TJMM
7 (2015), No. 1, 75-83

ON ČEBYŁEV-GRÜSS TYPE INEQUALITIES FOR DOUBLE INTEGRALS

MEHMET ZEKI SARIKAYA AND MEHMET EYUP KIRIS

Abstract. In this study, we establish some new inequalities of Čebyšev-Grüss type involving functions of two independent variables for double integrals. The analysis used in the proofs is elementary and our results provide new estimates on inequalities of this type.

1. Introduction

In 1935, G. Grüss [6] proved the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right|\leq\frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{1}$$

provided that f and g are two integrable function on $[a,b]$ satisfying the condition

$$\varphi\leq f(x)\leq\Phi \ \text{ and } \ \gamma\leq g(x)\leq\Gamma \text{ for all } x\in[a,b]. \tag{2}$$

The constant $\frac{1}{4}$ is best possible.

In 1882, P.L. Čebyšev [2] gave the following inequality:

$$|T(f,g)|\leq\frac{1}{12}(b-a)^2\left\|f'\right\|_\infty\left\|g'\right\|_\infty, \tag{3}$$

where $f,g:[a,b]\to\mathbb{R}$ are absolutely continuous function, whose first derivatives f' and g' are bounded,

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \tag{4}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty=ess\sup_{t\in[a,b]}|p(t)|$.

The following result of Grüss type was proved by Dragomir and Fedotov [3]:

Theorem 1. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is L-Lipshitzian on $[a,b]$, i.e,*

$$|u(x)-u(y)|\leq L|x-y| \text{ for all } x\in[a,b], \tag{5}$$

f is Riemann integrable on $[a,b]$ and there exist the real numbers m,M so that

$$m\leq f(x)\leq M \text{ for all } x\in[a,b]. \tag{6}$$

2010 *Mathematics Subject Classification.* 26D15.
Key words and phrases. Cebysev and Grüss type inequality, double integrals, two independent variables.

Then we have the inequality,

$$\left| \int_a^b f(x)du(x) - \frac{u(b)-u(a)}{b-a} \int_a^b f(x)dx \right| \leq \frac{1}{2} L(M-m)(b-a).$$

From [8], if $f : [a,b] \to \mathbb{R}$ is differentiable on (a,b) with the first derivative f' integrable on $[a,b]$, then Montgomery identity holds:

$$f(x) = \frac{1}{b-a} \int_a^b f(t)dt + \int_a^b P(x,t)f'(t)dt, \tag{7}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \leq t \leq x \\ \frac{t-b}{b-a}, & x < t \leq b. \end{cases}$$

In [9], Pachpatte established new inequalities of the Čebyšev type by using Pecaric's extension of the Montgomery identity [10].

In the last years, many papers were devoted to the generalization of Čebyšev-Grüss type inequalities, we can mention the works [1], [3]-[5], [7], [9]-[13]. Find new inequalities in the multidimensional case still an interesting problem. In this study, we establish some new inequalities of Čebyšev-Grüss type involving functions of two independent variables for double integrals. The analysis used in the proofs is elementary and our results provide new estimates on inequalities of this type.

2. Main results

Definition 1. *Consider a function $f : V \to \mathbb{R}$ defined on a subset V of $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $L = (L_1, L_2, ..., L_n)$ where $L_i \geq 0$, $i = 1, 2, ..., n$. We say that f is L-Lipschitzian function if*

$$|f(x) - f(y)| \leq \sum_{i=1}^{n} L\,|x_i - y_i|$$

for all $x, y \in V$.

Theorem 2. *Let $f, g : [a,b] \times [c,d] \to \mathbb{R}$ are two functions defined and integrable on $[a,b] \times [c,d]$. Then for*

$$\varphi \leq f(x,y) \leq \Phi \quad \text{and} \quad \gamma \leq g(x,y) \leq \Gamma \text{ for all } (x,y) \in [a,b] \times [c,d], \tag{8}$$

we have

$$\begin{aligned} &\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)g(x,y)dydx \right. \\ &\left. - \left(\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx \right) \left(\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(x,y)dydx \right) \right| \\ &\leq \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma). \end{aligned} \tag{9}$$

Proof. By using the well known Korkine's identity for mappings $f, g : [a,b] \times [c,d] \to \mathbb{R}$, we have

$$\int_a^b \int_c^d \int_a^b \int_c^d [f(x,s) - f(t,y)]\,[g(x,s) - g(t,y)]\, dsdtdydx \tag{10}$$

$$= \int_a^b \int_c^d \int_a^b \int_c^d [f(x,s)g(x,s) - f(x,s)g(t,y) - f(t,y)g(x,s) + f(t,y)g(t,y)]\, dsdtdydx$$

$$= 2\,(b-a)\,(d-c) \int_a^b \int_c^d f(x,s)g(x,s)dsdx - 2 \left(\int_a^b \int_c^d f(x,s)dsdx \right) \left(\int_a^b \int_c^d g(t,y)dydt \right)$$

Appling Cauchy-Schwarz's integral inequality, we obtain

$$\left[\frac{1}{2\,(b-a)^2\,(d-c)^2} \int_a^b \int_c^d \int_a^b \int_c^d [f(x,s) - f(t,y)]\,[g(x,s) - g(t,y)]\, dsdtdydx \right]^2$$

$$\le \left(\frac{1}{2\,(b-a)^4\,(d-c)^4} \int_a^b \int_c^d \int_a^b \int_c^d [f(x,s) - f(t,y)]^2\, dsdtdydx \right)$$

$$\times \left(\frac{1}{2\,(b-a)^4\,(d-c)^4} \int_a^b \int_c^d \int_a^b \int_c^d [fgx,s) - g(t,y)]^2\, dsdtdydx \right) \tag{11}$$

$$= \left[\frac{1}{(b-a)^2\,(d-c)^2} \int_a^b \int_c^d f^2(x,s)dsdx - \left(\frac{1}{(b-a)^2\,(d-c)^2} \int_a^b \int_c^d f(x,s)dsdx \right)^2 \right]$$

$$\times \left[\frac{1}{(b-a)^2\,(d-c)^2} \int_a^b \int_c^d g^2(x,s)dsdx - \left(\frac{1}{(b-a)^2\,(d-c)^2} \int_a^b \int_c^d g(x,s)dsdx \right)^2 \right]$$

It is easy to observe that

$$\frac{1}{(b-a)^2\,(d-c)^2} \int_a^b \int_c^d f^2(x,s)dsdx - \left(\frac{1}{(b-a)^2\,(d-c)^2} \int_a^b \int_c^d f(x,s)dsdx \right)^2 =$$

$$\left(\Phi - \frac{1}{(b-a)^2\,(d-c)^2} \int_a^b \int_c^d f(x,s)dsdx \right) \left(\frac{1}{(b-a)^2\,(d-c)^2} \int_a^b \int_c^d f(x,s)dsdx - \varphi \right)$$

$$- \frac{1}{(b-a)^2\,(d-c)^2} \int_a^b \int_c^d [\Phi - f(x,s)]\,[f(x,s) - \varphi]\, dsdx.$$

Since $[\Phi - f(x,s)]\,[f(x,s) - \varphi] \ge 0$ for each $(x,s) \in [a,b] \times [c,d]$, then we get

$$\frac{1}{(b-a)^2\,(d-c)^2} \int_a^b \int_c^d f^2(x,s)dsdx - \left(\frac{1}{(b-a)^2\,(d-c)^2} \int_a^b \int_c^d f(x,s)dsdx \right)^2 \tag{12}$$

$$= \left(\Phi - \frac{1}{(b-a)^2 (d-c)^2} \int_a^b \int_c^d f(x,s) ds dx \right) \left(\frac{1}{(b-a)^2 (d-c)^2} \int_a^b \int_c^d f(x,s) ds dx - \varphi \right).$$

Similarly, we have

$$\frac{1}{(b-a)^2 (d-c)^2} \int_a^b \int_c^d g^2(x,s) ds dx - \left(\frac{1}{(b-a)^2 (d-c)^2} \int_a^b \int_c^d g(x,s) ds dx \right)^2 \tag{13}$$

$$= \left(\Gamma - \frac{1}{(b-a)^2 (d-c)^2} \int_a^b \int_c^d g(x,s) ds dx \right) \left(\frac{1}{(b-a)^2 (d-c)^2} \int_a^b \int_c^d g(x,s) ds dx - \gamma \right).$$

Using (12) and (13) in (11), we get the following inequality

$$\left[\frac{1}{2(b-a)^2 (d-c)^2} \int_a^b \int_c^d \int_a^b \int_c^d [f(x,s) - f(t,y)] [g(x,s) - g(t,y)] ds dt dy dx \right]^2 \tag{14}$$

$$\leq \left(\Phi - \frac{1}{(b-a)^2 (d-c)^2} \int_a^b \int_c^d f(x,s) ds dx \right) \left(\frac{1}{(b-a)^2 (d-c)^2} \int_a^b \int_c^d f(x,s) ds dx - \varphi \right)$$

$$\times \left(\Gamma - \frac{1}{(b-a)^2 (d-c)^2} \int_a^b \int_c^d g(x,s) ds dx \right) \left(\frac{1}{(b-a)^2 (d-c)^2} \int_a^b \int_c^d g(x,s) ds dx - \gamma \right).$$

Now, using the elementary inequality for real numbers

$$4pq \leq (p+q)^2, \quad p, q \in \mathbb{R}$$

we get

$$\left[\frac{1}{2(b-a)^2 (d-c)^2} \int_a^b \int_c^d \int_a^b \int_c^d [f(x,s) - f(t,y)] [g(x,s) - g(t,y)] ds dt dy dx \right]^2$$

$$\leq \frac{1}{16} (\Phi - \varphi)^2 (\Gamma - \gamma)^2$$

which completes the proof. To prove the sharpness of (9), let choose

$$f(x,y) = g(x,y) = \begin{cases} 1, & a \leq x < \frac{a+b}{2}, \ c \leq y < \frac{c+d}{2} \\ -1, & a \leq x < \frac{a+b}{2}, \ \frac{c+d}{2} \leq y \leq d \\ -1, & \frac{a+b}{2} \leq x \leq b, \ c \leq y < \frac{c+d}{2} \\ 1, & \frac{a+b}{2} \leq x \leq b, \ \frac{c+d}{2} \leq y \leq d \end{cases}$$

then

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) g(x,y) dy dx = 1,$$

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d g(x,y) dy dx = 0$$

and

$$(\Phi - \varphi) = (\Gamma - \gamma) = 2$$

which the equality (9) is realized. □

Theorem 3. *Let $f,g:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ satisfies L-Lipschitzian conditions. That is, for (x,s) and (t,y) belong to $\Delta:=[a,b]\times[c,d]$, then we have*

$$|f(x,s)-f(t,y)|\le L_1|x-t|+L_2|s-y|$$
$$|g(x,s)-g(t,y)|\le L_3|x-t|+L_4|s-y|$$

where L_1, L_2, L_3 and L_4 are positive constants. Then, we have the following inequality:

$$|T(f,g;a,b,c,d)|\le\frac{(d-c)^2(b-a)^4}{6}L_1L_3+\frac{(d-c)^4(b-a)^2}{6}L_2L_4+\frac{(b-a)^3(d-c)^3}{9}(L_1L_4+L_2L_3)$$

where

$$T(f,g;a,b,c,d)=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx \tag{15}$$
$$-\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right)\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)dydx\right).$$

Proof. Since f,g are L-Lipschitzian, we have

$$|[f(x,s)-f(t,y)][g(x,s)-g(t,y)]|$$
$$\le L_1L_3(x-t)^2+L_2L_4(s-y)^2+(L_1L_4+L_2L_3)|x-t||s-y|$$

for all (x,s), $(t,y)\in\Delta:=[a,b]\times[c,d]$. Then by (10) we have that

$$\left|\int_a^b\int_c^d\int_a^b\int_c^d[f(x,s)-f(t,y)][g(x,s)-g(t,y)]\,dsdtdydx\right|$$
$$\le\int_a^b\int_c^d\int_a^b\int_c^d|[f(x,s)-f(t,y)][g(x,s)-g(t,y)]|\,dsdtdydx$$
$$\le\int_a^b\int_c^d\int_a^b\int_c^d\left[L_1L_3(x-t)^2+L_2L_4(s-y)^2+(L_1L_4+L_2L_3)|x-t||s-y|\right]dsdtdydx$$
$$=\frac{(d-c)^2(b-a)^4}{6}L_1L_3+\frac{(d-c)^4(b-a)^2}{6}L_2L_4+\frac{(b-a)^3(d-c)^3}{9}(L_1L_4+L_2L_3)$$

which completes the proof. □

Theorem 4. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ satisfy L-Lipschitzian conditions. Then, we have the following inequalities:*

$$|T(f,g;a,b,c,d)|\le$$
$$\begin{cases}\left[\dfrac{L_1}{4(d-c)}+\dfrac{L_2}{4(b-a)}\right]\|g\|_1 & \text{if } g\in L_1(\Delta)\\ 2^{\frac{1}{q}}\dfrac{(d-c)^{\frac{1}{q}-1}(b-a)^{\frac{1}{q}-1}}{[(q+1)(q+2)]^{\frac{1}{q}}}\left[L_1(d-c)^{\frac{1}{q}}+L_2(b-a)^{\frac{1}{q}}\right]\|g\|_p & \text{if } g\in L_p(\Delta)\\ \dfrac{2}{3}[(b-a)L_1+(d-c)L_2]\|g\|_\infty & \text{if } g\in L_\infty(\Delta)\end{cases}$$

where $T(f,g;a,b,c,d)$ is defined by (15).

Proof. We write that

$$\begin{aligned} &|f(x,s)g(x,s)-f(t,y)g(x,s)-f(x,s)g(t,y)+f(t,y)g(t,y)| \\ &\leq (L_1|x-t|+L_2|s-y|)\left[|g(x,s)|+|g(t,y)|\right] \end{aligned} \tag{16}$$

for all (x,s), $(t,y)\in\Delta:=[a,b]\times[c,d]$. Then, by integrating on Δ^2 and by using (16), we have that

$$\begin{aligned} &\left|\int_a^b\int_c^d\int_a^b\int_c^d [f(x,s)g(x,s)-f(t,y)g(x,s)-f(x,s)g(t,y)+f(t,y)g(t,y)]\,dsdtdydx\right| \\ &\leq \int_a^b\int_c^d\int_a^b\int_c^d (L_1|x-t|+L_2|s-y|)\left[|g(x,s)|+|g(t,y)|\right]dsdtdydx \\ &= L_1(d-c)\int_a^b\int_a^b\int_c^d |x-t|\,|g(x,s)|\,dsdtdx + L_2(b-a)\int_a^b\int_c^d\int_c^d |s-y|\,|g(x,s)|\,dsdydx \\ &+L_1(d-c)\int_a^b\int_c^d\int_a^b |x-t|\,|g(t,y)|\,dtdydx + L_2(b-a)\int_c^d\int_a^b\int_c^d |s-y|\,|g(t,y)|\,dsdtdy \\ &= L_1\frac{(d-c)}{2}\int_a^b\int_c^d \left[(t-a)^2+(b-t)^2\right]|g(x,s)|\,dtdy \\ &+L_2\frac{(b-a)}{2}\int_a^b\int_c^d \left[(s-c)^2+(d-s)^2\right]|g(x,s)|\,dsdx \\ &+L_1\frac{(d-c)}{2}\int_a^b\int_c^d \left[(t-a)^2+(b-t)^2\right]|g(t,y)|\,dtdy \\ &+L_2\frac{(b-a)}{2}\int_a^b\int_c^d \left[(y-c)^2+(d-y)^2\right]|g(t,y)|\,dydt \\ &\leq L_1\frac{(d-c)}{2}\max_{t\in[a,b]}\left\{(t-a)^2+(b-t)^2\right\}\int_a^b\int_c^d |g(x,s)|\,dsdx \\ &+L_2\frac{(b-a)}{2}\max_{s\in[c,d]}\left\{(s-c)^2+(d-s)^2\right\}\int_a^b\int_c^d |g(x,s)|\,dsdx \\ &+L_1\frac{(d-c)}{2}\max_{t\in[a,b]}\left\{(t-a)^2+(b-t)^2\right\}\int_a^b\int_c^d |g(t,y)|\,dydt \\ &+L_2\frac{(b-a)}{2}\max_{s\in[c,d]}\left\{(s-c)^2+(d-s)^2\right\}\int_a^b\int_c^d |g(t,y)|\,dydt \end{aligned}$$

$$= \left[L_1 \frac{(d-c)(b-a)^2}{4} + L_2 \frac{(b-a)(d-c)^2}{4} \right] \left[\int_a^b \int_c^d |g(x,s)|\, dsdx + \int_a^b \int_c^d |g(t,y)|\, dydt \right].$$

We note that

$$\int_a^b \int_c^d \int_a^b \int_c^d [f(x,s)g(x,s) - f(t,y)g(x,s) - f(x,s)g(t,y) + f(t,y)g(t,y)]\, dsdtdydx$$
$$= (b-a)(d-c) \int_a^b \int_c^d f(x,s)g(x,s)dsdx - \left(\int_a^b \int_c^d f(t,y)dydt \right) \left(\int_a^b \int_c^d g(x,s)dsdx \right)$$
$$- \left(\int_a^b \int_c^d f(x,s)dsdx \right) \left(\int_a^b \int_c^d g(t,y)dydt \right) + (b-a)(d-c) \int_a^b \int_c^d f(t,y)g(t,y)dydt.$$

Now, if $g \in L_1(\Delta)$, then we get

$$\left| \int_a^b \int_c^d \int_a^b \int_c^d [f(x,s)g(x,s) - f(t,y)g(x,s) - f(x,s)g(t,y) + f(t,y)g(t,y)]\, dsdtdydx \right|$$
$$\leq 2 \left[L_1 \frac{(d-c)(b-a)^2}{4} + L_2 \frac{(b-a)(d-c)^2}{4} \right] \|g\|_1 .$$

Now, assume that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $g \in L_p(\Delta)$, then we have

$$\left| \int_a^b \int_c^d \int_a^b \int_c^d [f(x,s)g(x,s) - f(t,y)g(x,s) - f(x,s)g(t,y) + f(t,y)g(t,y)]\, dsdtdydx \right|$$
$$\leq L_1 (d-c) \int_a^b \int_a^b \int_c^d |x-t|\, |g(x,s)|\, dsdtdx + L_2 (b-a) \int_a^b \int_c^d \int_c^d |s-y|\, |g(x,s)|\, dsdydx$$
$$+ L_1 (d-c) \int_a^b \int_c^d \int_a^b |x-t|\, |g(t,y)|\, dtdydx + L_2 (b-a) \int_c^d \int_a^b \int_c^d |s-y|\, |g(t,y)|\, dsdtdy$$
$$\leq L_1 (d-c) \left(\int_a^b \int_a^b \int_c^d |g(x,s)|^p\, dsdtdx \right)^{\frac{1}{p}} \left(\int_a^b \int_a^b \int_c^d |x-t|^q\, dsdtdx \right)^{\frac{1}{q}}$$
$$+ L_2 (b-a) \left(\int_a^b \int_c^d \int_c^d |g(x,s)|\, dsdydx \right)^{\frac{1}{p}} \left(\int_a^b \int_c^d \int_c^d |s-y|^q\, dsdydx \right)^{\frac{1}{q}}$$
$$+ L_1 (d-c) \left(\int_a^b \int_c^d \int_a^b |g(t,y)|^p\, dtdydx \right)^{\frac{1}{p}} \left(\int_a^b \int_c^d \int_a^b |x-t|^q\, dtdydx \right)^{\frac{1}{q}}$$
$$+ L_2 (b-a) \left(\int_c^d \int_a^b \int_c^d |g(t,y)|^p\, dsdtdy \right)^{\frac{1}{p}} \left(\int_c^d \int_a^b \int_c^d |s-y|^q\, dsdtdy \right)^{\frac{1}{q}}$$

$$= L_1 (d-c) \|g\|_p \left(\int_a^b \int_a^b \int_c^d |x-t|^q \, dsdtdx \right)^{\frac{1}{q}} + L_2 (b-a) \|g\|_p \left(\int_a^b \int_c^d \int_c^d |s-y|^q \, dsdydx \right)^{\frac{1}{q}}$$

$$+ L_1 (d-c) \|g\|_p \left(\int_a^b \int_c^d \int_a^b |x-t|^q \, dtdydx \right)^{\frac{1}{q}} + L_2 (b-a) \|g\|_p \left(\int_c^d \int_a^b \int_c^d |s-y|^q \, dsdtdy \right)^{\frac{1}{q}}$$

$$= 2^{1+\frac{1}{q}} \frac{(d-c)^{1+\frac{1}{q}} (b-a)^{1+\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}} \left[L_1 (d-c)^{\frac{1}{q}} + L_2 (b-a)^{\frac{1}{q}} \right] \|g\|_p$$

where

$$\int_a^b \int_a^b \int_c^d |x-t|^q \, dsdtdx = (d-c) \int_a^b \left[\int_a^x (x-t)^q \, dt + \int_x^b (t-x)^q \, dt \right] dx$$

$$= \frac{2(d-c)(b-a)^{q+2}}{(q+1)(q+2)}$$

and then we obtain that

$$\left| \int_a^b \int_c^d \int_a^b \int_c^d [f(x,s)g(x,s) - f(t,y)g(x,s) - f(x,s)g(t,y) + f(t,y)g(t,y)] \, dsdtdydx \right|$$

$$\leq 2^{1+\frac{1}{q}} \frac{(d-c)^{1+\frac{1}{q}} (b-a)^{1+\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}} \left[L_1 (d-c)^{\frac{1}{q}} + L_2 (b-a)^{\frac{1}{q}} \right] \|g\|_p .$$

Finally, assuming that $g \in L_\infty(\Delta)$, then we have

$$\left| \int_a^b \int_c^d \int_a^b \int_c^d [f(x,s)g(x,s) - f(t,y)g(x,s) - f(x,s)g(t,y) + f(t,y)g(t,y)] \, dsdtdydx \right|$$

$$\leq L_1 (d-c)^2 \|g\|_\infty \int_a^b \int_a^b |x-t| \, dtdx + L_2 (b-a)^2 \|g\|_\infty \int_c^d \int_c^d |s-y| \, dsdy$$

$$+ L_1 (d-c)^2 \|g\|_\infty \int_a^b \int_a^b |x-t| \, dtdx + L_2 (b-a)^2 \|g\|_\infty \int_c^d \int_c^d |s-y| \, dsdy$$

$$\leq \frac{4L_1 (d-c)^2 (b-a)^3}{3} \|g\|_\infty + \frac{4L_2 (d-c)^3 (b-a)^2}{3} \|g\|_\infty$$

which this completess the proof. □

References

[1] Boukerrioua, K. and Lakoud, A.G., *On generalization of Čebyšev type inequalities*, J. Inequal. Pure and Appl. Math. **8** (2) (2007), Art 55.

[2] Čebyšev, P.L., *Sur less expressions approximatives des integrales definies par les autres prises entre les memes limites*, Proc. Math. Soc. Charkov **2** (1882), 93–98.

[3] Dragomir, S.S. and Fedotov, I., *An inequality of Grüss type for Riemann-Stieltjes integral and applications for special means*, Tamkang J. of Math. **29** (4) (1998), 287–292.

[4] Dragomir, S.S., *Some integral inequalities of Grüss type*, Indian J. Pure Appl. Math. **31** (4) (2000), 397–415.

[5] Lakoud, A.G. and Aissaouinew, F., *Čebyšev type inequalities for double integrals*, Journal Math. Inequali. **5** (4) (2011), 453–462.

[6] Grüss, G., *Über das maximum des absoluten Betrages von* $\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$, Math. Z. **39** (1935), 215-226.

[7] Hussain, S. and Alomari, M.W., *Weighted Ostrowski and Čebyšev type inequalities with applications*, Konuralp Journal of Mathematics **1** (2) (2013), 1–16.

[8] Mitrinovic, D.S., Pecaric, J.E. and Fink, A.M., *Inequalities involving functions and their integrals and derivatives*, Kluwer Academic Publishers, Dordrecht, 1991.

[9] Pachpatte, B.G., *On Čebyšev-Grüss type inequalities via Pecaric's extension of the Montgomery identity*, J. Inequal. Pure and Appl. Math. **7** (1) (2006), Art 108.

[10] Pecaric, J.E., *On the Čebyšev inequality*, Bul. Sti. Tehn. Inst. Politehn "Traian Vuia" Timișora (Romania), **25** (39) (1) (1980), 5–9.

[11] Pecaric, J.E. and Tebes, B., *On Grüss type inequalities Dragomir and Fedotov*, J. Inequal. Pure and Appl. Math. **4** (5) (2003), Art 91.

[12] Sarikaya, M.Z., Aktan, N., Yıldırım, H., *Weighted Čebyšev-Grüss type inequalities on time scales*, J. Math. Inequal. **2** (2008), 185–195.

[13] Sarikaya, M.Z., *A Note on Grüss type inequalities on time scales*, Dynamic Systems and Applications, **17** (2008), 663–666.

Düzce University
Department of Mathematics
Konuralp Campus, Düzce, Turkey
E-mail address: sarikayamz@gmail.com

Afyon Kocatepe University
Department of Mathematics
Afyonkarahisar, Turkey
E-mail address: mkiris@gmail.com, kiris@aku.edu.tr