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ON SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY GENERALIZED SALAGEAN OPERATOR AND RUSCHEWEYH OPERATOR

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Abstract. A subclass of complex-valued harmonic univalent function defined by generalized Salagean operator and Ruscheweyh operator is introduced. Coefficient bounds, distortion theorem, and other properties of this class are obtained.

1. Introduction and Definitions

In any complex domain G a continuous function $f = u + iv$ is said to be harmonic in G if both u and v are real harmonic in G. A harmonic complex-valued function might be expressed in terms of analytic functions, h and g, in *simply* connected domain $D \subset G$ as $f = h + \overline{g}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| \geq |g'(z)|$ in D (see[3]).

Denote by $\mathcal{H}$ the family of functions $f = h + \overline{g}$, that are harmonic univalent and sense preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Thus for $f = h + \overline{g}$ in $\mathcal{H}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad 0 \leq b_1 < 1. \tag{1}$$

Note that the family of harmonic univalent functions $\mathcal{H}$, reduces to the class of analytic univalent functions S, which can be written in the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{2}$$

if the co-analytic part of $f = h + \overline{g}$ is identically zero that is $g \equiv 0$.

Also denote by $\mathcal{T}$(see [7]), the subclass of $\mathcal{H}$ consisting of all functions $f = h+\overline{g}$ where f and g are given by

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \qquad g(z) = -\sum_{k=1}^{\infty} |b_k| z^k, \quad 0 \leq b_1 < 1. \tag{3}$$

Definition 1 ([2]). *Let $f(z)$ be given by (2), $\lambda \geq 0$, and $n \in \mathbb{N}_0$. Then the differential operator D_λ^n is defined by*

$$D_\lambda^n = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)]^n a_k z^k. \qquad (z \in U)$$

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Definition 2 ([4]). *Let $f(z)$ be given by (2), $\beta \geq 0$, and $n,\ \alpha \in \mathbb{N}_0$. Then the operator $R^n_{\alpha,\beta}$ is defined by*

$$R^n_{\alpha,\beta}f(z) = z + \sum_{k=2}^{\infty}[1+\beta(k-1)]^n C(\alpha,k)a_k z^k, \qquad (z \in U)$$

where $C(\alpha,k) = \binom{k+\alpha-1}{\alpha}$.

Next definition provides a linear combination between D^n_λ and $R^n_{\alpha,\beta}$ which was introduced in [1].

Definition 3 ([1]). *Let $f(z)$ be given by (2), $\lambda,\beta,\gamma \geq 0$, and $n,\alpha \in \mathbb{N}_0$. Then the operator $D^n_{\lambda,\alpha,\beta,\gamma}$ is defined by*

$$D^n_{\lambda,\alpha,\beta,\gamma}f(z) = (1-\gamma)R^n_{\alpha,\beta}f(z) + \gamma D^n_\lambda f(z). \tag{4}$$

In 2002, Jahangiri et al. [5] introduced the modified Salagean operator of harmonic univalent function. In 2003, Murugusundaramoorthy [6] introduced the modified Ruscheweyh of harmonic univalent function. In the next definition we will modify the operator in Definition 3 to harmonic univalent function.

Definition 4. *Let $f = h + \overline{g}$ be given by (1), $\lambda,\beta,\gamma \geq 0$, and $n,\alpha \in \mathbb{N}_0$. We define the following differential operator*

$$\tilde{D}^n_{\lambda,\alpha,\beta,\gamma}f(z) = D^n_{\lambda,\alpha,\beta,\gamma}h(z) + \overline{D^n_{\lambda,\alpha,\beta,\gamma}g(z)},$$

where $D^n_{\lambda,\alpha,\beta,\gamma}f(z)$ given by(4).

We let $\mathcal{D}_{\mathcal{H}}(n,\lambda,\alpha,\beta,\gamma,\mu)$ denote the family of harmonic functions $f = h+\overline{g}$ for which

$$\Re\left\{\left(\tilde{D}^n_{\lambda,\alpha,\beta,\gamma}f(z)\right)'\right\} > \mu, \quad (z \in U)$$

We further denote by $\mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu)$, the subclass of $\mathcal{D}_{\mathcal{H}}(n,\lambda,\alpha,\beta,\gamma,\mu)$, where

$$\mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu) = \mathcal{T} \cap \mathcal{D}_{\mathcal{H}}(n,\lambda,\alpha,\beta,\gamma,\mu).$$

2. Coefficient Bounds

In this section, coefficient bound of the classes $\mathcal{D}_{\mathcal{H}}(n,\lambda,\alpha,\beta,\gamma,\mu)$ and $\mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu)$ are given.

Theorem 1. *Let $f = h+\overline{g}$ given by* (1), $0 \leq \mu < 1, n,\alpha \in \mathbb{N}_0, a_1 = 1, \lambda,\beta,\gamma \geq 0$. *If*

$$\sum_{k=2}^{\infty} \frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu}|a_k| + \frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu}|b_k| \leq 2, \tag{5}$$

where

$$\psi(n,k,\lambda,\alpha\,\beta,\gamma) = k\left(\gamma[1+\lambda(k-1)]^n + (1-\gamma)[1+\beta(k-1)]^n\right)C(\alpha,k). \tag{6}$$

Then f is sense preserving, harmonic univalent in U and $f \in \mathcal{D}_{\mathcal{H}}(n,\lambda,\alpha,\beta,\gamma,\mu)$.

Proof. Note first that

$$\begin{aligned}|h'(z)| &\geq 1 - \sum_{k=2}^{\infty}\psi(n,k,\lambda,\alpha\,\beta,\gamma)|a_k||z|^{k-1}\\ &> \sum_{k=2}^{\infty}\psi(n,k,\lambda,\alpha\,\beta,\gamma)|b_k||z|^{k-1}\\ &\geq |g'(z)|,\end{aligned}$$

so that f is locally univalent and sense preserving.

To show that f is univalent in U, we consider that (5) holds. If $g(z)$ vanish, then f is analytic. And then, the univalence of f comes from its close-to-convexity. If $g(z) \neq 0$ and z_1, z_2 are any distant points in U, then

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\ &= 1-\left|\sum_{k=2}^{\infty}\frac{b_k(z_1^k-z_2^k)}{(z_1-z_2)+\sum_{k=2}^{\infty}a_k(z_1^k-z_2^k)}\right| \\ &> 1-\frac{\sum_{k=2}^{\infty}k|b_k|}{1-\sum_{k=2}^{\infty}k|a_k|} \\ &\geq 1-\frac{\sum_{k=2}^{\infty}\frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu}|b_k|}{1-\sum_{k=2}^{\infty}\frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu}|a_k|} \\ &\geq 0.\end{aligned}$$

Therefore, f is univalent.

Using the fact that $\Re\, w \geq \mu$ if and only if $|1-\mu+w| \geq |1+\mu-w|$, it is suffices to show that

$$\left|1-\mu+\left(\tilde{D}^n_{\lambda,\alpha,\beta,\gamma}f(z)\right)'\right|-\left|1-\mu+\left(\tilde{D}^n_{\lambda,\alpha,\beta,\gamma}f(z)\right)'\right| \geq 0. \tag{7}$$

Substituting for $\left(\tilde{D}^n_{\lambda,\alpha,\beta,\gamma}f(z)\right)'$ in (7) yields

$$\begin{aligned}&\left|1-\mu+\left(\tilde{D}^n_{\lambda,\alpha,\beta,\gamma}f(z)\right)'\right|-\left|1-\mu+\left(\tilde{D}^n_{\lambda,\alpha,\beta,\gamma}f(z)\right)'\right| \\ &\geq 2(1-\mu)-2\sum_{k=2}^{\infty}\psi(n,k,\lambda,\alpha\,\beta,\gamma)|a_k||z|^{k-1}-2\sum_{k=2}^{\infty}\psi(n,k,\lambda,\alpha\,\beta,\gamma)|b_k||z|^{k-1} \\ &= 2(1-\mu)\left\{1-\sum_{k=2}^{\infty}\frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu}|a_k|-\sum_{k=2}^{\infty}\frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu}|b_k|\right\} \\ &> 2(1-\mu)\left\{1-\left(\sum_{k=2}^{\infty}\frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu}|a_k|+\sum_{k=2}^{\infty}\frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu}|b_k|\right)\right\}\end{aligned}$$

This last expression is non-negative by (5), and so $f \in \mathcal{D}_{\mathcal{H}}(n,\lambda,\alpha,\beta,\gamma,\mu)$. □

Theorem 2. *Let $f = h + \overline{g}$ be given by* (3). *Then $f \in \mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu)$ in and only if*

$$\sum_{k=2}^{\infty}\frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu}|a_k|+\frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu}|b_k| \leq 2. \tag{8}$$

Proof. Since $\mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu) \subset \mathcal{D}_{\mathcal{H}}(n,\lambda,\alpha,\beta,\gamma,\mu)$, we only need to prove the "only if" part of the theorem. To do so, assume that $f \in \mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu)$. Then by (8) we have

$$\Re\{\left(\tilde{D}^n_{\lambda,\alpha,\beta,\gamma}f(z)\right)'\} = \Re\{\left(\tilde{D}^n_{\lambda,\alpha,\beta,\gamma}h(z)\right)'+\overline{\left(\tilde{D}^n_{\lambda,\alpha,\beta,\gamma}g(z)\right)'}\}$$

$$= \Re\left\{1-\sum_{k=2}^{\infty}\psi(n,k,\lambda,\alpha\,\beta,\gamma)|a_k|z_k^{k-1}-\sum_{k=1}^{\infty}\psi(n,k,\lambda,\alpha\,\beta,\gamma)|b_k|\overline{z_k{}^{k-1}}\right\} > \mu.$$

If we choose z to be real and let $z \to 1^-$, we get

$$1 - \sum_{k=2}^{\infty} \psi(n,k,\lambda,\alpha\,\beta,\gamma)|a_k| - \sum_{k=1}^{\infty} \psi(n,k,\lambda,\alpha\,\beta,\gamma)|b_k| > \mu.$$

Which is precisely the assertion of Theorem 2. □

3. Distortion Theorem and Extreme Points

In this section, distortion theorem and extreme points of the class $\mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu)$ are obtained.

Theorem 3. *If* $f \in \mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu), 0 \leq \mu < 1, n, \alpha \in \mathbb{N}_0,\ a_1 = 1, \lambda,\beta,\gamma \geq 0,$ *and* $|z| = r < 1$, *then*

$$|f(z)| \leq (1+|b_1|)r + \left(\frac{1-\mu}{\psi(n,2,\lambda,\alpha\,\beta,\gamma)} - \frac{\psi(n,1,\lambda,\alpha\,\beta,\gamma)}{\psi(n,2,\lambda,\alpha\,\beta,\gamma)}|b_1| \right) r^2$$

and

$$|f(z)| \leq (1-|b_1|)r - \left(\frac{1-\mu}{\psi(n,2,\lambda,\alpha\,\beta,\gamma)} - \frac{\psi(n,1,\lambda,\alpha\,\beta,\gamma)}{\psi(n,2,\lambda,\alpha\,\beta,\gamma)}|b_1| \right) r^2,$$

where $\psi(n,1,\lambda,\alpha\,\beta,\gamma)$ *given by* (6).

Proof. We will only prove the right hand inequality. The argument for the left hand inequality is similar. Let $f \in \mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu)$ take the absolute value of f, we obtain

$$\begin{aligned} |f(z)| &\leq (1+|b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|)\, r^k \\ &\leq (1+|b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|)\, r^2 \end{aligned}$$

That is,

$$\begin{aligned} |f(z)| &\leq (1+|b_1|)r + \frac{1-\mu}{\psi(n,2,\lambda,\alpha\,\beta,\gamma)} \left(\sum_{k=2}^{\infty} \frac{\psi(n,2,\lambda,\alpha\,\beta,\gamma)}{1-\mu}|a_k| + \frac{\psi(n,2,\lambda,\alpha\,\beta,\gamma)}{1-\mu}|b_k| \right) r^2 \\ &\leq (1+|b_1|)r + \frac{1-\mu}{\psi(n,2,\lambda,\alpha\,\beta,\gamma)} \left(1 - \frac{\psi(n,1,\lambda,\alpha\,\beta,\gamma)}{1-\mu}|b_1| \right) r^2 \\ &\leq (1+|b_1|)r + \frac{1-\mu}{\psi(n,2,\lambda,\alpha\,\beta,\gamma)} \left(\frac{1-\mu}{\psi(n,2,\lambda,\alpha\,\beta,\gamma)} - \frac{\psi(n,1,\lambda,\alpha\,\beta,\gamma)}{\psi(n,2,\lambda,\alpha\,\beta,\gamma)}|b_1| \right) r^2. \end{aligned}$$

□

Corollary 1. *Let* f *be of the form* (3) *so that* $f \in \mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu)$. *Then*

$$\{w : |w| < \frac{\psi(n,2,\lambda,\alpha\,\beta,\gamma)+\mu-1}{\psi(n,2,\lambda,\alpha\,\beta,\gamma)} - \frac{\psi(n,2,\lambda,\alpha\,\beta,\gamma)-\psi(n,1,\lambda,\alpha\,\beta,\gamma)}{\psi(n,2,\lambda,\alpha\,\beta,\gamma)}|b_1|\} \subset f(U).$$

Theorem 4. *Let* $f = h + \overline{g}$ *be given by*(3). *Then* $f \in \mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu)$ *if and only if*

$$f(z) = \sum_{k=1}^{\infty} (\delta_k h_k(z) + \sigma_k g_k(z))$$

where

$$
\begin{aligned}
h_1(z) &= z\\
h_k(z) &= z - \frac{1-\mu}{\psi(n,2,\lambda,\alpha\,\beta,\gamma)} z^k, \qquad (k = 2,3,...),\\
g_k(z) &= z - \frac{1-\mu}{\psi(n,2,\lambda,\alpha\,\beta,\gamma)} \overline{z}^k, \qquad (k = 1,2,...),\\
&\sum_{k=1}^{\infty} \delta_k + \sigma_k = 1,
\end{aligned}
$$

$\delta_k \geq 0, \sigma_k \geq 0$. *In particular, the extreme points of* $\mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu)$ *are* $\{h_k\}$ *and* $\{g_k\}$.

Proof. Any function f in $\mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu)$ can be expressed as

$$
\begin{aligned}
f(z) &= \sum_{k=1}^{\infty} (\delta_k h_k(z) + \sigma_k g_k(z))\\
&= \sum_{k=1}^{\infty} (\delta_k + \sigma_k) z - \sum_{k=2}^{\infty} \frac{1-\mu}{\psi(n,k,\lambda,\alpha\,\beta,\gamma)} \delta_k z^k - \sum_{k=1}^{\infty} \frac{1-\mu}{\psi(n,k,\lambda,\alpha\,\beta,\gamma)} \sigma_k \overline{z}^k.
\end{aligned}
$$

Then

$$
\sum_{k=2}^{\infty} \frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu} |a_k| + \sum_{k=1}^{\infty} \frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu} |b_k| = \sum_{k=2}^{\infty} \delta_k + \sum_{k=1}^{\infty} \sigma_k = 1 - \delta_1 \leq 1.
$$

Therefore, $f \in \mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu)$.

Conversely, suppose that $f \in \mathcal{D}_{\mathcal{T}}(n,\lambda,\alpha,\beta,\gamma,\mu)$. Setting

$$
\begin{aligned}
\delta_k &= \frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma}{1-\mu} a_k, \qquad (k = 2,3,...),\\
\sigma_k &= \frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma}{1-\mu} b_k, \qquad (k = 1,2,...),
\end{aligned}
$$

where $\sum_{k=1}^{\infty} \delta_k + \sigma_k = 1$, we obtain

$$
f(z) = \sum_{k=1}^{\infty} (\delta_k h_k(z) + \sigma_k g_k(z))
$$

as required. □

4. Convolution Property

The convolution of two harmonic functions

$$
f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| \overline{z}^k,
$$

and

$$
F(z) = z - \sum_{k=2}^{\infty} |A_k| z^k - \sum_{k=1}^{\infty} |B_k| \overline{z}^k,
$$

where $|A_k| \leq 1$ and $|B_k| \leq 1$, define as

$$
\begin{aligned}
(f * F)(z) &= f(z) * F(z)\\
&= z - \sum_{k=2}^{\infty} |a_k||A_k| z^k - \sum_{k=1}^{\infty} |b_k||B_k| \overline{z}^k.
\end{aligned}
$$

The convolution property of $\mathcal{D}_{\mathcal{T}}(n, \lambda, \alpha, \beta, \gamma, \mu)$ is given in the next theorem.

Theorem 5. *For* $0 \leq \mu_1 \leq \mu_2 < 1$, *let* $f(z) \in \mathcal{D}_{\mathcal{T}}(n, \lambda, \alpha, \beta, \gamma, \mu_2)$ *and* $F(z) \in \mathcal{D}_{\mathcal{T}}(n, \lambda, \alpha, \beta, \gamma, \mu_1)$. *Then*

$$(f * F)(z) \in \mathcal{D}_{\mathcal{T}}(n, \lambda, \alpha, \beta, \gamma, \mu_2) \subset \mathcal{D}_{\mathcal{T}}(n, \lambda, \alpha, \beta, \gamma, \mu_1).$$

Proof. Since f and F are in $\mathcal{D}_{\mathcal{T}}$. Then

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu_1}|a_k||A_k| - \sum_{k=1}^{\infty} \frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu_1}|b_k||B_k| \\
&\leq \sum_{k=2}^{\infty} \frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu_1}|a_k| - \sum_{k=1}^{\infty} \frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu_1}|b_k| \\
&\leq \sum_{k=2}^{\infty} \frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu_2}|a_k| - \sum_{k=1}^{\infty} \frac{\psi(n,k,\lambda,\alpha\,\beta,\gamma)}{1-\mu_2}|b_k| \\
&\leq 1.
\end{aligned}$$

Thus,

$$(f * F)(z) \in \mathcal{D}_{\mathcal{T}}(n, \lambda, \alpha, \beta, \gamma, \mu_2).$$

Since $0 \leq \mu_1 \leq \mu_2 < 1$, we get $\mathcal{D}_{\mathcal{T}}(n, \lambda, \alpha, \beta, \gamma, \mu_2) \subset \mathcal{D}_{\mathcal{T}}(n, \lambda, \alpha, \beta, \gamma, \mu_1)$. □

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