(i) there exists $\eta > 0$ *such that*
$|K(t,s,u,v,w) - H(t,s,u,v,w)| \leq \eta$, *for all* $t, s \in [a,b]$, $u, v, w \in J \subset \mathbf{B}$ *compact subset.*

Under these conditions, if $y^* \in \overline{B}(f;r) \subset C([a,b], \mathbf{B})$ *is a solution of the perturbed integral equation (4'), then we have:*

$$\|x^* - y^*\|_{(C[a,b],\mathbf{B})} \leq \frac{\eta(b-a)}{1 - 3L_K(b-a)}. \tag{6}$$

3.2. **Data dependence of the solution of integral equations system (1').**

In the particular case $\mathbf{B} = \mathbb{R}^m$, we have the system of nonlinear integral equations

$$x(t) = \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t), \ t \in [a,b] \tag{1'}$$

where $t \in [a,b]$, $K : [a,b] \times [a,b] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ and $f : [a,b] \longrightarrow \mathbb{R}^m$, that have the form:

$$\begin{cases} x_1(t) = \int_a^b K_1(t,s,x(s),x(a),x(b))ds + f_1(t) \\ x_2(t) = \int_a^b K_2(t,s,x(s),x(a),x(b))ds + f_2(t) \\ \cdots\cdots\cdots\cdots\cdots \\ x_m(t) = \int_a^b K_m(t,s,x(s),x(a),x(b))ds + f_m(t) \end{cases}, \ t \in [a,b] \tag{1''}$$

and we consider the perturbed system of integral equations

$$y(t) = \int_a^b K(t,s,y(s),y(a),y(b))ds + h(t), \ t \in [a,b], \tag{7}$$

where $H \in C([a,b] \times [a,b] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$ and $h \in C([a,b], \mathbb{R}^m)$, that have the form:

$$\begin{cases} y_1(t) = \int_a^b K_1(t,s,y(s),y(a),y(b))ds + h_1(t) \\ y_2(t) = \int_a^b K_2(t,s,y(s),y(a),y(b))ds + h_2(t) \\ \cdots\cdots\cdots\cdots\cdots \\ y_m(t) = \int_a^b K_m(t,s,y(s),y(a),y(b))ds + h_m(t) \end{cases}, \ t \in [a,b]. \tag{7'}$$

In order to obtain a result of continuous data dependence of the solution of integral equations system (1') it was used the following result (see [13], [16], [23] and [27]).

Theorem 12. *(Dobriţoiu M.,* [14]*) We suppose that:*

(i) $K \in C([a,b] \times [a,b] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$*;*

(ii) $f \in C([a,b], \mathbb{R}^m)$*;*

(iii) there exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$ *such that*

$$\begin{pmatrix} |K_1(t,s,u_1,u_2,u_3) - K_1(t,s,v_1,v_2,v_3)| \\ \cdots\cdots\cdots\cdots\cdots \\ |K_m(t,s,u_1,u_2,u_3) - K_m(t,s,v_1,v_2,v_3)| \end{pmatrix} \leq$$

$$\leq Q \begin{pmatrix} |u_{11} - v_{11}| + |u_{21} - v_{21}| + |u_{31} - v_{31}| \\ \cdots\cdots\cdots\cdots\cdots \\ |u_{1m} - v_{1m}| + |u_{2m} - v_{2m}| + |u_{3m} - v_{3m}| \end{pmatrix}$$

for all $t, s \in [a,b]$, $u_i, v_i \in \mathbb{R}^m$, $i = 1,2,3$*;*

(iv) $[3(b-a)Q]^n \to 0$ *as* $n \to \infty$.

Then the system of integral equations (1') has a unique solution $x^* \in C([a,b],\mathbb{R}^m)$*. This solution can be obtained by the successive approximations method, starting at any element* $x_0 \in C([a,b],\mathbb{R}^m)$*. Moreover, if* x_n *is the* n^{th} *successive approximation, then we have the following estimate:*

$$\|x^* - x_n\|_{R^m} \le [3(b-a)Q]^n \cdot [I - 3(b-a)Q]^{-1} \|x_0 - x_1\|_{\mathbb{R}^m} . \tag{8}$$

Now, for the system of integral equations (1'), we suppose that the conditions of Theorem 12 are satisfied and then, it results that this system has a unique solution $x^* \in C([a,b],\mathbb{R}^m)$.

We have the following data dependence theorem:

Theorem 13. *(Dobriţoiu M.,* [14]*) We suppose that:*

(i) the conditions of the Theorem 12 are satisfied and we denote by x^* *the unique solution of the system of integral equations (6) in the Banach space* $C([a,b],\mathbb{R}^m)$*;*

(ii) there exists $T_1, T_2 \in M_{m1}(\mathbb{R}_+)$ *such that*
$\|K(t,s,u_1,u_2,u_3) - H(t,s,u_1,u_2,u_3)\|_C \le T_1$*, for all* $t,s \in [a,b]$*,* $u_i \in \mathbb{R}^m$*,* $i = 1,2,3$*,*
and
$\|f(t) - h(t)\|_C \le T_2$*, for all* $t \in [a,b]$ *.*

Under these conditions, if $y^* \in C([a,b],\mathbb{R}^m)$ *is a solution of the perturbed system of integral equations (6), then the following estimate is true:*

$$\|x^* - y^*\|_C \le [I - 3\,(b-a)\,Q]^{-1}\,[(b-a)T_1 + T_2] . \tag{9}$$

4. The differentiability of the solution

In this section we will prove two theorems of differentiability of the solution of the integral equation (1).

4.1. The differentiability of the solution with respect to a and b.

We consider the integral equation with modified argument (1)

$$x(t) = \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t),$$

where $t \in [\alpha,\beta]$, $\alpha,\beta \in \mathbb{R}$, $\alpha \le \beta$, $a,b \in [\alpha,\beta]$ and $K \in C([\alpha,\beta] \times [\alpha,\beta] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$, $f \in C([\alpha,\beta],\mathbb{R}^m)$ and $x \in C([\alpha,\beta],\mathbb{R}^m)$. We have:

Theorem 14. *Suppose that there exists a matrix* $Q \in M_{m\times m}(\mathbb{R}_+)$ *such that:*

(i) $[3(\beta-\alpha)Q]^n \to 0$ *as* $n \to \infty$*;*

(ii)
$$\begin{pmatrix} |K_1(t,s,u_1,u_2,u_3) - K_1(t,s,v_1,v_2,v_3)| \\ \cdot\ \cdot\ \cdot \\ |K_m(t,s,u_1,u_2,u_3) - K_m(t,s,v_1,v_2,v_3)| \end{pmatrix} \le$$
$$\le Q \begin{pmatrix} |u_{11}-v_{11}| + |u_{21}-v_{21}| + |u_{31}-v_{31}| \\ \cdot\ \cdot\ \cdot \\ |u_{1m}-v_{1m}| + |u_{2m}-v_{2m}| + |u_{3m}-v_{3m}| \end{pmatrix},$$
for all $t,s \in [\alpha,\beta]$*,* $u_i, v_i \in \mathbb{R}^m$*,* $i = \overline{1,3}$*.*

Then

(a) the integral equation (1) has a unique solution, $x^*(\,\cdot,a,b) \in C([\alpha,\beta],\mathbb{R}^m)$*;*

(b) *for all* $x_0 \in C([\alpha,\beta],\mathbb{R}^m)$, *the sequence* $(x^n)_{n\in\mathbb{N}}$, *defined by the relation:*

$$x^{n+1}(t;a,b) := \int_a^b K(t,s,x^n(s;a,b),x^n(a;a,b),x^n(b;a,b))ds + f(t),$$

converges uniformly to x^*, *for all* $t,a,b \in [\alpha,\beta]$ *and*

$$\begin{pmatrix} \|x_1^n - x_1^*\|_C \\ . \quad . \quad . \\ \|x_m^n - x_m^*\|_C \end{pmatrix} \leq \left[I_m - 3(\beta-\alpha)Q\right]^{-1}\left[3(\beta-\alpha)Q\right]^n \begin{pmatrix} \left\|x_1^1 - x_1^0\right\|_C \\ . \quad . \quad . \\ \left\|x_m^1 - x_m^0\right\|_C \end{pmatrix};$$

(c) *the function* $x^* : [\alpha,\beta]\times[\alpha,\beta]\times[\alpha,\beta] \longrightarrow \mathbb{R}^m$, $(t,a,b) \mapsto x^*(t;a,b)$ *is continuous;*

(d) *if* $K(t,s,.,.,.) \in C^1(\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m,\mathbb{R}^m)$ *for all* $t,s \in [\alpha,\beta]$, *then*

$$x^*(t;.,.) \in C^1([\alpha,\beta]\times[\alpha,\beta],\mathbb{R}^m) \ for \ all \ t \in [\alpha,\beta]..$$

Proof. Denote $X := C([\alpha,\beta]^3,\mathbb{R}^m)$. We consider on X *the generalized norm* defined by the relation (3).

Also, we consider the operator $B : X \longrightarrow X$ defined by the relation:

$$B(x)(t;a,b) := \int_a^b K(t,s,x(s;a,b),x(a;a,b),x(b;a,b))ds, \tag{10}$$

for all $t,a,b \in [\alpha,\beta]$.

Using the conditions (i), (ii) and applying *the Perov's theorem* 4, it results that the conclusions (a), (b) and (c) are fulfilled.

(d) We prove that there exists $\frac{\partial x^*}{\partial a}$, $\frac{\partial x^*}{\partial b} \in X$.

If we assume that there exists $\frac{\partial x^*}{\partial a}$, then from (1) it results that:

$$\frac{\partial x^*(t;a,b)}{\partial a} = -K(t,a,x^*(a;a,b),x^*(a;a,b),x^*(b;a,b)) +$$
$$+\int_a^b \left[\left(\frac{\partial K_j(t,s,x^*(s;a,b),x^*(a;a,b),x^*(b;a,b))}{\partial x_i^*(s;a,b)}\right)\cdot\left(\frac{\partial x^*(s;a,b)}{\partial a}\right)+\right.$$
$$+\left(\frac{\partial K_j(t,s,x^*(s;a,b),x^*(a;a,b),x^*(b;a,b))}{\partial x_i^*(a;a,b)}\right)\cdot\left(\frac{\partial x^*(a;a,b)}{\partial a}\right)+$$
$$\left.+\left(\frac{\partial K_j(t,s,x^*(s;a,b),x^*(a;a,b),x^*(b;a,b))}{\partial x_i^*(b;a,b)}\right)\cdot\left(\frac{\partial x^*(b;a,b)}{\partial a}\right)\right]ds.$$

This relation leads us to consider the operator $C : X\times X \longrightarrow X$ defined by the relation:

$$C(x,y)(t;a,b) := -K(t,a,x(a;a,b),x(a;a,b),x(b;a,b)) +$$
$$+\int_a^b \left[\left(\frac{\partial K_j(t,s,x(s;a,b),x(a;a,b),x(b;a,b))}{\partial x_i(s;a,b)}\right)\cdot y(s;a,b)+\right.$$
$$+\left(\frac{\partial K_j(t,s,x(s;a,b),x(a;a,b),x(b;a,b))}{\partial x_i(a;a,b)}\right)\cdot y(a;a,b)+$$
$$\left.+\left(\frac{\partial K_j(t,s,x(s;a,b),x(a;a,b),x(b;a,b))}{\partial x_i(b;a,b)}\right)\cdot y(b;a,b)\right]ds. \tag{11}$$

Using the condition (ii) we obtain:

$$\left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3)}{\partial u_{1i}}\right|\right)_{i,j=1}^m \leq Q, \tag{12}$$

$$\left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3)}{\partial u_{2i}}\right|\right)_{i,j=1}^m \leq Q, \tag{13}$$

$$\left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3)}{\partial u_{3i}}\right|\right)_{i,j=1}^{m} \leq Q, \tag{14}$$

for all $t,s \in [\alpha,\beta]$, $u_1,u_2,u_3 \in \mathbb{R}^m$.

Using (11), (12), (13) and (14) it results that:

$$\|C(x,y_1) - C(x,y_2)\| \leq 3(\beta-\alpha)Q \cdot \|y_1 - y_2\|$$

for all $x, y_1, y_2 \in X$.

Now, if we consider the operator $A : X \times X \longrightarrow X \times X$, $A = (B,C)$ then we observe that the conditions of *the Fiber Generalized Contractions Theorem* 5, are fulfilled and therefore it results that A is a PO and the sequence $\left(x^{n+1}(t;a,b),\ y^{n+1}(t;a,b)\right)$, defined by the relations:

$$x^{n+1}(t;a,b) := \int_a^b K(t,s,x^n(s;a,b),x^n(a;a,b),x^n(b;a,b))ds + f(t),$$

$$y^{n+1}(t;a,b) := -K(t,a,x^n(s;a,b),x^n(a;a,b),x^n(b;a,b)) +$$
$$+\int_a^b \left[\left(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(a;a,b),x^n(b;a,b))}{\partial u_{1i}}\right)\cdot y^n\,(s;a,b) +\right.$$
$$+\left(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(a;a,b),x^n(b;a,b))}{\partial u_{2i}}\right)\cdot y^n\,(a;a,b) +$$
$$\left.+\left(\frac{\partial K_j(t,s,x^n(s;a,b),x^n(a;a,b),x^n(b;a,b))}{\partial u_{3i}}\right)\cdot y^n\,(b;a,b)\right] ds$$

converges uniformly (with respect to $t,a,b \in [\alpha,\beta]$) to $(x^*,y^*) \in F_A$, for all $(x^0,y^0) \in X \times X$.

If we take $x^0 = y^0 = 0$, then $y^1 = \frac{\partial x^1}{\partial a}$ and we prove through induction that $y^n = \frac{\partial x^n}{\partial a}$. Thus, we have:

$$x^n \overset{uniformly}{\longrightarrow} x^* \ as\ n \to \infty,$$
$$\frac{\partial x^n}{\partial a} \overset{uniformly}{\longrightarrow} y^* \ as\ n \to \infty,$$

and it results that there exists $\frac{\partial x^*}{\partial a}$ (i.e. x^* is differentiable with respect to a) and $\frac{\partial x^*}{\partial a} = y^*$.

By an analogous reasoning we prove that there exists $\frac{\partial x^*}{\partial b}$. □

4.2. **The differentiability of the solution with respect to a parameter.**

In what follows we apply *the Fiber Generalized Contractions Theorem* 5, to study the differentiability with respect to a parameter of the solution of the integral equation (1):

$$x(t) = \int_a^b K(t,s,x(s),x(a),x(b);\lambda)ds + f(t),\ t \in [a,b]$$

where $K \in C([a,b]\times[a,b]\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^m\times J,\mathbb{R}^m)$, $J \subset \mathbb{R}$ is a compact interval, $f \in C([a,b],\mathbb{R}^m)$ and $x \in C([a,b],\mathbb{R}^m)$.

The following theorem of differentiability of the solution is true.

Theorem 15. *Suppose that there exists a matrix* $Q \in M_{m\times m}(\mathbb{R}_+)$ *such that:*

(i) $[3(b-a)Q]^n \longrightarrow 0$ *as* $n \to \infty$*;*

(ii) $$\begin{pmatrix} |K_1(t,s,u_1,u_2,u_3) - K_1(t,s,v_1,v_2,v_3)| \\ \cdot\ \cdot\ \cdot \\ |K_m(t,s,u_1,u_2,u_3) - K_m(t,s,v_1,v_2,v_3)| \end{pmatrix} \leq$$
$$\leq Q \begin{pmatrix} |u_{11}-v_{11}| + |u_{21}-v_{21}| + |u_{31}-v_{31}| \\ \cdot\ \cdot\ \cdot \\ |u_{1m}-v_{1m}| + |u_{2m}-v_{2m}| + |u_{3m}-v_{3m}| \end{pmatrix},$$

for all $t, s \in [a, b]$, $u_i, v_i \in \mathbb{R}^m$, $i = \overline{1,3}$.

Then

(a) *for all* $\lambda \in J$, *the integral equation (1) has a unique solution*, $x^*(.,\lambda) \in C([a, b], \mathbb{R}^m)$;

(b) *for all* $x_0 \in C([a, b] \times J, \mathbb{R}^m)$, *the sequence* $(x^n)_{n\in\mathbb{N}}$, *defined by the relation:*

$$x^{n+1}(t; \lambda) := \int_a^b K(t, s, x^n(s; \lambda), x^n(a; \lambda), x^n(b; \lambda))ds + f(t),$$

converges uniformly to x^*, *for all* $t \in [a, b]$, $\lambda \in J$ *and*

$$\begin{pmatrix} \|x_1^n - x_1^*\|_C \\ . \quad . \quad . \\ \|x_m^n - x_m^*\|_C \end{pmatrix} \leq \left[I_m - 3\,(b-a)\,Q\right]^{-1} \left[3\,(b-a)\,Q\right]^n \begin{pmatrix} \left\|x_1^1 - x_1^0\right\|_C \\ . \quad . \quad . \\ \left\|x_m^1 - x_m^0\right\|_C \end{pmatrix};$$

(c) *the function* $x^* : [a, b] \times J \longrightarrow \mathbb{R}^m$, $(t; \lambda) \mapsto x^*(t; \lambda)$ *is continuous ;*

(d) *if* $K(t, s, ., ., ., .) \in C^1(\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times J, \mathbb{R}^m)$ *for all* $t, s \in [a, b]$, *then* $x^*(t; .) \in C^1(J, \mathbb{R}^m)$ *for all* $t \in [a, b]$.

Proof. Denote $X := C([a, b] \times J, \mathbb{R}^m)$. We consider *the generalized norm* on X, defined by the relation (3).

Also, we consider the operator $B : X \longrightarrow X$ defined by the relation:

$$B(x)(t; \lambda) := \int_a^b K(t, s, x(s; \lambda), x(a; \lambda), x(b; \lambda))ds + f(t), \tag{15}$$

for all $t \in [a, b]$, $\lambda \in J$.

From conditions (i), (ii) and applying *the Perov's Theorem* 4, it results that the conclusions (a), (b) and (c) are fulfilled.

(d) We prove that there exists $\frac{\partial x^*}{\partial \lambda}$ and $\frac{\partial x^*}{\partial \lambda} \in X$.

We assume that there exists $\frac{\partial x^*}{\partial \lambda}$. Then using (1) we obtain:

$$\begin{aligned}\frac{\partial x^*(t;\lambda)}{\partial \lambda} = \int_a^b \Bigg[&\left(\frac{\partial K_j(t, s, x^*(s; \lambda), x^*(a; \lambda), x^*(b; \lambda); \lambda)}{\partial x_i^*(s; \lambda)}\right)_{i,j=1}^m \cdot \left(\frac{\partial x^*(s;\lambda)}{\partial \lambda}\right) + \\ &+ \left(\frac{\partial K_j(t, s, x^*(s; \lambda), x^*(a; \lambda), x^*(b; \lambda); \lambda)}{\partial x_i^*(a; \lambda)}\right)_{i,j=1}^m \cdot \left(\frac{\partial x^*(a;\lambda)}{\partial \lambda}\right) + \\ &+ \left(\frac{\partial K_j(t, s, x^*(s; \lambda), x^*(a; \lambda), x^*(b; \lambda); \lambda)}{\partial x_i^*(b; \lambda)}\right)_{i,j=1}^m \cdot \left(\frac{\partial x^*(b;\lambda)}{\partial \lambda}\right) + \\ &+ \left(\frac{\partial K_j(t, s, x^*(s; \lambda), x^*(a; \lambda), x^*(b; \lambda); \lambda)}{\partial \lambda}\right)_{i=1}^m \Bigg] ds. \end{aligned} \tag{16}$$

This relation leads us to consider the operator $C : X \times X \longrightarrow X$ defined by the relation:

$$\begin{aligned} C(x, y)(t, \lambda) := \int_a^b \Bigg[&\left(\frac{\partial K_j(t, s, x(s; \lambda), x(a; \lambda), x(b; \lambda); \lambda)}{\partial x_i(s; \lambda)}\right)_{i,j=1}^m \cdot y(s; \lambda) + \\ &+ \left(\frac{\partial K_j(t, s, x(s; \lambda), x(a; \lambda), x(b; \lambda); \lambda)}{\partial x_i(a; \lambda)}\right)_{i,j=1}^m \cdot y(a, \lambda) + \\ &+ \left(\frac{\partial K_j(t, s, x(s; \lambda), x(a; \lambda), x(b; \lambda); \lambda)}{\partial x_i(b; \lambda)}\right)_{i,j=1}^m \cdot y(b; \lambda) + \\ &+ \left(\frac{\partial K_j(t, s, x(s; \lambda), x(a; \lambda), x(b; \lambda); \lambda)}{\partial \lambda}\right)_{j=1}^m \Bigg] ds, \end{aligned} \tag{17}$$

for all $x, y \in X$.

From condition (ii) we obtain:

$$\left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3)}{\partial u_{1i}}\right|\right)_{i,j=1}^{m} \le Q, \tag{18}$$

$$\left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3)}{\partial u_{2i}}\right|\right)_{i,j=1}^{m} \le Q, \tag{19}$$

$$\left(\left|\frac{\partial K_j(t,s,u_1,u_2,u_3)}{\partial u_{3i}}\right|\right)_{i,j=1}^{m} \le Q, \tag{20}$$

for all $t, s \in [a,b]$, $u_1, u_2, u_3 \in \mathbb{R}^m$.

From (17), (18), (19) and (20) it results that

$$\|C(x,y_1) - C(x,y_2)\| \le 3(b-a)Q \cdot \|y_1 - y_2\|,$$

for all $x, y_1, y_2 \in X$.

Now, if we consider the operator $A : X \times X \longrightarrow X \times X$, $A = (B,C)$, $A(x,y) = (B(x), C(x,y))$, then we observe that the conditions of *the Fiber Generalized Contractions Theorem* 5 are fulfilled and therefore it results that A is a PO and the sequences:

$$x^{n+1}(t,\lambda) := B(x^n(t,\lambda))$$

$$y^{n+1}(t,\lambda) := C(x^n(t,\lambda), y^n(t,\lambda)),$$

converge uniformly (with respect to $t \in [a,b]$ and $\lambda \in J$) to $(x^*, y^*) \in F_A$, for all $(x^0, y^0) \in X \times X$.

If we take $x^0 \in X$, $y^0 \in X$, such that $y^0 = \frac{\partial x^0}{\partial \lambda}$, then we prove by induction that $y^n = \frac{\partial x^n}{\partial \lambda}$. Thus, we have:

$$x^n \overset{uniformly}{\longrightarrow} x^* \; as \; n \to \infty,$$

$$\frac{\partial x^n}{\partial \lambda} \overset{uniformly}{\longrightarrow} y^* \; as \; n \to \infty.$$

Using *the Weierstrass's Theorem* it results that there exists $\frac{\partial x^*}{\partial \lambda}$ (x^* is differentiable with respect to λ) and $\frac{\partial x^*}{\partial \lambda} = y^*$. □

5. Examples

Example 5.1. We consider the integral equation with modified argument:

$$x(t) = \int_0^1 \left[\frac{\sin(x(s))}{7} + \frac{x(0) + x(1)}{5}\right] ds + 2\cos t + 1, \; t \in [0,1] \tag{21}$$

where $K \in C([0,1] \times [0,1] \times \mathbb{R}^3)$, $K(t,s,u_1,u_2,u_3) = \frac{\sin(u_1)}{7} + \frac{u_2 + u_3}{5}$,

$f \in C[0,1]$, $f(t) = 2cost + 1$, $x \in C[0,1]$ and the perturbed integral equation:

$$y(t) = \int_0^1 \left[\frac{\sin(y(s))}{7} + \frac{y(0) + y(1)}{5} - t - 2\right] ds + \cos t, \; t \in [0,1] \tag{22}$$

where $H \in C([0,1] \times [0,1] \times \mathbb{R}^3)$, $H(t,s,v_1,v_2,v_3) = \frac{\sin(v_1)}{7} + \frac{v_2 + v_3}{5} - t - 2$,

$h \in C[0,1]$, $h(t) = cost$, and $y \in C[0,1]$.

The operator $A : C[0,1] \longrightarrow C[0,1]$, attached to equation (21 and defined by the relation:

$$A(x)(t) = \int_0^1 \left[\frac{\sin(x(s))}{7} + \frac{x(0) + x(1)}{5}\right] ds + 2\cos t + 1, \; t \in [0,1] \tag{23}$$

is an α-contraction with the coefficient $\alpha = \frac{19}{35}$.

Since the conditions of theorem 10 (Dobritoiu M., [16]), of existence and uniqueness of the solution in the space $C[0,1]$ are fulfilled, it results that the integral equation (21) has a unique solution $x^* \in C[0,1]$.

We have:

$|\,K(t,s,u_1,u_2,u_3) - H(t,s,u_1,u_2,u_3)\,| = |\,t+2\,| \leq 3$, for all $t,s \in [0,1]$

and

$|\,f(t) - h(t)\,| = |\cos t + 1\,| \leq 2$, for all $t \in [0,1]$.

The conditions of theorem 8 are fulfilled and therefore, if $y^* \in C[0,1]$ is a solution of the integral equation (22), then the following estimate is true:

$$\|x^* - y^*\|_{C[0,1]} \leq \frac{175}{16}.$$

Example 5.2. In what follows we consider the system of integral equations:

$$\begin{cases} x_1(t) = \int_0^1 \left[\frac{t+2}{15}x_1(s) + \frac{t}{5}x_1(0) + \frac{t}{5}x_1(1)\right] ds + 2t + 1 \\ x_2(t) = \int_0^1 \left[\frac{t+2}{21}x_2(s) + \frac{t}{7}x_2(0) + \frac{t}{7}x_2(1)\right] ds + \sin t \end{cases}, \ t \in [0,1], \tag{24}$$

where $K \in C([0,1] \times [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2)$,

$K(t,s,u_1,u_2,u_3) = (K_1(t,s,u_1,u_2,u_3)\,,\ K_2(t,s,u_1,u_2,u_3))$,

$K_1(t,s,u_1,u_2,u_3) = \frac{t+2}{15}u_{11} + \frac{1}{5}u_{21} + \frac{1}{5}u_{31}$,

$K_2(t,s,u_1,u_2,u_3) = \frac{t+2}{21}u_{12} + \frac{1}{7}u_{22} + \frac{1}{7}u_{32}$,

$f \in C([0,1], \mathbb{R}^2)$, $f(t) = (f_1(t), f_2(t))$, $f_1(t) = 2t+1$, $f_2(t) = \sin t$, $x \in C([0,1], \mathbb{R}^2)$,

and the perturbed system of integral equations:

$$\begin{cases} y_1(t) = \int_0^1 \left[\frac{s+3}{15}y_1(s) + \frac{1}{5}y_1(0) + \frac{1}{5}y_1(1) - 3\right] ds + 2t - 1 \\ y_2(t) = \int_0^1 \left[\frac{s+3}{21}y_2(s) + \frac{1}{7}y_2(0) + \frac{1}{7}y_2(1) - 1\right] ds + \cos t \end{cases}, \ t \in [0,1] \tag{25}$$

where $H \in C([0,1] \times [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2)$,

$H(t,s,v_1,v_2,v_3) = (H_1(t,s,v_1,v_2,v_3)\,,\ H_2(t,s,v_1,v_2,v_3))$,

$H_1(t,s,v_1,v_2,v_3) = \frac{s+3}{15}v_{11} + \frac{1}{5}v_{21} + \frac{1}{5}v_{31} - 3$,

$H_2(t,s,v_1,v_2,v_3) = \frac{s+3}{21}v_{12} + \frac{1}{7}v_{22} + \frac{1}{7}v_{32} - 1$,

$h \in C([0,1], \mathbb{R}^2)$, $h(t) = (h_1(t), h_2(t))$, $h_1(t) = 2t-1$, $h_2(t) = \cos t$ and $x \in C([0,1], \mathbb{R}^2)$.

The operator $A : C([0,1], \mathbb{R}^2) \longrightarrow C([0,1], \mathbb{R}^2)$, $A(x)(t) = (A_1(x)(t), A_2(x)(t))$, attached to system (24) and defined by the relation:

$$\begin{matrix} A_1(x)(t) = \int_0^1 \left[\frac{t+2}{15}x_1(s) + \frac{t}{5}x_1(0) + \frac{t}{5}x_1(1)\right] ds + 2t + 1 \\ A_2(x)(t) = \int_0^1 \left[\frac{t+2}{21}x_2(s) + \frac{t}{7}x_2(0) + \frac{t}{7}x_2(1)\right] ds + \sin t \end{matrix}, \ t \in [0,1], \tag{26}$$

satisfies a generalized Lipschitz condition with the matrix $Q = \begin{pmatrix} 1/5 & 0 \\ 0 & 1/7 \end{pmatrix}$ and according to theorem 2, it results that the matrix $3(1-0)Q = \begin{pmatrix} 3/5 & 0 \\ 0 & 3/7 \end{pmatrix}$ converges to zero. So, the operator A is a contraction with the matrix $\begin{pmatrix} 3/5 & 0 \\ 0 & 3/7 \end{pmatrix}$.

The conditions of theorem 12 (Dobritoiu M., [14]) of existence and uniqueness of the solution of a system of integral equations, being satisfied, it results that the system of integral equations (24) has a unique solution $x^* \in C([0,1], \mathbb{R}^2)$ and the following estimates are true:

$$\|\,K(t,s,u_1,u_2,u_3) - H(t,s,u_1,u_2,u_3))\,\|_{\mathbb{R}^2} \leq \begin{pmatrix} 3 \\ 1 \end{pmatrix},$$

for all $t, s \in [0,1]$ and

$$\|f(t) - h(t)\|_{R^2} \leq \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \ for\ all\ t \in [0,1].$$

Under these conditions, if $y^* \in C([0,1], \mathbb{R}^2)$ is a solution of the system of integral equations (24), then according to theorem 13 (Dobritoiu M., [14]), the following estimate is true:

$$\|x^* - y^*\|_{R^2} \leq \begin{pmatrix} 25/2 \\ 21/4 \end{pmatrix}.$$

Example 5.3. We consider the system of integral equations:

$$\begin{cases} x_1(t) = \int_a^b \left[\frac{1}{10}(t+s)x_1(s) + \frac{2t+1}{15}x_1(a) + \frac{t+2}{15}x_1(b)\right] ds \ + \ 1 - \cos t \\ x_2(t) = \int_a^b \left[\frac{1}{2}x_1(s) + \frac{2t+s}{24}x_2(s) + \frac{2t+1}{24}x_2(a) + \frac{t+2}{24}x_2(b)\right] ds \ + \ \sin t \end{cases} \tag{27}$$

where $t, a, b \in [0,1]$, $K \in C([0,1] \times [0,1] \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2)$,
$K(t,s,u_1,u_2,u_3) = (K_1(t,s,u_1,u_2,u_3)\,,\ K_2(t,s,u_1,u_2,u_3))$,
$K_1(t,s,u_1,u_2,u_3) = \frac{1}{10}(t+s)u_{11} + \frac{2t+1}{15}u_{21} + \frac{t+2}{15}u_{31}$,
$K_2(t,s,u_1,u_2,u_3,u_4) = \frac{1}{2}u_{11} + \frac{2t+s}{24}u_{12} + \frac{2t+1}{24}u_{22} + \frac{t+2}{24}u_{32}$,
$f \in C([0,1], \mathbb{R}^2)$, $f(t) = (f_1(t), f_2(t))$, $f_1(t) = 1 - \cos t$, $f_2(t) = \sin t$, $x \in C([0,1], \mathbb{R}^2)$, and applying the theorem 14 it was studied the differentiability of the solution of this system with respect to a and b.

From the condition (ii) of the theorem 14, we have:

$$\begin{pmatrix} |K_1(t,s,u_1,u_2,u_3) - K_1(t,s,v_1,v_2,v_3)| \\ |K_2(t,s,u_1,u_2,u_3) - K_2(t,s,v_1,v_2,v_3)| \end{pmatrix} \leq$$

$$\leq \begin{pmatrix} 1/5 & 0 \\ 1/2 & 1/8 \end{pmatrix} \begin{pmatrix} |\,u_{11} - v_{11}\,| + |\,u_{21} - v_{21}\,| + |\,u_{31} - v_{31}\,| \\ |\,u_{12} - v_{12}\,| + |\,u_{22} - v_{22}\,| + |\,u_{32} - v_{32}\,| \end{pmatrix}, t, s \in [0,1].$$

According to theorem 2, it results that the matrix $3(b-a)Q = (b-a)\begin{pmatrix} 3/5 & 0 \\ 3/2 & 3/8 \end{pmatrix}$, $0 < b - a < 1$, $Q \in M_{2\times 2}(\mathbb{R}_+)$ converges to zero.

Hence, the conditions of theorem 14 being satisfied, it results that:

- the system of integral equations (27) has a unique solution $x^*(., a, b)$ in the space $C([0,1], \mathbb{R}^2)$;
- for all $x^0 \in C([0,1], \mathbb{R}^2)$, the sequence $(x^n)_{n\in\mathbb{N}}$, defined by the relation:
  $x^{n+1}(t;a,b) := \int_a^b K(t,s,x^n(s;a,b), x^n(a;a,b), x^n(b;a,b))ds + f(t)$
  converges uniformly to x^*, for all $t, a, b \in [0,1]$ and
  $\begin{pmatrix} \|x_1^n - x_1^*\|_C \\ \|x_2^n - x_2^*\|_C \end{pmatrix} \leq \begin{pmatrix} 5/2 & 0 \\ 6 & 8/5 \end{pmatrix} \cdot \begin{pmatrix} 3/5 & 0 \\ 3/2 & 3/8 \end{pmatrix}^n \cdot \begin{pmatrix} \left\|x_1^1 - x_1^0\right\|_C \\ \left\|x_2^1 - x_2^0\right\|_C \end{pmatrix}$;
- the function $x^* : [0,1] \times [0,1] \times [0,1] \longrightarrow \mathbb{R}^2)$, $(t;a,b) \mapsto x^*(t;a,b)$ is continuous;
- if $K(t,s,.,.,.,.) \in C^1(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}^2)$ for all $t, s \in [0,1]$, then $x^*(t;.,.) \in C^1([0,1] \times [0,1], \mathbb{R}^2)$ for all $t \in [0,1]$.

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