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OSCILLATION OF SECOND ORDER QUASILINEAR DIFFERENCE EQUATIONS WITH SEVERAL NEUTRAL TERMS

E. THANDAPANI, D. SEGHAR, AND S. SELVARANGAM

Abstract. This paper deals with the oscillatory properties of a certain second order difference equation with several neutral terms. Some new sufficient conditions are established which simplify the examination of the equation studied. The results obtained here extend those in [4] for second order difference equation.

1. Introduction

In this paper we are concerned with the oscillation of all solutions of a second order neutral difference equation of the form

$$\Delta\left(a_n(\Delta z_n)^{\alpha}\right) + q_n x_{n-\sigma}^{\beta} = 0,\ n \geq n_0 \tag{1}$$

where $z_n = x_n + \sum\limits_{i=1}^{m} p_{in} x_{\tau_i(n)},\ m > 0$ is an integer, subject to the following conditions:

(H_1) α and β are ratios of odd positive integers;
(H_2) $\{a_n\}$ and $\{q_n\}$ are positive real sequences;
(H_3) $\{p_{in}\}$ is a nonnegative real sequences such that $0 \leq p_{in} \leq p_i < \infty$;
(H_4) $\{\tau_i(n)\}$ is a sequence of integers and σ is a positive integer with $\tau_i(n-\sigma) = \tau_i(n)-\sigma$ for $i = 1, 2, \ldots, m$;
(H_5) $R_n = \sum\limits_{s=n_0}^{n-1} \dfrac{1}{a_s^{1/\alpha}} \to \infty$ as $n \to \infty$.

By a solution of equation (1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$ where $\theta = \max\left\{\sigma, \min\limits_{1\leq i\leq m} \tau_i(n)\right\}$, and satisfying the equation (1) for all $n \geq n_0$. A nontrivial solution of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

It is well-known that second order neutral difference equations have applications in various problems of population dynamics, economics, biology, etc. Therefore, there has been much interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of various types of second order difference equations, see, for example [1, 2, 3, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18],and the references cited therein. In the following, we present some background details that motivate our study. In [9], the authors proved that $0 \leq p_n \leq 1$ together with $\sum\limits_{n=n_0}^{\infty} q_n(1 - p_{n-\sigma}) = \infty$ guarantee the oscillation of all

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solutions of the neutral difference equation

$$\Delta^2(x_n + p_n x_{n-\tau}) + q_n x_{n-\sigma} = 0.$$

For the same equation, the authors in [1, 3] established an oscillation criterion provided that $q_n \geq q > 0,\ p_1 \leq p_n \leq p_2$ and $\{p_n\}$ is not eventually negative. This result has been improved and extended by many authors. We mention Lalli and Grace [8] who studied oscillation of nonlinear difference equation

$$\Delta\left(a_n \Delta(x_n + p_n x_{n-\tau})\right) + q_n f(x_{n-\sigma}) = 0,$$

under the assumptions

$$\frac{f(u)}{u} \geq M > 0,\ \sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty \text{ and } 0 \leq p_n < 1.$$

Sun and Saker [11], and Saker [10] established several oscillation results for the equation

$$\Delta\left(a_n \left(\Delta(x_n + p_n x_{n-\tau})\right)^{\gamma}\right) + f(n, x_{n-\tau}) = 0,$$

under the assumptions

$$f(n,u)\text{sgn}u \geq q_n u^{\gamma},\ \sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\gamma}} = \infty \text{ and } 0 \leq p_n < 1.$$

Grace and El-Morshedy [6], and Thandapani and Selvarangam [17] studied neutral difference equation

$$\Delta\left(a_n \left(\Delta(x_n + p_n x_{\tau(n)})\right)^{\alpha}\right) + q_n x^{\beta}_{\sigma(n)} = 0,$$

and presented some new oscillation criteria in the case

$$0 \leq p_n \leq p < \infty \text{ and } \tau \circ \sigma = \sigma \circ \tau.$$

In [13], the authors studied a mixed-type neutral difference equation of the form

$$\Delta(a_n \Delta(x_n + b_n x_{n-\sigma_1} + c_n x_{n+\sigma_2})) + q_n x^{\alpha}_{n-\sigma_3} + p_n x^{\beta}_{n+\sigma_4} = 0 \tag{2}$$

when $\sum\limits_{n=n_0}^{\infty} \dfrac{1}{a_n} = \infty$ and $0 \leq b_n + c_n < 1$.

Motivated by the above observation, in this paper we study the oscillatory behavior of equation (1). The results obtained here reduced to those presented in [5, 8, 9, 10] for the particular case $m = 1$. In Section 2, we obtain some sufficient conditions for oscillation of all solutions of equation (1), and in Section 3 we present some examples to illustrate the main results.

2. Oscillation Results

We begin with the following lemmas which will be useful in proving the main theorems.

Lemma 1. *Let $\{x_n\}$ be a positive solution of equation* (1)*. Then the corresponding $\{z_n\}$ satisfies*

$$z_n > 0,\ \Delta z_n > 0,\ \textit{and}\ \Delta(a_n(\triangle z_n)^{\alpha}) < 0$$

eventually.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (1). Then from equation (1), we have

$$\Delta(a_n(\Delta z_n)^{\alpha}) = -q_n x^{\beta}_{n-\sigma} < 0.$$

Hence $\{a_n(\Delta z_n)^{\alpha}\}$ is decreasing, and thus either $\Delta z_n > 0$ or $\Delta z_n < 0$ eventually. If $\Delta z_n < 0$ then, by (H_5) we have $z_n < 0$ eventually. This contradiction completes the proof. □

Lemma 2. *Assume that* $y_i \geq 0$ *for* $i = 1, 2, 3, \dots m$. *Then*

(a) $\sum_{i=1}^{m} y_i^{\alpha} \geq \frac{l}{m^{\alpha-1}} \left(\sum_{i=1}^{m} y_i \right)^{\alpha}$ *for* $\alpha \geq 1$;

(b) $\sum_{i=1}^{m} y_i^{\alpha} \geq \left(\sum_{i=1}^{m} y_i \right)^{\alpha}$ *for* $0 < \alpha < 1$.

Proof. The proof is similar to that of in [13] and hence the details are omitted. □

In what follows, we use the notations

$$\begin{aligned} Q(n) &= \min\{q_n, q_{\tau_1(n)}, \dots, q_{\tau_m(n)}\} \\ Q_{n,n_1} &= Q_n (R_n - R_{n_1})^{\beta} \end{aligned}$$

where $n \geq n_1$ is sufficiently large.

Theorem 1. *Assume that* $\beta \geq 1$. *If the first order difference inequality*

$$\Delta \left(y_n + \sum_{i=1}^{m} c_i^{\beta} y_{\tau_i(n)} \right) + \frac{Q_{n,n_1}}{(m+1)^{\beta-1}} y_{n-\sigma}^{\beta/\alpha} \leq 0 \tag{3}$$

has no positive decreasing solution, then every solution of equation (1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a positive solution of equation (1). Then from equation (1), we have for $n \geq n_0$

$$\Delta(a_n(\Delta z_n)^{\alpha}) + q_n x_{n-\sigma}^{\beta} = 0, \tag{4}$$

and

$$\sum_{i=1}^{m} c_i^{\beta} \Delta(a_{\tau_i(n)}(\Delta z_{\tau_i(n)})^{\alpha}) + \sum_{i=1}^{m} c_i^{\beta} q_{\tau_i(n)} x_{\tau_i(n)-\sigma}^{\beta} = 0. \tag{5}$$

Combining (4) and (5), we are led to

$$\Delta(a_n(\Delta z_n)^{\alpha}) + \sum_{i=1}^{m} c_i^{\beta} \Delta(a_{\tau_i(n)}(\Delta z_{\tau_i(n)})^{\alpha}) + Q_n \left(x_{n-\sigma}^{\beta} + \sum_{i=1}^{m} c_i^{\beta} x_{\tau_i(n)-\sigma}^{\beta} \right) \leq 0. \tag{6}$$

Using Lemma 2 (a) in (6), we obtain

$$\Delta(a_n(\Delta z_n)^{\alpha}) + \sum_{i=1}^{m} c_i^{\beta} \Delta(a_{\tau_i(n)}(\Delta z_{\tau_i(n)})^{\alpha}) + \frac{Q_n}{(m+1)^{\beta-1}} z_{n-\sigma}^{\beta} \leq 0, \tag{7}$$

where we have used $\tau_i(n) - \sigma = \tau_i(n - \sigma)$ for $i = 1, 2, \dots, m$. It follows from Lemma 1 that $y_n = a_n(\Delta z_n)^{\alpha} \geq 0$ is decreasing, and so

$$z_n = z_{n_1} + \sum_{s=n_1}^{n-1} \frac{a_s^{1/\alpha} \Delta z_s}{a_s^{1/\alpha}} \geq y_n^{1/\alpha}(R_n - R_{n_1}). \tag{8}$$

From (7) and (8), we have

$$\Delta \left(y_n + \sum_{i=1}^{m} c_i^{\beta} y_{\tau_i(n)} \right) + \frac{Q_{n,n_1}}{(m+1)^{\beta-1}} y_{n-\sigma}^{\beta/\alpha} \leq 0, \ n \geq n_1. \tag{9}$$

Thus, $\{y_n\}$ is a positive decreasing solution of the inequality (3), which is a contradiction. This completes the proof. □

Theorem 2. *Assume that $\beta \geq 1$ and $\tau_i(n) \geq n$ for $i = 1, 2, \ldots, m$. If the first order difference inequality*

$$\Delta w_n + \frac{Q_{n,n_1}}{(m+1)^{\beta-1}\left(1+\sum\limits_{i=1}^{m} c_i^{\beta}\right)^{\beta/\alpha}} w_{n-\sigma}^{\beta/\alpha} \leq 0 \tag{10}$$

has no positive decreasing solution, then every solution of equation (1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a positive solution of equation (1). Then it follows from the proof of Theorem 1 that $y_n = a_n(\Delta z_n)^{\alpha}$ is positive decreasing and satisfying (9). We now define

$$w_n = y_n + \sum_{i=1}^{m} c_i^{\beta} y_{\tau_i(n)}.$$

Then $w_n > 0$ and in view of $\tau_i(n) \geq n$, we have

$$w_n \leq y_n \left(1 + \sum_{i=1}^{m} c_i^{\beta}\right).$$

Substituting the last inequality into (9), we see that $\{w_n\}$ is a positive solution of inequality (10). This contradiction completes the proof. □

Remark 1. *Theorem 1 reduces to that of in* [4]*, when $\alpha = \beta$ and $m = 1$. Theorem 2 extends some results in* [15]*, in the case where $m = 1$ and $\alpha = 1$.*

Next, by adding additional assumptions on α and β, we can deduce explicit oscillation criteria for the equation (1) from Theorem 2.

Corollary 1. *In addition to assumptions of Theorem 2, let $\alpha = \beta$. If*

$$\liminf_{n\to\infty} \sum_{s=n-\sigma}^{n-1} Q_{s,n_1} > (m+1)^{\beta-1}\left(1+\sum_{i=1}^{m} c_i^{\beta}\right)\left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1} \tag{11}$$

then every solution of equation (1) *is oscillatory.*

Proof. By Theorem 2 of [7], assumption (11) ensures that the inequality (10) has no positive solution. The result now follows from Theorem 2. □

Corollary 2. *In addition to assumptions of Theorem 2, let $\alpha > \beta$. If*

$$\sum_{n=n_0}^{\infty} Q_{n,n_1} = \infty, \tag{12}$$

then every solution of equation (1) *is oscillatory.*

Proof. By Theorem 1 of [12], assumption (12) ensures that the inequality (10) has no positive solutions. The result now follows from Theorem 2. □

Corollary 3. *In addition to assumptions of Theorem 2, let $\alpha < \beta$. If there exists a $\lambda > \frac{1}{\sigma}\log \beta/\alpha$ such that*

$$\liminf_{n\to\infty}\left[Q_{n,n_1}\exp(-e^{\lambda n})\right] > 0 \tag{13}$$

then every solution of equation (1) *is oscillatory.*

Proof. By Theorem 2 of [12], assumption (13) ensures that the inequality (10) has no positive solutions. The result now follows from Theorem 2. □

In the following, we use the notation $\tau = \min\{\tau_i(n) : i = 1, 2, \dots, m\}$ and τ^{-1} stands for the inverse of τ.

Theorem 3. *Assume that $\beta \geq 1$ and $\tau(n) \leq n$. If the first order difference inequality*

$$\Delta w_n + \frac{Q_{n,n_1}}{(m+1)^{\beta-1}\left(1 + \sum\limits_{i=1}^{m} c_i^{\beta}\right)^{\beta/\alpha}} w_{\tau^{-1}(n-\sigma)}^{\beta/\alpha} \leq 0 \tag{14}$$

has no positive solution, then every solution of equation (1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a positive solution of equation (1). Then it follows from the proof of Theorem 1 that $y_n = a_n(\Delta z_n)^{\alpha} > 0$ is decreasing and satisfying 3. Now define

$$0 < w_n = y_n + \sum_{i=1}^{m} c_i^{\beta} y_{\tau_i(n)} \leq y_{\tau(n)} \left(1 + \sum_{i=1}^{m} c_i^{\beta}\right). \tag{15}$$

Using (15) in (3), we obtain that $\{w_n\}$ is a positive solution of the inequality (14). This contradiction completes the proof. □

Remark 2. *Theorem 3 extends some results of* [6] *in the case $\alpha = \beta = 1$ and $m = 1$.*

Corollary 4. *Assume that $\alpha = \beta \geq 1$ and $\tau(n) = n - k$, where k is a positive integer such that $k < \sigma$. If*

$$\liminf_{n\to\infty} \sum_{s=n+k-\sigma}^{n-1} Q_{s,n_1} > (m+1)^{\beta-1}\left(1 + \sum_{i=1}^{m} c_i^{\beta}\right)\left(\frac{\sigma - k}{\sigma - k + 1}\right)^{\sigma-k+1} \tag{16}$$

then every solution of equation (1) *is oscillatory.*

Proof. By Theorem 2 of [7], the condition (16) implies that the difference inequality (14) has no positive solution. The result now follows from Theorem 3. □

Corollary 5. *Assume that $\alpha > \beta \geq 1$ and $\tau(n) = n - k$ where k is a positive integer such that $k < \sigma$. If*

$$\sum_{n=n_0}^{\infty} Q_{n,n_1} = \infty, \tag{17}$$

then every solution of equation (1) *is oscillatory.*

Proof. By Theorem 1 of [12], assumption (17) ensures that the inequality (14) has no positive solutions. The result now follows from Theorem 3. □

Corollary 6. *Assume that $\alpha < \beta \geq 1$ and $\tau(n) = n - k$ where k is a positive integer such that $k < \sigma$. If there exists a $\lambda > \dfrac{1}{\sigma - k}\log \beta/\alpha$ such that*

$$\liminf_{n\to\infty} \left[Q_{n,n_1} \exp(-e^{n\lambda})\right] > 0, \tag{18}$$

then every solution of equation (1) *is oscillatory.*

Proof. By Theorem 2, of [12], the condition (18) ensure that the difference inequality (14) has no positive solutions. The result now follows from Theorem 3. □

Next we turn our attention to the case $0 < \beta < 1$.

Theorem 4. *Assume that* $0 < \beta < 1$. *If the first order difference inequality*

$$\Delta\left(y_n + \sum_{i=1}^{m} c_i^{\beta} y_{\tau_i(n)}\right) + Q_{n,n_1} y_{n-\sigma}^{\beta/\alpha} \le 0$$

has no positive decreasing solution, then every solution of equation (1) *is oscillatory.*

Proof. The proof is exactly same as that of Theorem 1 except by using Lemma 2 (b) instead of Lemma 2 (a). Hence the details are omitted. □

Theorem 5. *Assume that* $0 < \beta < 1$ *and* $\tau_i(n) \ge n$ *for* $i = 1, 2, \ldots, m$. *If the first order difference inequality*

$$\Delta w_n + \frac{Q_{n,n_1}}{\left(1 + \sum\limits_{i=1}^{m} c_i^{\beta}\right)^{\beta/\alpha}} w_{n-\sigma}^{\beta/\alpha} \le 0$$

has no positive solution, then every solution of equation (1) *is oscillatory.*

Proof. The proof is similar to that of Theorem 2 and hence the details are omitted. □

Corollary 7. *Assume* $0 < \beta < 1$ *and* $\tau_i(n) \ge n$ *for* $i = 1, 2, \ldots, m$.

(a) *If* $\alpha = \beta$ *and*

$$\liminf_{n\to\infty} \sum_{s=n-\sigma}^{n-1} Q_{s,n_1} > \left(1 + \sum_{i=1}^{m} c_i^{\beta}\right)\left(\frac{\sigma}{\sigma+1}\right)^{\sigma+1},$$

then every solution of equation (1) *is oscillatory.*

(b) *If* $\alpha > \beta$ *and*

$$\sum_{n=n_0}^{\infty} Q_{n,n_1} = \infty,$$

then every solution of equation (1) *is oscillatory.*

(c) *If* $\alpha < \beta$ *and there exists* $\lambda > \dfrac{1}{\sigma} \log \beta/\alpha$ *such that*

$$\liminf_{n\to\infty} \left[Q_{n,n_1} \exp(-e^{n\lambda})\right] > 0,$$

then every solution of equation (1) *is oscillatory.*

Proof. The proof is similar to that of Corollaries 1 to 3 by using Theorem 5 instead of Theorem 2 and hence the details are omitted. □

Theorem 6. *Assume that* $0 < \beta < 1$ *and* $\tau(n) \le n$. *If the first order difference inequality*

$$\Delta w_n + \frac{Q_{n,n_1}}{\left(1 + \sum\limits_{i=1}^{m} c_i^{\beta}\right)^{\beta/\alpha}} w_{\tau^{-1}(n-\sigma)}^{\beta/\alpha} \le 0$$

has no positive solution, then every solution of equation (1) *is oscillatory.*

Proof. The proof is similar to that of Theorem 3 except using Lemma 2 (b) instead of Lemma 2 (a). So the details are omitted. □

Corollary 8. *Assume that* $0 < \beta < 1$ *and* $\tau(n) = n - k$ *where* k *is a positive integer such that* $k < \sigma$.

(*a*) *If* $\alpha = \beta$ *and*

$$\liminf_{n\to\infty} \sum_{s=n+k-\sigma}^{n-1} Q_{s,n_1} > \left(1 + \sum_{i=1}^{m} c_i^{\beta}\right)\left(\frac{\sigma - k}{\sigma - k + 1}\right)^{\sigma-k+1},$$

then every solution of equation (1) *is oscillatory.*

(*b*) *If* $\alpha > \beta$ *and*

$$\sum_{n=n_0}^{\infty} Q_{n,n_1} = \infty,$$

then every solution of equation (1) *is oscillatory.*

(*c*) *If* $\alpha < \beta$ *and* $\tau(n) = n - k$ *where* k *is a positive integer such that* $k < \sigma$. *If there exists a* $\lambda > \dfrac{1}{\sigma - k}\log \beta/\alpha$ *such that*

$$\liminf_{n\to\infty}\left[Q_{n,n_1}\exp(-e^{\lambda n})\right] > 0,$$

then every solution of equation (1) *is oscillatory.*

Proof. The proof is similar to that of Corollaries 4 to 6 by using Theorem 6 instead of Theorem 3 and hence the details are omitted. □

3. Examples

In this section we present some examples to illustrate the main results.

Example 1. *Consider the second order neutral difference equation*

$$\Delta(n(\Delta(x_n + 2x_{n+1} + x_{n+2}))^3) + \frac{8}{27}(9n + 8)x_{n-1}^3 = 0,\ n \ge 1 \tag{19}$$

It is easy to see that all conditions of Corollary 1 are satisfied and hence every solution of equation (19) *is oscillatory. In fact* $\{x_n\} = \{(-1)^n 2^n\}$ *is one such oscillatory solution of equation* (19).

Example 2. *Consider the second order neutral difference equation*

$$\Delta((\Delta(x_n + x_{n+2} + 2x_{n+4}))^3) + 1024\ x_{n-3}^{5/3} = 0,\ n \ge 1 \tag{20}$$

It is easy to see that all conditions of Corollary 2 are satisfied and hence every solution of equation (20) *is oscillatory. In fact* $\{x_n\} = \{(-1)^n\}$ *is one such oscillatory solution of equation* (20).

Example 3. *Consider the second order neutral difference equation*

$$\Delta^2(x_n + 2x_{n+2} + x_{n+3}) + \exp(e^n)x_{n-3}^3 = 0,\ n \ge 1. \tag{21}$$

Here $R_n = n$, $\beta = 3$, $\sigma = 3$ *and* $\alpha = 1$. *By choosing* $\lambda = 1$, *we see that conditions of Corollary 3 are satisfied and hence every solution of equation* (21) *is oscillatory.*

We conclude this paper with the following remark.

Remark 3. *One can construct examples similar to the above to illustrate the other theorems and the details are left to the reader. Further, it is interesting to extend the results of this paper to the difference equation with several negative neutral terms of the form*

$$\Delta\left(a_n\left(\Delta\left(x_n - \sum_{i=1}^{m} p_{in}x_{\tau_i(n)}\right)\right)^{\alpha}\right) + q_n x_{n-\sigma}^{\beta} = 0,\ n \ge n_0.$$

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Ramanujan Institute For Advanced Study in Mathematics
University of Madras
Chennai-600005, India.
E-mail address: `ethandapani@yahoo.co.in`

Ramanujan Institute For Advanced Study in Mathematics
University of Madras
Chennai-600005, India.
E-mail address: `d.seghar@gmail.com`

Department of Mathematics
Presidency College
Chennai-600005, India.
E-mail address: `selvarangam.9962@gmail.com`