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PERIODIC SOLUTIONS FOR TOTALLY NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS

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Abstract. We obtain sufficient conditions under which solutions of certain classes of totally nonlinear neutral delay difference equations are periodic. A reformulated version of a fixed point theorem of Krasnoselskii is used to arrive at the main results. The results obtained in the paper generalizes the work in [8].

1. Introduction

Periodic solutions of difference equations has been studied extensively in recent times. We refer to [1]-[2], [5]-[9] and the references therein for a wealth of information on this subject.

In this paper we study the existence of periodic solutions of the equation

$$\begin{aligned}\Delta x(n) &= -a(n)h(x(n+1)) + c(n)\Delta x(n-\tau(n)) \\ &\quad + G(n, x(n), x(n-\tau(n))),\ \forall n \in \mathbb{Z},\end{aligned} \tag{1}$$

where

$$G : \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

with $\mathbb{Z}$ and $\mathbb{R}$ being the set of integers and real numbers respectively. Throughout this paper Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ for any sequence $\{x(n), n \in \mathbb{Z}\}$. In [8] the authors considered (1) when $h(x(n+1)) = x(n)$.

2. Preliminaries

Let T be an integer such that $T \geq 1$. Define $P_T = \{\varphi \in C(\mathbb{Z}, \mathbb{R}) : \varphi(n+T) = \varphi(n)\}$ where $C(\mathbb{Z}, \mathbb{R})$ is the space of all real valued functions. Then $(P_T, ||.||)$ is a Banach space with the maximum norm

$$||\varphi|| = \max_{n \in [0, T-1]} |\varphi(n)|.$$

Also, for any $L > 0$, define

$$\mathbb{M} = \{\varphi \in P_T \ :\ ||\varphi|| \leq L\}.$$

In this paper we assume that

$$a(n+T) = a(n),\ c(n+T) = c(n),\ \tau(n+T) = \tau(n),\ \tau(n) \geq \tau^* > 0, \tag{2}$$

for some constant τ^*. Suppose further that

$$a(n) > 0, \tag{3}$$

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and

$$G(n+T,x,y) = G(n,x,y). \tag{4}$$

Moreover, we also assume that G is Lipschitz continuous in x and y. That is, there are positive constants k_1, k_2 such that

$$|G(n,x,y) - G(n,z,w)| \le k_1||x-z|| + k_2||y-w||, \text{ for } x,y,z,w \in \mathbb{R}. \tag{5}$$

Lemma 1. *Suppose that* (2) *and* (3) *hold. If $x \in P_T$, then x is a solution of equation* (1) *if and only if*

$$\begin{aligned} x(n) \quad &= \quad \frac{c(n-1)}{1+a(n-1)}x(n-\tau(n)) + \Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times\Big[\sum_{r=n-T}^{n-1} a(r)(x(r+1) - h(x(r+1)))\prod_{s=r}^{n-1}(1+a(s))^{-1} \\ &\quad + \sum_{r=n-T}^{n-1}\{x(r-\tau(r))\phi(r) + G(r,x(r),x(r-\tau(r)))\}\prod_{s=r}^{n-1}(1+a(s))^{-1}\Big], \end{aligned} \tag{6}$$

where

$$\phi(r) = \frac{c(r-1)}{1+a(r-1)} - c(r). \tag{7}$$

Proof. Let $x \in P_T$ be a solution of (1). Rewrite (1) as

$$\begin{aligned} &\Delta x(n) + a(n)x(n+1) \\ &= a(n)x(n+1) - a(n)h(x(n+1)) + c(n)\Delta x(n-\tau(n)) + G(n,x(n),x(n-\tau(n))). \end{aligned}$$

We consider two cases; $n \ge 1$ and $n \le 0$. Considering first the case when $n \ge 1$ by multiplying both sides of the above equation by $\prod_{s=0}^{n-1}(1+a(s))$ and summing from $(n-T)$ to $(n-1)$ we obtain

$$\begin{aligned} &\sum_{r=n-T}^{n-1}\Delta\Big[\prod_{s=0}^{r-1}(1+a(s))x(r)\Big] \\ &= \sum_{r=n-T}^{n-1} a(r)\{x(r+1) - h(x(r+1))\}\prod_{s=0}^{r-1}(1+a(s)) \\ &\quad + \sum_{r=n-T}^{n-1}\{c(r)\Delta x(r-\tau(r)) + G(r,x(r),x(r-\tau(r)))\}\prod_{s=0}^{r-1}(1+a(s)). \end{aligned}$$

Which gives

$$\begin{aligned} &\prod_{s=0}^{n-1}(1+a(s))x(n) - \prod_{s=0}^{n-T-1}(1+a(s))x(n-T) \\ &= \sum_{r=n-T}^{n-1} a(r)\{x(r+1) - h(x(r+1))\}\prod_{s=0}^{r-1}(1+a(s)) \\ &\quad + \sum_{r=n-T}^{n-1}\{c(r)\Delta x(r-\tau(r)) + G(n,x(r),x(r-\tau(r)))\}\prod_{s=0}^{r-1}(1+a(s)). \end{aligned}$$

By dividing both sides of the above expression by $\prod_{s=0}^{n-1}(1+a(s))$ and the fact that $x(n)=x(n-T)$, we obtain

$$\begin{aligned}
x(n) &= \Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \qquad (8)\\
&\times\Big[\sum_{r=n-T}^{n-1} a(r)(x(r+1)-h(x(r+1)))\prod_{s=r}^{n-1}(1+a(s))^{-1}\\
&+\sum_{r=n-T}^{n-1}\{c(r)\Delta x(r-\tau(r))+G(r,x(r),x(r-\tau(r)))\}\prod_{s=r}^{n-1}(1+a(s))^{-1}\Big].
\end{aligned}$$

By performing a summation by parts on the above equation we obtain

$$\begin{aligned}
&\sum_{r=n-T}^{n-1} c(r)\Delta x(r-\tau(r))\prod_{s=r}^{n-1}(1+a(s))^{-1} \qquad (9)\\
&=\frac{c(n-1)}{1+a(n-1)}x(n-\tau(n))\Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)\\
&+\sum_{r=n-T}^{n-1} x(r-\tau(r))\phi(r)\prod_{s=r}^{n-1}(1+a(s))^{-1},
\end{aligned}$$

where ϕ is given by (7). Finally, substituting (9) into (8) completes the proof.

Now for $n\leq 0$, equation (1) is equivalent to

$$\begin{aligned}
&\Delta\Big[\prod_{s=n-1}^{0}(1+a(s))x(n)\Big]\\
&=a(t)\{x(n+1)-h(x(n+1))\}\prod_{s=n-1}^{0}(1+a(s))\\
&+\{c(n)\Delta x(n-\tau(n))+G(n,x(n),x(n-\tau(n)))\}\prod_{s=n-1}^{0}(1+a(s)).
\end{aligned}$$

Summing the above equation from $(n-T)$ to $n-1$ we obtain (6). □

In the proof of our main theorem, we employ a fixed point theorem in which the notion of a large contraction is required as one of the sufficient conditions. First, we give the following definition which can be found in [4].

Definition 1. *Let* $(\mathbb{M},d)$ *be a metric space and* $B:\mathbb{M}\to\mathbb{M}$. B *is said to be a large contraction if* $\psi,\varphi\in\mathbb{M}$, *with* $\psi\neq\varphi$ *then* $d(B\varphi,B\psi)<d(\varphi,\psi)$ *and if for all* $\epsilon>0$ *there exists* $\delta<1$ *such that*

$$[\psi,\varphi\in\mathbb{M},d(\varphi,\psi)\geq\epsilon]\Rightarrow d(B\varphi,B\psi)\leq\delta d(\varphi,\psi).$$

The next theorem, which constitutes a basis for our main result, is a reformulated version of Krasnoselskii's fixed point theorem.

Theorem 1 (Krasnoselskii-Burton [4])**.** *Let* $\mathbb{M}$ *be a bounded convex non-empty subset of a Banach space* $(S,||.||)$. *Suppose that* A, B *map* $\mathbb{M}$ *into* $\mathbb{M}$ *and that*

(i) for all $x,y\in\mathbb{M}\Rightarrow Ax+By\in\mathbb{M}$,

(ii) A *is continuous and* AM *is contained in a compact subset of* M,

(iii) B is a large contraction.

Then there is a $z \in \mathbb{M}$ with $z = Az + Bz$.

For the next lemma we make the following assumptions on the function $h : \mathbb{R} \to \mathbb{R}$.

(H1) h is continuous on $U_L = [-L, L]$.
(H2) h is strictly increasing on U_L.
(H3) $\sup_{s \in U_L \cap \mathbb{Z}} \Delta h(s) \leq 1$.
(H4) $(s-r)\Big\{\sup_{i \in U_L \cap \mathbb{Z}} \Delta h(i)\Big\} \geq h(s) - h(r) \geq (s-r)\Big\{\inf_{i \in U_L \cap \mathbb{Z}} \Delta h(i)\Big\} \geq 0$ for $s, r \in U_L$ with $s \geq r$.

Lemma 2. *Let L be a positive constant and $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying $(H1)-(H4)$. If $(H\varphi)(n) = \varphi(n+1) - h(\varphi(n+1))$, then H is a large contraction on the set $\mathbb{M}$.*

Proof. Let $\phi, \varphi \in \mathbb{M}$ with $\phi \neq \varphi$. Then $\phi(n+1) \neq \varphi(n+1)$ for some $n \in \mathbb{Z}$. Define the set

$$D(\phi, \varphi) = \Big\{\ n \in \mathbb{Z} : \phi(n+1) \neq \varphi(n+1)\Big\}.$$

Note that $\varphi(n+1) \in U_L$ for all $n \in \mathbb{Z}$ whenever $\varphi \in \mathbb{M}$. Since h is strictly increasing

$$\frac{h(\varphi(n+1)) - h(\phi(n+1))}{\varphi(n+1) - \phi(n+1)} = \frac{h(\phi(n+1)) - h(\varphi(n+1))}{\phi(n+1) - \varphi(n+1)} > 0 \tag{10}$$

holds for all $n \in D(\phi, \varphi)$. By (H3) we have

$$1 \geq \sup_{i \in U_L \cap \mathbb{Z}} \Delta h(i) \geq \inf_{s \in U_L \cap \mathbb{Z}} \Delta h(s) \geq 0. \tag{11}$$

Define the set $U_n \subset U_L$ by $U_n = [\varphi(n+1), \phi(n+1)] \cap U_L$ if $\phi(n+1) > \varphi(n+1)$, and $U_n = [\phi(n+1), \varphi(n+1)] \cap U_L$ if $\phi(n+1) < \varphi(n+1)$, for $n \in D(\phi, \varphi)$. Hence, for a fixed $n_0 \in D(\phi, \varphi)$ we get by (H4) and (10) that

$$\sup\{\Delta h(u) : u \in U_{n_0} \cap \mathbb{Z}\} \geq \frac{h(\phi(n_0+1)) - h(\varphi(n_0+1))}{\phi(n_0+1) - \varphi(n_0+1)} \geq \inf\{\Delta h(u) : u \in U_{n_0} \cap \mathbb{Z}\}.$$

Since $U_n \subset U_L$ for every $n \in D(\phi, \varphi)$, we find

$$\sup_{u \in U_L \cap \mathbb{Z}} \Delta h(u) \geq \sup\{\Delta h(u) : u \in U_{n_0} \cap \mathbb{Z}\} \geq \inf\{\Delta h(u) : u \in U_{n_0} \cap \mathbb{Z}\} \geq \inf_{u \in U_L \cap \mathbb{Z}} \Delta h(u),$$

and therefore,

$$1 \geq \sup_{u \in U_L \cap \mathbb{Z}} \Delta h(u) \geq \frac{h(\varphi(n+1)) - h(\phi(n+1))}{\varphi(n+1) - \phi(n+1)} \geq \inf_{u \in U_L \cap \mathbb{Z}} \Delta h(u) \geq 0 \tag{12}$$

for all $n \in D(\phi, \varphi)$. So, (12) yields

$$\begin{aligned}
|(H\phi)(n) - (H\varphi)(n)| &= |\phi(n+1) - h(\phi(n+1)) - \varphi(n+1) + h(\varphi(n+1))| \\
&= |\phi(n+1) - \varphi(n+1)|\Big|1 - \Big(\frac{h(\phi(n+1)) - h(\varphi(n+1))}{\phi(n+1) - \varphi(n+1)}\Big)\Big| \\
&\leq |\phi(n+1) - \varphi(n+1)|\Big(1 - \inf_{u \in U_L \cap \mathbb{Z}} \Delta h(u)\Big) \qquad (13)
\end{aligned}$$

for all $n \in D(\phi, \varphi)$. Thus, (12) and (13) imply that H is a large contraction in the supremum norm. To see this choose a fixed $\epsilon \in (0,1)$ and assume that ϕ and φ are two functions in $\mathbb{M}$ satisfying

$$\|\phi - \varphi\| = \sup_{n \in [-L, L] \cap \mathbb{Z}} |\phi(n+1) - \varphi(n+1)| \geq \epsilon.$$

If $|\phi(n+1)-\varphi(n+1)| \leq \epsilon/2$ for some $n \in D(\phi,\varphi)$, then from (13)

$$|(H\phi)(n)-(H\varphi)(n)| \leq |\phi(n+1)-\varphi(n+1)| \leq \frac{1}{2}\|\phi-\varphi\|. \tag{14}$$

Since h is continuous and strictly increasing, the function $h(u+\frac{\epsilon}{2})-h(u)$ attains its minimum on the closed and bounded interval $[-L,L]$. Thus, if $\frac{\epsilon}{2} < |\phi(n+1)-\varphi(n+1)|$ for some $n \in D(\phi,\varphi)$, then from (12) and (H3) we conclude that

$$1 \geq \frac{h(\phi(n+1))-h(\varphi(n+1))}{\phi(n+1)-\varphi(n+1)} > \lambda,$$

and therefore,

$$\begin{aligned} |(H\phi)(n)-(H\varphi)(n)| &\leq |\phi(n+1)-\varphi(n+1)|\Big\{1-\frac{h(\phi(n+1))-h(\varphi(n+1))}{\phi(n+1)-\varphi(n+1)}\Big\} \\ &\leq (1-\lambda)\|\phi(n+1)-\varphi(n+1)\|, \end{aligned} \tag{15}$$

where

$$\lambda := \frac{1}{2L}\min\Big\{h(u+\frac{\epsilon}{2})-h(u), u \in [-L,L]\Big\} > 0.$$

Consequently, it follows from (14) and (15) that

$$|(H\phi(n)-(H\varphi)(n)| \leq \delta\|\phi-\varphi\|,$$

where $\delta = \max\Big\{\frac{1}{2}, 1-\lambda\Big\} < 1$. The proof is complete. □

3. Existence of periodic solutions

In this section we prove our main results. We begin by defining the maps $A, B : \mathbb{M} \to \mathbb{M}$ as follows

$$\begin{aligned} (A\varphi)(n) &= \frac{c(n-1)}{1+a(n-1)}\varphi(n-\tau(n)) + \Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n-T}^{n-1}\{\varphi(r-\tau(r))\phi(r)+G(r,\varphi(r),\varphi(r-\tau(r)))\}\prod_{s=r}^{n-1}(1+a(s))^{-1}, \end{aligned} \tag{16}$$

and

$$\begin{aligned} (B\varphi)(n) &= \Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n-T}^{n-1} a(r)(x(r+1)-h(x(r+1)))\prod_{s=r}^{n-1}(1+a(s))^{-1}. \end{aligned} \tag{17}$$

For the rest of the paper we make the following assumptions.

$$(k_1+k_2)L + |G(n,0,0)| \leq \beta L a(n), \tag{18}$$

$$|\phi(n)| \leq \delta a(n), \tag{19}$$

$$\max_{n\in[0,T-1]}\Big|\frac{c(n-1)}{1+a(n-1)}\Big| = \alpha, \tag{20}$$

$$J(\beta+\alpha+\delta) \leq 1, \tag{21}$$

where α, β, δ and J are positive constants with $J \geq 3$.

Lemma 3. *Suppose* (2)-(5) *and* (18)-(21) *hold. Then the mapping* $A : \mathbb{M} \to \mathbb{M}$ *defined in* (16) *is continuous in the maximum norm and maps* $\mathbb{M}$ *into compact subsets of* $\mathbb{M}$.

Proof. We first show that $A : \mathbb{M} \to \mathbb{M}$. Let $\varphi \in \mathbb{M}$. Then

$$\begin{aligned}(A\varphi)(n+T) &= \frac{c(n+T-1)}{1+a(n+T-1)}\varphi(n+T-\tau(n+T)) + \Big(1-\prod_{s=n}^{n+T-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n}^{n+T-1}\{\varphi(r-\tau(r))\phi(r) + G(r,\varphi(r),\varphi(r-\tau(r)))\}\prod_{s=r}^{n+T-1}(1+a(s))^{-1} \\ &= \frac{c(n-1)}{1+a(n-1)}\varphi(n-\tau(n)) + \Big(1-\prod_{s=n}^{n+T-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n}^{n+T-1}\{\varphi(r-\tau(r))\phi(r) + G(r,\varphi(r),\varphi(r-\tau(r)))\}\prod_{s=r}^{n+T-1}(1+a(s))^{-1}\end{aligned}$$

Let $j = r - T$, then

$$\begin{aligned}(A\varphi)(n+T) &= \frac{c(n-1)}{1+a(n-1)}\varphi(n-\tau(n)) + \Big(1-\prod_{s=n}^{n+T-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{j=n-T}^{n+T-1}\{\varphi(j+T-\tau(j+T))\phi(j+T) \\ &\quad + G(j+T,\varphi(j+T),\varphi(j+T-\tau(j+T)))\}\prod_{s=j+T}^{n+T-1}(1+a(s))^{-1}.\end{aligned}$$

Now let $k = s - T$, then

$$\begin{aligned}(A\varphi)(n+T) &= \frac{c(n-1)}{1+a(n-1)}\varphi(n-\tau(n)) + \Big(1-\prod_{k=n-T}^{n-1}(1+a(k))^{-1}\Big)^{-1} \\ &\quad \times \sum_{j=n-T}^{n-1}\{\varphi(j-\tau(j))\phi(j) \\ &\quad + G(j,\varphi(j),\varphi(j-\tau(j)))\}\prod_{k=j}^{n-1}(1+a(k))^{-1} = (A\varphi)(n).\end{aligned}$$

Consequently, $A : P_T \to P_T$.
In view of (5) we have that

$$\begin{aligned}|G(n,x,y)| &= |G(n,x,y) - G(n,0,0) + G(n,0,0)| \\ &\leq |G(n,x,y) - G(n,0,0)| + |G(n,0,0)| \\ &\leq k_1\|k_1\| + k_2\|y\| + |G(n,0,0)|.\end{aligned}$$

Also it follows from (3) that $1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}>0$. So, for any $\varphi\in\mathbb{M}$, we obtain

$$\begin{aligned}
|(A\varphi)(n)| &\leq \Big|\frac{c(n-1)}{1+a(n-1)}\Big||\varphi(n-\tau(n))| + \Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\
&\quad \times \sum_{r=n-T}^{n-1}\{|\varphi(r-\tau(r))||\phi(r)| + |G(r,\varphi(r),\varphi(r-\tau(r)))|\}\prod_{s=r}^{n-1}(1+a(s))^{-1}, \\
&\leq \alpha L + \Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\
&\quad \times \sum_{r=n-T}^{n-1}\{\delta L a(r) + (k_1+k_2)L + |G(r,0,0)|\}\prod_{s=r}^{n-1}(1+a(s))^{-1} \\
&\leq \alpha L + \Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1}\sum_{r=n-T}^{n-1}\{(\delta+\beta)La(r)\}\prod_{s=r}^{n-1}(1+a(s))^{-1} \\
&\leq \alpha L + \Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1}(\delta+\beta)L\sum_{r=n-T}^{n-1}\Delta_r\Big[\prod_{s=r}^{n-1}(1+a(s))^{-1}\Big] \\
&= (\alpha+\delta+\beta)L \leq \frac{L}{J} < L.
\end{aligned}$$

Thus $A\varphi\in\mathbb{M}$. Consequently, we have $A:\mathbb{M}\to\mathbb{M}$.

We next show that A is continuous in the maximum norm. Let $\varphi,\psi\in\mathbb{M}$, and let

$$\mu_1 = \max_{n\in[0,T-1]}\Big|\frac{c(n-1)}{1+a(n-1)}\Big|,\quad \mu_2 = \max_{n\in[0,T-1]}\Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1},$$
$$\mu_3 = \max_{r\in[n-T,T-1]}|\phi(r)|.$$

Let $\epsilon>0$ be given. Choose $\eta=\epsilon/\rho$ where $\rho=\mu_1+\mu_2T(\mu_3+k_1+k_2)$ such that $\|\varphi-\psi\|<\eta$. Note that from (3), we have $\max_{r\in[n-T,T-1]}\prod_{s=r}^{n-1}(1+a(s))^{-1}\leq 1$. Thus,

$$\begin{aligned}
&|(A\varphi)(n)-(A\psi)(n)| \\
&\leq \Big|\frac{c(n-1)}{1+a(n-1)}\Big|\|\varphi-\psi\| \\
&+\Big(1-\prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1}\sum_{r=n-T}^{n-1}\Big\{\|\varphi-\psi\||\phi(r)| \\
&+|G(r,\varphi(r),\varphi(r-\tau(r)))-G(r,\psi(r),\psi(r-\tau(r)))|\Big\}\prod_{s=r}^{n-1}(1+a(s))^{-1} \\
&\leq \mu_1\|\varphi-\psi\| + \mu_2\sum_{r=n-T}^{n-1}\Big\{\mu_3\|\varphi-\psi\| + (k_1+k_2)\|\varphi-\psi\|\Big\} \\
&\leq \Big\{\mu_1+\mu_2T(\mu_3+k_1+k_2)\Big\}\|\varphi-\psi\| < \epsilon.
\end{aligned}$$

Therefore showing that A is continuous.

Next, we show that A maps bounded subsets into compact sets. Since M is bounded and A is continuous, $A\mathbb{M}$ is a subset of $\mathbb{R}^T$ which is bounded. So, $A\mathbb{M}$ is contained in a compact subset of $\mathbb{M}$. The proof is complete. □

Lemma 4. *Suppose* (2)-(5) *and* (18) *hold. Also, suppose that*

$$\max(|H(-L)|, |H(L)|) \leq \frac{(J-1)L}{J}. \tag{22}$$

For A, B defined by (16) *and* (17) *respectively, if $\varphi, \psi \in \mathbb{M}$ are arbitrary, then*

$$A\varphi + B\psi : \mathbb{M} \to \mathbb{M}.$$

Proof. Let $\varphi, \psi \in \mathbb{M}$ be arbitrary. Using the result of Lemma 3 we obtain

$$\begin{aligned}
&|(A\varphi)(n) + (B\psi)(n)| \\
&\leq \Big|\frac{c(n-1)}{1+a(n-1)}\Big||\varphi(n-\tau(n))| + \Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\
&\quad \times \sum_{r=n-T}^{n-1}\{|\varphi(r-\tau(r))||\phi(r)| + |G(r,\varphi(r),\varphi(r-\tau(r)))|\}\prod_{s=r}^{n-1}(1+a(s))^{-1} \\
&\quad + \Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\
&\quad \times \max(|H(-L)|, |H(L)|)\sum_{r=n-T}^{n-1}a(r)\prod_{s=r}^{n-1}(1+a(s))^{-1} \\
&\leq \frac{L}{J} + \Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\
&\quad \times \max(|H(-L)|, |H(L)|)\sum_{r=n-T}^{n-1}\Delta\Big[\prod_{s=r}^{n-1}(1+a(s))^{-1}\Big] \\
&\leq \frac{L}{J} + \frac{(J-1)L}{J} = L.
\end{aligned}$$

Thus $A\varphi + B\psi \in \mathbb{M}$. This completes the proof. □

The next result gives a relationship between the mappings H and B in the sense of a large contraction.

Lemma 5. *Let B be defined by* (17) *and assume that* (2)-(3) *and* (22) *hold. If H is a large contraction on $\mathbb{M}$ then so is the mapping $B : \mathbb{M} \to \mathbb{M}$.*

Proof. We will first show that B maps $\mathbb{M}$ into itself. Let $\varphi \in \mathbb{M}$ then

$$\begin{aligned}
(B\varphi)(n+T) &= \Big(1 - \prod_{s=n}^{n+T-1}(1+a(s))^{-1}\Big)^{-1} \\
&\quad \times \sum_{r=n}^{n+T-1}a(r)(x(r+1) - h(x(r+1)))\prod_{s=r}^{n+T-1}(1+a(s))^{-1}
\end{aligned}$$

Let $j = r - T$, then

$$\begin{aligned}
(B\varphi)(n+T) &= \Big(1 - \prod_{s=n}^{n+T-1}(1+a(s))^{-1}\Big)^{-1} \\
&\quad \times \sum_{r=n}^{n+T-1}a(j+T)(x(j+T+1) - h(x(j+T+1)))\prod_{s=j+T}^{n+T-1}(1+a(s))^{-1}
\end{aligned}$$

Now let $k = s - T$, then

$$\begin{aligned}(B\varphi)(n+T) &= \Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n-T}^{n-1} a(j)(x(j+1) - h(x(j+1))) \prod_{k=j}^{n-1}(1+a(s))^{-1} \\ &= (B\varphi)(n).\end{aligned}$$

That is, $B : P_T \to P_T$.
In view of (22), we have

$$\begin{aligned}|(B\varphi)(n)| &\le \Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n-T}^{n-1} |a(r)||H(\varphi(r+1))| \prod_{s=r}^{n-1}(1+a(s))^{-1} \qquad (23)\\ &\le \frac{(J-1)L}{J} < L. \qquad (24)\end{aligned}$$

That is $B\varphi \in \mathbb{M}$ and consequently we have $B : \mathbb{M} \to \mathbb{M}$.

We next show that B is a large contraction. If H is a large contraction on $\mathbb{M}$, for $x, y \in \mathbb{M}$, with $x \ne y$, we have $\|Hx - Hy\| \le \|x - y\|$. Thus, it follows from the equality

$$a(r)\prod_{s=r}^{n-1}(1+a(s))^{-1} = \Delta\Big[\prod_{s=r}^{n-1}(1+a(s))^{-1}\Big]$$

that

$$\begin{aligned}|Bx(n) - By(n)| &\le \Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n-T}^{n-1} a(r)|H(x(r+1)) - H(y(r+1))| \prod_{s=r}^{n-1}(1+a(s))^{-1} \\ &\le \|x-y\|\Big(1 - \prod_{s=n-T}^{n-1}(1+a(s))^{-1}\Big)^{-1} \\ &\quad \times \sum_{r=n-T}^{n-1} a(r)\prod_{s=r}^{n-1}(1+a(s))^{-1} = \|x-y\|.\end{aligned}$$

Thus

$$\|Bx - By\| \le \|x - y\|.$$

One may also show in a similar way that

$$\|Bx - By\| \le \delta\|x - y\|$$

holds if we know the existence of a $\delta \in (0,1)$ and that for all $\epsilon > 0$

$$[x, y \in \mathbb{M}, \|x - y\| > 0] \Rightarrow \|Hx - Hy\| \le \delta\|x - y\|.$$

The proof is complete. □

Theorem 2. *Let $(P_T, \|.\|)$ be the Banach space of T-periodic real valued functions and $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq L\}$, where L is a positive constant. Suppose that (2)-(5) and (18)-(21) hold. Then equation (1) has a T-periodic solution φ in $\mathbb{M}$.*

Proof. By Lemma 1, φ is a solution of (1) if

$$\varphi = A\varphi + B\varphi,$$

where A and B are given by (16) and (17) respectively. By Lemma 3, $A : \mathbb{M} \to \mathbb{M}$ is completely continuous. By Lemma 4, $A\varphi + B\psi \in \mathbb{M}$ whenever $\varphi, \psi \in \mathbb{M}$. Moreover, $B : \mathbb{M} \to \mathbb{M}$ is a large contraction by lemma 5. Thus all the hypotheses of Theorem 1 of Krasnoselskii are satisfied. Thus, there exists a fixed point $\varphi \in \mathbb{M}$ such that $\varphi = A\varphi + B\varphi$. Hence (1) has a $T-$ periodic solution. $\square$

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