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GLOBAL EXISTENCE OF SOLUTION FOR REACTION DIFFUSION SYSTEMS WITH A FULL MATRIX

K. BOUKERRIOUA

Abstract. The aim of this paper is to study the global existence in time of solutions for some class of reaction-diffusion systems with full matrix.Our techniques are based on invariant regions and Lyapunov functional methods.Our goal is to show, under suitable assumptions, that the proposed model have a global solution for a large class of the functions f and g.

1. introduction

We consider the following reaction-diffusion system

$$\frac{\partial u}{\partial t} - a\Delta u - b\Delta v = f\left(u,v\right) \quad \text{in} \quad]0,+\infty[\times \Omega, \tag{1}$$

$$\frac{\partial v}{\partial t} - c\Delta u - a\Delta v = g\left(u,v\right) \quad \text{in} \quad]0,+\infty[\times \Omega, \tag{2}$$

with the following boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \qquad \text{in} \quad]0,+\infty[\times \partial\Omega, \tag{3}$$

and the initial data

$$u(0,x) = u_0, \quad v(0,x) = v_0 \quad \text{in } \Omega. \tag{4}$$

where Ω is an open bounded domain of classe C^1 in $\mathbb{R}^n$ with boundary $\partial\Omega$, $\frac{\partial}{\partial\eta}$ denotes the outward normal derivative on $\partial\Omega$, Δ denotes the Laplacian operator with respect to the x variable, a,b,c are positive constants satisfying the condition $b+c<2a$ which reflects the parabolicity of the system.The initial data are assumed to be in the following region

$$\Sigma = \left\{ (u_0,v_0) \in \mathbb{R}^2 \text{ such that } v_0 \geq \sqrt{\frac{c}{b}} \mid u_0 \mid \right\}. \tag{5}$$

The components $u(t,x)$ and $v(t,x)$ represent either chemical concentrations or biological population densities and system (1)-(4) is a mathematical model describing various chemical and biological phenomena (see E. L. Cussler [2], J.Savchik [10]).We suppose that the reaction terms f and g are continuousely differentiable on Σ, $(f\left(r,s\right), g\left(r,s\right))$ is in Σ, for all (r,s) in $\partial\Sigma$ such that

$$g(s,\sqrt{\frac{c}{b}}s) \geq \sqrt{\frac{c}{b}}f(s,\sqrt{\frac{c}{b}}s) \text{ and } g(-s,\sqrt{\frac{c}{b}}s) \geq -\sqrt{\frac{c}{b}}f(-s,\sqrt{\frac{c}{b}}s), \text{for all } \ s \geq 0. \tag{6}$$

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Assume further that

$$\sup(\left|(-\sqrt{c}f+\sqrt{b}g)(r,s)\right|,\left|(\sqrt{c}f+\sqrt{b}g)(r,s)\right|)\leq C(|r|+s+1)^{m}, \tag{7}$$

where C is a positive constant and $m \geq 1$.
We suppose that, one of the following conditions is satisfied:
1-There exist $p \geq 2$, $c(p) > 0$ and positive numbers $(Bi(p))_{0 \leq i \leq p}$ such that

$$\sqrt{c}(B_i(p)-B_{i-1}(p))f(r,s)+\sqrt{b}(B_i(p)+B_{i-1}(p))g(r,s)\leq C(p)(r+s+1), \tag{8}$$

where

$$a^2B_i^2(p)\leq .(a^2-bc)B_{i-1}(p)B_{i+1}(p). \tag{9}$$

2-There exist $c(1) > 0$ and $B_i(1)$, $0 \leq i \leq 1$ such that

$$\sqrt{c}(B_1(1)-B_0(1))f(r,s)+\sqrt{b}(B_1(1)+B_0(1))g(r,s)\leq C(1)(r+s+1), B_1(1), B_0(1)>0. \tag{10}$$

In the case when the coefficient $-\Delta u$ of (1) is different of the one of $-\Delta v$ in (2), N. Alikakos [1] established global existence and L^∞-bounds of solutions for positive initial data for $b = c = 0, f(u, v) = -g(u, v) = -uv^\beta$ and $1 < \beta < (n + 2)/n$. In [6], K.Masuda showed that solutions of the system established by N. Alikakos [1] exist globally for every $\beta > 1$. A. Haraux and A. Youkana [3] have generalized the method of K.Masuda to established a global existence result of system (1)-(4) for a large class of the function f and g: More precisely, they showed that for

$$f(u,v)=-g(u,v)=-u\varphi(v), \tag{11}$$

where φ satisfied the following condition

$$\lim_{v\to+\infty}\frac{\log\left(1+\varphi(v)\right)}{v}=0. \tag{12}$$

The present investigation is a continuation of results obtained in [7, 8] where $b = c = 0$ in [7] and $c = 0$ in [8]. In this study, we will treat the case of a general full matrix where $a = d, b > 0, c > 0$.

2. Existence of local solutions

The usual norms in spaces $L^p(\Omega), L^\infty(\Omega)$ and $C(\overline{\Omega})$ are respectively denoted by :

$$\|u\|_p^p=\frac{1}{|\Omega|}\int_\Omega |u(x)|^p\,dx,$$

$$\|u\|_\infty = ess\sup_{x\in\Omega}|u(x)|,$$

$$\|u\|_{C(\overline{\Omega})}=\max_{x\in\overline{\Omega}}|u(x)|.$$

If For any initial data in $C(\overline{\Omega})$ or $L^p(\Omega)$, $p \in (1, \infty)$, local existence and uniqueness of solutions to the initial value problem (1)-(4) follow from the basic existence theory for abstract semilinear differential equations (see D. Henry [4] and A. Pazy [11]). The solutions are classical on$]0, T^*[$,where T^* denotes the eventual blowing-up time in $L^\infty(\Omega)$.
Furthermore,if $T^* < +\infty$, then

$$\lim_{t\uparrow T^*}(\|u(t)\|_\infty+\|v(t)\|_\infty)=+\infty.$$

Therefore, if there exists a positive constant C such that

$$\|u(t)\|_\infty + \|v(t)\|_\infty \le C, \forall t \in]0, T^*[\,,$$

then $T^* = +\infty$.

3. Existence of global solutions

If we multiplying equation (1) through by $\sqrt{c}$ and equation (2) by $\sqrt{b}$, subtracting the resulting equations one time and adding them an other time we have

$$\frac{\partial w}{\partial t} - \left(a + \sqrt{bc}\right)\Delta w = F(w,z) \quad \text{in }]0,T^*[\times\Omega, \tag{13}$$

$$\frac{\partial z}{\partial t} - \left(a - \sqrt{bc}\right)\Delta z = G(w,z) \text{ in }]0,T^*[\times\Omega, \tag{14}$$

with the boundary conditions

$$\frac{\partial w}{\partial \eta} = \frac{\partial z}{\partial \eta} = 0 \text{ in }]0,T^*[\times\partial\Omega, \tag{15}$$

and the initial data

$$w(0,x) = w_0(x), z(0,x) = z_0(x) \text{ in } \Omega, \tag{16}$$

where,

$$\begin{aligned} w(t,x) &= \sqrt{c}u(t,x) + \sqrt{b}v(t,x), \\ z(t,x) &= -\sqrt{c}u(t,x) + \sqrt{b}v(t,x), \end{aligned} \tag{17}$$

for all (t,x) in$]0,T^*[\times\Omega$ and

$$\begin{aligned} F(w,z) &= (\sqrt{c}f + \sqrt{b}g)(u,v), \\ G(w,z) &= (-\sqrt{c}f + \sqrt{b}g)(u,v), \text{ for all } (u,v) \text{ in } \Sigma. \end{aligned} \tag{18}$$

To prove global existence of solutions for (1)-(4), one needs to prove it for problem (13)-(16).To this subject, it is well known that, it suffices to derive an uniform estimate of the quantity

$$\sup(\|(F(w,z)\|_q, \|(G(w,z)\|_q), \tag{19}$$

for some $q > \frac{n}{2}$.

Now, we present the main result

Theorem 1. *Let $(w(t.), z(t,))$ be a solution of (13)-(16). If one of the conditions (8) or (10) has been satisfied,there would exist an integer $p \ge 1$ and a continuous function ω_p :$\mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\sup(\|(w(t,.)\|_p, \|z(t,.)\|_p) \le \omega_p(t), t < T^*. \tag{20}$$

Proof. Let us consider the function defined by Lp

$$L_p(t) = \int_\Omega \left(\sum_{i=0}^{i=p} C_p^i B_i(p) w^i z^{p-i}\right)dx = \int_\Omega \left(\sum_{i=0}^{i=p} \alpha_i(p) w^i z^{p-i}\right)dx, \tag{21}$$

where

$$\alpha_i(p) = C_p^i B_i(p), i = 0, \ldots, p. \tag{22}$$

Differentiating L_p with respect to t , we obtain

$$L_p^{'}(t)=\int_\Omega(\sum_{i=1}^{i=p}i\alpha_i(p)w^{i-1}z^{p-i})\frac{\partial w}{\partial t}dx+\int_\Omega(\sum_{i=0}^{i=p-1}(p-i)\alpha_i(p)w^iz^{p-i-1})\frac{\partial z}{\partial t}dx. \quad (23)$$

Consequently,

$$L_p^{'}(t)=\int_\Omega(\sum_{i=1}^{i=p}i\alpha_i(p)w^{i-1}z^{p-i})\frac{\partial w}{\partial t}dx+\int_\Omega(\sum_{i=1}^{i=p}(p-i+1)\alpha_{i-1}(p)w^{i-1}z^{p-i})\frac{\partial z}{\partial t}dx. \quad (24)$$

using (13) and (14) , we gets

$$\begin{aligned} L_p^{'}(t) &= \int_\Omega(\sum_{i=1}^{i=p}i\alpha_i(p)w^{i-1}z^{p-i})(F(w,z)+\left(a+\sqrt{bc}\right)\Delta w)dx+ \quad (25)\\ + & \int_\Omega(\sum_{i=1}^{i=p}(p-i+1)\alpha_{i-1}(p)w^{i-1}z^{p-i})(G(w,z)+\left(a-\sqrt{bc}\right)\Delta z\ )dx. \end{aligned}$$

which implies

$$\begin{aligned} L_p^{'}(t) &= \int_\Omega\sum_{i=1}^{i=p}i\alpha_i(p)F(w,z)\,w^{i-1}z^{p-i}dx+ \quad (26)\\ &\int_\Omega(\sum_{i=1}^{i=p}(p-i+1)\alpha_{i-1}(p)G(w,z)\,w^{i-1}z^{p-i}dx+\\ + &\int_\Omega\sum_{i=1}^{i=p}(\left(a+\sqrt{bc}\right)i\alpha_i(p)\Delta w)w^{i-1}z^{p-i})dx+\\ &+\int_\Omega\sum_{i=1}^{i=p}(\left(a-\sqrt{bc}\right)(p-i+1)\alpha_{i-1}(p)\Delta z)w^{i-1}z^{p-i}dx, \end{aligned}$$

we distinguish two cases:

1-when $p=1$, we obtain from (26)

$$\begin{aligned} L_1^{'}(t) &= \int_\Omega(\left(a+\sqrt{bc}\right)\alpha_1(1)\Delta w+\left(a-\sqrt{bc}\right)\alpha_0(1)\Delta z)dx+ \quad (27)\\ &\int_\Omega(\alpha_1(1)F(w,z)+\alpha_0(1)G(w,z))dx, \end{aligned}$$

By applying Green's formula, we obtain

$$L_1^{'}(t)=\int_\Omega(B_1(1)F(w,z)+B_0(1)G(w,z))dx, \quad (28)$$

then

$$\begin{aligned} L_1^{'}(t) &= \int_\Omega(B_1(1)\left[(\sqrt{c}f+\sqrt{b}g)(u,v)\right]+B_0(1)\left[(-\sqrt{c}f+\sqrt{b}g)(u,v)\right])dx,\\ &= \int_\Omega\sqrt{c}(B_1(1)-B_0(1))f(u,v)+\sqrt{b}(B_1(1)+B_0(1))g(u,v). \end{aligned}$$

Using condition (10),(17) and (18), we deduce

$$\begin{aligned} L_1'(t) &\leq C(1)\int_\Omega (u+v+1)dx = \\ & C(1)\int_\Omega \frac{1}{2\sqrt{bc}}((\sqrt{c}+\sqrt{b})w+(\sqrt{c}-\sqrt{b})z)dx + C(1)mes(\Omega), \end{aligned}$$

It follows that

$$L_1'(t) \leq \omega_1 L_1(t) + \omega_2, t < T^*, \tag{29}$$

where

$$\begin{aligned} \omega_1 &= C(1)\frac{1}{2\sqrt{bc}}(\sqrt{b}+\sqrt{c})\max(\frac{1}{B_1(1)},\frac{1}{B_0(1)}), \\ \omega_2 &= C(1)mes(\Omega). \end{aligned}$$

By a simple integration of (29) for all $t < T^*$, we have

$$L_1(t) \leq (L_1(0)+\frac{\omega_2}{\omega_1})\exp(\omega_1 t) - \frac{\omega_2}{\omega_1}), \tag{30}$$

from (21) , we obtain

$$\begin{aligned} L_1(t) &\geq \min(B_1(1),B_0(1))\int_\Omega (w+z)dx \geq \\ & \min(B_1(1),B_0(1)) \sup \|w(t,.),z(t,.)\|_1 , \end{aligned} \tag{31}$$

Then we get

$$\sup \|w(t,.),z(t,.)\|_1 \leq \omega_3(t), \text{for } t \leq T^*,$$

where

$$\omega_3(t) = \frac{1}{\min(B_1(1),B_0(1))} \times \left\{(L_1(0)+\frac{\omega_2}{\omega_1})\exp(\omega_1 t) - \frac{\omega_2}{\omega_1})\right\},$$

2-If $p \geq 2$. We set

$$\begin{aligned} T = \int_\Omega \sum_{i=1}^{i=p}(\left(a+\sqrt{bc}\right) i\alpha_i(p)\Delta w)w^{i-1}z^{p-i}dx + \\ \int_\Omega \sum_{i=1}^{i=p}(\left(a-\sqrt{bc}\right)(p-i+1)\alpha_{i-1}(p)\Delta z)w^{i-1}z^{p-i})dx, \end{aligned} \tag{32}$$

then (32) can be written as

$$\begin{aligned} T &= \sum_{i=1}^{i=p}\int_\Omega \Delta(\left(a+\sqrt{bc}\right) i\alpha_i(p)w)w^{i-1}z^{p-i}dx + \\ \sum_{i=1}^{i=p} & \int_\Omega \Delta(\left(a-\sqrt{bc}\right)(p-i+1)\alpha_{i-1}(p)z)w^{i-1}z^{p-i}dx, \end{aligned}$$

Then, Green's formula, gives

$$T = -\sum_{i=1}^{i=p}\int_\Omega (\nabla(\left(a+\sqrt{bc}\right) i\alpha_i(p)w)\nabla(w^{i-1}z^{p-i})dx \quad (33)$$
$$-\sum_{i=1}^{i=p}\int_\Omega \nabla(\left(a-\sqrt{bc}\right)(p-i+1)\alpha_{i-1}(p)z)\nabla(w^{i-1}z^{p-i})dx,$$

then

$$\begin{aligned} T &= -\int_\Omega \sum_{i=2}^{i=p}\left(a+\sqrt{bc}\right) i(i-1)\alpha_i(p)w^{i-2}z^{p-i}\nabla^2 w dx + \\ &\int_\Omega \sum_{i=1}^{i=p-1}\left(a+\sqrt{bc}\right) i(p-i\ \ )w^{i-1}z^{p-i-1}\nabla w\nabla z dx + \\ &\int_\Omega \sum_{i=2}^{i=p}\left(a-\sqrt{bc}\right)(i-1)(p-i+1)\alpha_{i-1}(p)w^{i-2}z^{p-i}\nabla w\nabla z dx + \\ &\int_\Omega \sum_{i=1}^{i=p-1}\left(a-\sqrt{bc}\right)(p-i+1)(p-i)\alpha_{i-1}(p)w^{i-1}z^{p-i-1}\nabla^2 z dx, \end{aligned}$$

By a simple computation and from (26), it follows that

$$\begin{aligned} L_p^{'}(t) &= \int_\Omega \sum_{i=1}^{i=p}(iC_p^i B_i(p))F(w,z))w^{i-1}z^{p-i}dx + \\ &\int_\Omega \left(\sum_{i=1}^{i=p}(p-i+1)C_p^{i-1}B_{i-1}(p)G(w,z)\right) w^{i-1}z^{p-i})dx - \\ &\int_\Omega \sum_{i=1}^{i=p-1}\left(w^{i-1}z^{p-i-1}\left\{\left(a+\sqrt{bc}\right) i(i+1)C_p^{i+1}B_{i+1}(p)\nabla^2 w + \right.\right. \\ &2ai(p-i)C_p^i B_i(p)\nabla w\nabla z + \\ &\left.\left.\left(a-\sqrt{bc}\right)(p-i)(p-i+1)C_p^{i-1}B_{i-1}(p)\nabla^2 z\right\}\right) dx, \end{aligned}$$

Using the fact that

$$\begin{aligned} iC_p^i &= (p-i+1)C_p^{i-1} = pC_{p-1}^{i-1}, \\ i(i+1)C_p^{i+1} &= i(p-i)C_p^i = (p-i)(p-i+1)C_p^{i-1} = p(p-1)C_{p-2}^{i-1}, \end{aligned}$$

we conclude

$$\begin{aligned} L_p^{'}(t) &= \int_\Omega (\sum_{i=1}^{i=p}(pC_{p-1}^{i-1}\left[B_i(p)F(w,z)+B_{i-1}(p)G(w,z)\right]w^{i-1}z^{p-i})dx - \\ &p(p-1)\int_\Omega (\sum_{i=1}^{i=p-1} C_{p-2}^{i-1}\left[\left(a+\sqrt{bc}\right)B_{i+1}(p)\nabla^2 w\right. \\ &\left.+(2a)B_i(p)\nabla w\nabla z+\left(a-\sqrt{bc}\right)B_{i-1}(p)\nabla^2 z\right] w^{i-1}z^{p-i-1})dx, \end{aligned}$$

it follows that the quadratic forms

$$\left(a+\sqrt{bc}\right)B_{i+1}(p)\triangledown^2 w+2aB_i(p)\triangledown w\triangledown z+\left(a-\sqrt{bc}\right)B_{i-1}(p)\triangledown^2 z,$$

are positive since from (9)

$$(2aB_i(p))^2-4\left(a+\sqrt{bc}\right)B_{i+1}\left(a-\sqrt{bc}\right)B_{i-1}=(2a)^2(B_i(p))^2-4(a^2-bc)B_{i+1}B_{i-1}\leq 0,$$

Consequently, □

$$L_p^{'}(t)\leq p(\int_\Omega\sum_{i=1}^{i=p}C_{p-1}^{i-1}\left[B_i(p))F(w,z)+B_{i-1}(p)G(w,z)\right]w^{i-1}z^{p-i}dx.$$

Using condition (8), (17) and (18) we get

$$L_p^{'}(t)\leq p\int_\Omega(\sum_{i=1}^{i=p}C_{p-1}^{i-1}(\frac{1}{2\sqrt{bc}}(\sqrt{b+}\sqrt{c})w+(\sqrt{c}-\sqrt{b})z+1)w^{i-1}z^{p-i})dx,$$

by a simple computation, we obtain, for an appropriate constant $w_0(p)$

$$L_p^{'}(t)\leq\omega_0(p)(\int_\Omega\sum_{i=0}^{i=p}C_p^iw^i.z^{p-i}dx+\int_\Omega(\sum_{i=0}^{i=p-1}C_{p-1}^iw^i.z^{p-i-1}dx),\qquad(34)$$

Using the fact that

$$\sum_{i=0}^{i=p-1}C_{p-1}^iw^i.z^{p-i-1}=(w+z)^{p-1},$$

then the inequality (34) can be written as

$$L_p^{'}(t)\leq\omega_1(p)L_p(t)+\omega_0(p)\int_\Omega(w+z)^{p-1}dx,\qquad(35)$$

Applying Hölder's inequality to the second term in the right hand side of the above inequality, we obtain

$$L_p^{'}(t)\leq\omega_1(p)L_p(t)+\omega_0(p)(mes(\Omega))^{\frac{1}{p}}(\int_\Omega(w+z)^pdx)^{\frac{p-1}{p}},\qquad(36)$$

Since the following inequality holds,

$$(w+z)^p=\sum_{i=0}^{i=p}C_p^iw^iz^{p-i}\leq\frac{\sup_{0\leq i\leq p}C_p^i}{\min_{0\leq i\leq p}\alpha_i(p)}\sum_{i=0}^{i=p}\alpha_i(p)w^iz^{p-i}$$

Hence, the functional L_p satisfies the following diferential inequality

$$L_p^{'}(t)\leq\omega_1(p)L_p(t)+\omega_2(p)(L_p(t))^{\frac{p-1}{p}},\forall t<T^*,\qquad(37)$$

where

$$\omega_2(p)=\omega_0(p)(mes(\Omega))^{\frac{1}{p}}(\frac{\sup_{1\leq i\leq p}C_p^i}{\min_{1\leq i\leq p}\alpha_i(p)})^{\frac{p-1}{p}},$$

while putting

$$q(t)=(L_p(t))^{\frac{1}{p}}.$$

One gets

$$pq^{'}(t)\leq\omega_1(p)q(t)+\omega_2(p),$$

which gives us, by a simple integration

$$(L_p(t))^{\frac{1}{p}} \leq \left[(L_p(0))^{\frac{1}{p}} + \frac{\omega_2(p)}{\omega_1(p)}\right] \exp(\frac{\omega_1(p)}{p}t) - \frac{\omega_2(p)}{\omega_1(p)}, \tag{38}$$

By using the inequality

$$L_p(t) = \int_\Omega (\sum_{i=0}^{i=p} \alpha_i(p) w^i z^{p-i}) dx \geq \int_\Omega (\alpha_p(p) w^p + \alpha_0(p) z^p) dx, \tag{39}$$

then

$$(L_p(t))^{\frac{1}{p}} \geq \min(\alpha_p(p), \alpha_0(p))^{\frac{1}{p}} \times \sup((\int_\Omega w^p dx)^{\frac{1}{p}}, (\int_\Omega z^p dx)^{\frac{1}{p}})), \tag{40}$$

and therefore, for all $t < T^*$,

$$\sup(\|(w(t,.)\|_p, \|, z(t,.)\|_p) \leq \frac{(L_p(t))^{\frac{1}{p}}}{\min(\alpha_p(p), \alpha_0(p))^{\frac{1}{p}}}, \tag{41}$$

from (38), (40) and (41), we obtain

$$\sup(\|(w(t,.)\|_p, \|z(t,.)\|_p) \leq \omega_p(t), \forall t < T^*, \tag{42}$$

where

$$\omega_p(t) = \frac{\left[L_p(0) + \frac{\omega_2(p)}{\omega_1(p)}\right] \exp(\omega_1 t) - \frac{\omega_2(p)}{\omega_1(p)}}{\min(\alpha_p(p), \alpha_0(p))^{\frac{1}{p}}}.$$

The proof of Lemma is completed.

Theorem 2. *Let $(w(t,.), z(t,.))$ be a solution of problem (13)-(16). We assume that the condition (7) holds and one of the conditions (8) or (10) are satisfied . In addition if $\frac{p}{m} > \frac{n}{2}$, then the solution $(w(t,.), z(t,.))$ exists globally in time.*

Proof. From (7), we have

$$\sup(\left|(\sqrt{c}f + \sqrt{b}g)(u,v)\right|^{\frac{p}{m}}, \left|(-\sqrt{c}f + \sqrt{b}g)(u,v)\right|^{\frac{p}{m}}) \leq C^{\frac{p}{m}}(|u| + v + 1)^p, \forall v \geq 0,$$

then

$$\begin{aligned} \sup(\left\|(\sqrt{c}f + \sqrt{b}g)(u,v)\right\|_{\frac{p}{m}}^{\frac{p}{m}}, \left\|(-\sqrt{c}f + \sqrt{b}g)(u,v)\right\|_{\frac{p}{m}}^{\frac{p}{m}}) &\leq C^{\frac{p}{m}} \int_\Omega (|u| + v + 1)^p dx \\ &\leq C' \int_\Omega (w + z + 1)^p dx, \end{aligned}$$

where

$$C' = C^{\frac{p}{m}} \max(\frac{1}{2\sqrt{bc}}(\sqrt{b} + \sqrt{c}), 1)^p,$$

using the following formula

$$\begin{aligned} \int_\Omega (w + z + 1)^p dx &= \int_\Omega (\sum_{i=0}^{i=p} C_p^i (w+z)^i dx = \int_\Omega ((w+z)^p + 1) dx + \\ &+ \sum_{i=1}^{i=p-1} C_p^i \int_\Omega (w+z)^k dx, \end{aligned} \tag{43}$$

An application of Hölder's inequality from (43), gives

$$\begin{aligned}\int_\Omega (w+z+1)^p dx &\leq mes(\Omega) + \int_\Omega (w+z)^p dx + \\ & \sum_{i=1}^{i=p-1} C_p^i (mes(\Omega))^{\frac{p-i}{p}} (\int_\Omega (w+z)^p dx)^{\frac{k}{p}},\end{aligned} \tag{44}$$

using (42), we get

$$(\int_\Omega (w+z)^p dx)^{\frac{1}{p}} = \|w(t,.)+z(t,.)\|_p \leq \|w(t,.)\|_p + \|z(t,.)\|_p \leq 2\omega_p(t), \tag{45}$$

and the inequality (44) can be written as follows

$$\begin{aligned}\int_\Omega (w+z+1)^p dx &\leq mes(\Omega) + 2^p(\omega_p(t))^p + \\ \sum_{i=1}^{i=p-1} C_p^i 2^i (\omega_p(t))^i (mes(\Omega))^{\frac{p-i}{p}} &= \sum_{k=0}^{k=p} C_p^i 2^i (\omega_p(t))^i (mes(\Omega))^{\frac{p-i}{p}}.\end{aligned} \tag{46}$$

□

Therefore

$$\begin{aligned}&\sup(\|F(w,z)\|_{\frac{p}{m}}^{\frac{p}{m}}, \|G(w,z)\|_{\frac{p}{m}}^{\frac{p}{m}}) \\ &= \sup(\left\|(\sqrt{c}f+\sqrt{b}g)(u,v)\right\|_{\frac{p}{m}}^{\frac{p}{m}}, \left\|(-\sqrt{c}f+\sqrt{b}g)(u,v)\right\|_{\frac{p}{m}}^{\frac{p}{m}}) \\ &\leq C' \int_\Omega (w+z+1)^p dx, \forall t < T^*, \ \frac{p}{m} > \frac{n}{2} \\ &\leq C' \sum_{k=0}^{k=p} C_p^i 2^i (\omega_p(t))^i (mes(\Omega))^{\frac{p-i}{p}} = C_{p,m}(t), \forall t < T^*, \frac{p}{m} > \frac{n}{2},\end{aligned}$$

where

$$C_{p,m}(t) = C'(\sum_{i=0}^{i=p} C_p^i 2^i (\omega_p(t))^i (mes(\Omega))^{\frac{p-i}{p}}).$$

It' s clear that if we take $q = \frac{p}{m}$, we obtain an uniform estimate of $\sup(\|(F(w,z)\|_q, \|(G(w,z)\|_q)$, for some $q > \frac{n}{2}$.

It has been mentioned above, if we derive an uniform estimate of $\|(F(w,z)\|_q$ and $\|(G(w,z)\|_q$ on $]0,T^*[$, then the solution $(w(t,.),z(t,.))$ of (13)-(16) exists globally in time.

Now, by (17), it is easy to see that the solution of problem (1)-(4) exists globally in time.

Remark 1. *Applying invariant region's method see* [10]*, from* (6)*, we obtain the positivity of the solution of* (13)-(16)*, which shows that the solution of the system remains in the region* Σ*. For more details, one can consult* [5].

References

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University of 8 Mai 1945 Guelma
Guelma, Algeria.
E-mail address: khaledv2004@yahoo.fr