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SOME INEQUALITIES FOR THE GROWTH OF SELF-RECIPROCAL POLYNOMIALS

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Abstract. In this paper we prove a property of self reciprocal polynomials and use it to obtain few inequalities for the growth of such polynomials.

1. Introduction and statement of results

Throughout this paper $\mathcal{P}_n$ will denote the class of all polynomials of degree at most n whose coefficients are complex numbers and $R > 1$ a real number unless otherwise stated. If $f \in \mathcal{P}_n$, then by applying the maximum modulus principle to the polynomial $f^{\sim}(z) := z^n \overline{f(1/\overline{z})}$ one concludes that

$$\max_{|z|=R>1} |f(z)| \leq R^n \max_{|z|=1} |f(z)| . \tag{1}$$

Equality holds for polynomials of the form cz^n where $c \in \mathbb{C}$ (the set of complex numbers).

It is known ([7], p. 428) that for any z such that $|z| = R > 1$

$$|f(z)| + |f^{\sim}(z)| \leq (R^n + 1) \max_{|z|=1} |f(z)| . \tag{2}$$

A polynomial f in $\mathcal{P}_n$ is called self-inversive if $f(z) \equiv \mathrm{e}^{\mathrm{i}\gamma} z^n \overline{f(1/\overline{z})}$ where $\gamma \in \mathbb{R}$. Let $\mathcal{P}_n^{\sim}$ denote the subclass of $\mathcal{P}_n$ consisting of self-inversive polynomials and f belongs to $\mathcal{P}_n^{\sim}$. From (2), it follows that

$$\max_{|z|=R>1} |f(z)| \leq \frac{R^n+1}{2} \max_{|z|=1} |f(z)| . \tag{3}$$

The inequality is sharp as $f(z) = z^n + 1$ is a polynomial in $\mathcal{P}_n^{\sim}$ for which equality holds in (3). It is noteworthy that the bound for $|f(z)|$ at a point outside the unit disk is greatly improved for self-inversive polynomials.

There is another subclass of $\mathcal{P}_n$ which has attracted the attention of many mathematicians for over forty years. It consists of those polynomials f in $\mathcal{P}_n$ which satisfy the condition $f(z) \equiv z^n f(1/z)$. Following Rahman and Schmeisser [7], we will call them self-reciprocal polynomials and will denote the class of such polynomials by $\mathcal{P}_n^{\vee}$. The condition that defines the subclass $\mathcal{P}_n^{\vee}$ looks very similar to the one used in the definition of $\mathcal{P}_n^{\sim}$, and indeed, the two classes do share some similar properties. For example, polynomials in $\mathcal{P}_n^{\sim}$ as well as in $\mathcal{P}_n^{\vee}$ have at least half of their zeroes outside the open unit disk (it is understood that a polynomial f belonging to the class $\mathcal{P}_n$ but of degree $m < n$ has $n - m$ of its zeroes at ∞).

Let $f \in \mathcal{P}_n^{\vee}$ where n is odd. From the definition, we have $f(-1) = -f(-1)$ which gives $f(-1) = 0$. Thus, a self-reciprocal polynomial of odd degree will necessarily have a zero

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at -1. For this reason, if $f \in \mathcal{P}_1^{\vee}$, then it is of the form $f(z) := c(z+1)$ for some constant $c \neq 0$. Clearly,

$$\max_{|z|=1}|f(z)| = 2|c| \quad \text{and} \quad \max_{|z|=R>1}|p(z)| = |c|\,(R+1) = \frac{R^n+1}{2}\max_{|z|=1}|f(z)|.$$

Thus, inequality (3) holds for all polynomials in $\mathcal{P}_1^{\vee}$ as well. It was shown by Rahman ([6], pp. 229 – 230) that it also holds for polynomials in $\mathcal{P}_2^{\vee}$. This makes one wonder if (3) holds for all polynomials in $\mathcal{P}_n^{\vee}$ whatever n may be. Rahman and Schmeisser ([7], pp. 431 – 432) proved that (3) is not true in general. In fact, they proved that for $n \geq 3$, there exists a polynomial $f \in \mathcal{P}_n^{\vee}$ such that

$$\max_{|z|=R>1}|f(z)| \geq \frac{R^n+R^{n-2}}{2}\max_{|z|=1}|f(z)| \geq \frac{R^n+1}{2}\max_{|z|=1}|f(z)|. \tag{4}$$

The sharp analogue of (3) for polynomials in $\mathcal{P}_n^{\vee}$ remains unknown for any $n \geq 3$.

For $f \in \mathcal{P}_n$, $R > 0$ and $p > 0$, let

$$M_p(f;R) := \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(Re^{i\theta})|^p d\theta\right)^{1/p}$$

denote the L^p mean of the polynomial f on the circle $|z| = R$. It is well known (see Hardy ([4], p. 143)) that as a function of p, M_p is non decreasing and

$$\lim_{p\to\infty} M_p(f;R) = \max_{|z|=R}|f(z)|.$$

Let us denote $\max_{|z|=R}|f(z)|$ by $M_\infty(f;R)$. In view of this, (1) may be written as

$$M_\infty(f;R) \leq R^n M_\infty(f;1)\,. \tag{5}$$

If f belongs to $\mathcal{P}_n$, then so does $g(z) = z^n f(1/z)$. Moreover, f may be written in terms of g as $f(z) = z^n g(1/z)$ and for $R > 1$, $p > 0$

$$\int_{-\pi}^{\pi}|f(Re^{i\theta})|^p d\theta = R^{np}\int_{-\pi}^{\pi}|g(e^{-i\theta}/R)|^p d\theta$$
$$\leq R^{np}\int_{-\pi}^{\pi}|g(e^{-i\theta})|^p d\theta = R^{np}\int_{-\pi}^{\pi}|f(e^{i\theta})|^p d\theta.$$

Thus, we have the following generalization of (1)

$$M_p(f;R) \leq R^n M_p(f;1) \qquad (R > 1, p > 0). \tag{6}$$

The above inequality is sharp as equality holds for $f(z) = cz^n$ where $c \in \mathbb{C}$.

Since self-reciprocal polynomials of exact degree n cannot be of the form cz^n, it implies that strict inequality will hold in (6) for such polynomials. It would be interesting to know what improvement in (6) would result if f belonged to $\mathcal{P}_n^{\vee}$, the class of self-reciprocal polynomials. In this direction, we have the following result, which is valid for $p = 2$.

Theorem 1. *If f is a polynomial in $\mathcal{P}_n^{\vee}$, then for $R > 0$*

$$M_2(f;R) \leq \left(\frac{R^{2n}+1}{2}\right)^{1/2} M_2(f;1). \tag{7}$$

The result is sharp as the equality holds for $f(z) = c(1+z^n)$.

The estimation of $M_p(f;R)$ in terms of $M_q(f;1)$, where $0 < p \leq \infty$, $0 < q \leq \infty$ are real numbers, is an interesting problem in its own right. The fact that $M_p(f;R)$ is an increasing function of p, the following inequality is obvious from (6)

$$M_p(f;R) \leq R^n M_\infty(f;1) \qquad (p > 0). \tag{8}$$

Since the extremal polynomials for (1) and (6) are the same, we conclude that inequality (8) is sharp and $f(z) = cz^n$ serves as the extremal polynomial. So it is natural to ask: ***What will be the corresponding inequality if we assume that $f \in \mathcal{P}_n^{\vee}$?***

Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ belong to $\mathcal{P}_n^{\vee}$ and z_k be the k^{th} root of unity. It is easy to see that

$$\begin{aligned} |f(z_k)|^2 &= \sum_{\nu=0}^{n} a_\nu (\mathrm{e}^{2k\pi\mathrm{i}/n})^\nu \sum_{\mu=0}^{n} \overline{a}_\mu (\mathrm{e}^{-2k\pi\mathrm{i}/n})^\mu \\ &= \sum_{\nu=0}^{n} |a_\nu|^2 + \sum_{\nu=0}^{n} \sum_{\mu=0,\ \mu\neq\nu}^{n} a_\nu \overline{a}_\mu (\mathrm{e}^{2(\nu-\mu)\pi\mathrm{i}/n})^k. \end{aligned}$$

We observe that $(\mathrm{e}^{2(\nu-\mu)\pi\mathrm{i}/n})^k = 1$ if and only if $\nu - \mu = \pm n$, and this happens if $\nu = 0, \mu = n$ or $\nu = n, \mu = 0$. In view of this,

$$|f(z_k)|^2 = \sum_{\nu=0}^{n} |a_\nu|^2 + a_0\overline{a}_n + a_n\overline{a}_0 + \sum_{\nu=1}^{n-1} \sum_{\mu=1\ \mu\neq\nu}^{n-1} a_\nu \overline{a}_\mu (\mathrm{e}^{2(\nu-\mu)\pi\mathrm{i}/n})^k. \tag{9}$$

Summing (9) over $1 \le k \le n$, we get

$$\begin{aligned} \sum_{k=1}^{n} |f(z_k)|^2 &= \sum_{k=1}^{n} \{ \sum_{\nu=0}^{n} |a_\nu|^2 + a_0\overline{a}_n + a_n\overline{a}_0 \} + \sum_{\nu=1}^{n-1} \sum_{\mu=1}^{n-1} a_\nu \overline{a}_\mu \sum_{k=1}^{n} (\mathrm{e}^{2\pi(\nu-\mu)\mathrm{i}/n})^k \\ &= n\{ \sum_{\nu=0}^{n} |a_\nu|^2 + 2|a_0|^2 \} \end{aligned} \tag{10}$$

as $\sum_{k=1}^{n} (\mathrm{e}^{2\pi(\nu-\mu)\mathrm{i}/n})^k = 0$ and $a_\nu = a_{n-\nu}$ for all $0 \le \nu \le n$. Thus, Theorem 1 and (10) give us

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(R\mathrm{e}^{\mathrm{i}\theta})|^2 \le \frac{R^{2n}+1}{2} \left(\frac{1}{n} \sum_{k=1}^{n} |f(z_k)|^2 - 2|a_0|^2 \right). \tag{11}$$

Once again, from the result of Hardy ([4], p. 143), which states that the mean value is an increasing function of p, we get the following

Theorem 2. *If $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial in $\mathcal{P}_n^{\vee}$ then for $R > 0$ and $0 < p \le 2$*

$$M_p(f;R) \le \left(\frac{R^{2n}+1}{2} \right)^{1/2} \left(\frac{1}{n} \sum_{k=1}^{n} |f(z_k)|^2 - 2|a_0|^2 \right)^{1/2} \tag{12}$$

where z_k is the k^{th} root of unity.

Since $|f(z_k)| \le M_\infty(f;1)$ for $1 \le k \le n$, Theorem 2 immediately gives the following

Corollary 1. *If f is a polynomial in $\mathcal{P}_n^{\vee}$, then for $R > 0$ and $0 < p \le 2$*

$$M_p(f;R) \le \left(\frac{R^{2n}+1}{2} \right)^{1/2} \left(M_\infty^2(f;1) - 2|a_0|^2 \right)^{1/2}. \tag{13}$$

As discussed earlier, in general, (4) holds true for polynomials in $\mathcal{P}_n^{\vee}$. However, if we impose some restrictions on the location of their zeroes, the bound in (4) can be significantly improved. For example, if f has all its zeroes in the left half plane or in the right half plane, Govil and others [3] prove that

$$\max_{|z|=R>1} |f(z)| \le \frac{R^n + \sqrt{2} - 1}{\sqrt{2}} \max_{|z|=1} |f(z)|.$$

In [2], readers may find various estimates for $\max_{|z|=R>1} |f(z)|$ in terms of $\max_{|z|=1} |f(z)|$ where f belongs to $\mathcal{P}_n^\vee$.

Next, we present a property of polynomials in $\mathcal{P}_n^\vee$ which have all their zeroes in the left half plane. Despite its simplicity, it does not seem to have been noted before. We will use it to obtain some interesting consequences for this class of polynomials. We prove the following

Theorem 3. *Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial in $\mathcal{P}_n^\vee$ having all its zeroes in the left half plane. Suppose, in addition, that its zeroes in the third quadrant are of modulus at most 1. Then, for any real number $R > 1$,*

$$\left|f(Re^{i\theta})\right| \leq \left|f(Re^{-i\theta})\right| \qquad (0 \leq \theta \leq \pi). \tag{14}$$

As an application of above theorem, we prove the following L^p inequality for $\mathcal{P}_n^\vee$ valid on the half interval $(0, \pi)$.

Corollary 2. *Let f be a polynomial in $\mathcal{P}_n^\vee$ having all its zeroes in the left half plane. Suppose, in addition, that its zeroes in the third quadrant are of modulus at most 1. Then, for $R > 1$ and $p > 0$*

$$\int_0^\pi \left|f(Re^{i\theta})\right|^p d\theta \leq \frac{\int_{-\pi}^{\pi} |R^n e^{i\alpha} + 1|^p d\alpha}{\int_{-\pi}^{\pi} |e^{i\alpha} + 1|^p d\alpha} \int_0^\pi \left|f(e^{i\theta})\right|^p d\theta. \tag{15}$$

As the second application of Theorem 3, we have the following result.

Corollary 3. *Let f be a polynomial in $\mathcal{P}_n^\vee$ having all its zeroes in the left half plane. Suppose, in addition, that its zeroes in the third quadrant are of modulus at most 1. Then, for $R > 1$ and $p > 0$*

$$\left(\frac{1}{\pi}\int_0^\pi \left|f(Re^{i\theta})\right|^p d\theta\right)^{1/p} \leq \left(\frac{R^{np}+1}{2}\right)^{1/p} \max_{0\leq\theta\leq\pi} \left|f(e^{i\theta})\right|. \tag{16}$$

This may be seen as an analogue of 8 for $\mathcal{P}_n^\vee$ over the half interval $(0, \pi)$. We do not know if the result is sharp.

As yet another application of Theorem 3, we state the following

Corollary 4. *Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial in $\mathcal{P}_n$ such that all its zeroes are real and lie in $[-1, 0]$. Let $\kappa := \{x + iy |\ y \geq 0\}$ be the upper half of the complex plane and $\xi \in \kappa - [-1, 0]$. Furthermore, let*

$$\Omega_R := \{x + iy | \frac{x^2}{(R + R^{-1}/2)^2} + \frac{y^2}{(R - R^{-1}/2)^2} = 1\}$$

be an ellipse passing through ξ having foci -1 and 1. Then

$$|f(\xi)| \leq \left|f(\bar{\xi})\right|. \tag{17}$$

2. Lemmas

Next, we will list all the known results needed in our proofs. We will start with the following

Definition 1. *A function $\phi : (0, \infty) \to \mathbb{R}$ is said to belong to $\mathfrak{F}$, if it is non-decreasing and $\psi(u) := \phi(e^u)$ is a convex function on $(-\infty, \infty)$.*

It can be easily verified that u^p belongs to $\mathfrak{F}$ when $p > 0$.

Definition 2. *Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial in $\mathcal{P}_n$ and $\Lambda := (\lambda_0, \lambda_1, \cdots, \lambda_n) \in \mathbb{C}^{n+1}$. Define a linear operator $\Phi_\Lambda : \mathcal{P}_n \to \mathcal{P}_n$ as follows.*

$$\Phi_\Lambda f(z) = \sum_{\nu=0}^{n} a_\nu \lambda_\nu z^\nu.$$

The operator Φ_Λ is said to be admissible, if it preserves one of the following two properties.

- *$f(z)$ has all its zeroes in the unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$.*
- *$f(z)$ has all its zeroes outside the unit disk $\{z \in \mathbb{C} : |z| \geq 1\}$.*

For admissible operators, Arestov [1] proved the following

Lemma 1. *For $f \in \mathcal{P}_n$, $\phi \in \mathfrak{F}$, and Φ_Λ, an admissible operator given above, we have*

$$\int_{-\pi}^{\pi} \phi\left(|\Phi_\Lambda f(\mathrm{e}^{\mathrm{i}\theta})|\right) d\theta \leq \int_{-\pi}^{\pi} \phi\left(c(\Lambda, n)|f(\mathrm{e}^{\mathrm{i}\theta})|\right) d\theta, \tag{18}$$

where

$$c(\Lambda, n) = max(|\lambda_0|, |\lambda_n|).$$

Definition 3. *Let $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ and $g(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} b_\nu z^\nu$ be polynomials in $\mathcal{P}_n$. The composition of f and g, denoted by $f * g$, is defined as*

$$(f * g)(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu b_\nu z^\nu.$$

We will need the following result of Schur-Szegö ([7], p.109) about the location of the zeroes of composite polynomials.

Lemma 2. *Let $f(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu z^\nu$ and $g(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} b_\nu z^\nu$ be polynomials in $\mathcal{P}_n$. Furthermore, assume that all the zeroes of f lie in $\mathbf{D} := \{z : |z| \geq 1\}$. Then, each zero ω of $(f * g)(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} a_\nu b_\nu z^\nu$ will be of the form*

$$\omega = \alpha\beta \tag{19}$$

where α belongs to $\mathbf{D}$ and β is a zero of g.

The next result is given in Rahman and Schmeisser [7] and stated earlier in the introduction.

Lemma 3. *Let f be polynomials in $\mathcal{P}_n$ and let $f^{\sim}(z) := z^n \overline{f(1/\overline{z})}$. Then, for any z such that $|z| = R > 1$*

$$|f(z)| + |f^{\sim}(z)| \leq (R^n + 1) \max_{|z|=1} |f(z)|. \tag{20}$$

We will also make use of the following lemma which is well known. However, for the sake of completeness, we will prove it.

Lemma 4. *Let f belong to $\mathcal{P}_n$ and $R > 1$. If f has all its zeroes in $\mathbf{D} := \{z : |z| \geq 1\}$ then for any $\alpha \in \mathbb{R}$, $F(z) := f(Rz) + \mathrm{e}^{\mathrm{i}\alpha} R^n f(z/R)$ will also have all its zeroes in $\mathbf{D}$.*

Proof. Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial in $\mathcal{P}_n$ such that its zeroes lie in the region $\mathbf{D} := \{z : |z| \geq 1\}$. For an arbitrary real number α and $R > 1$, consider the polynomial

$$g(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} \{R^\nu + \mathrm{e}^{\mathrm{i}\alpha} R^{n-\nu}\} z^\nu.$$

It can be easily seen that $g(z) = (1 + Rz)^n + \mathrm{e}^{\mathrm{i}\alpha} R^n (1 + z/R)$ is a polynomial in $\mathcal{P}_n$ whose zeroes are $\beta_k = \left(R\mathrm{e}^{\mathrm{i}\alpha+\pi+\mathrm{k}\pi} - 1\right) / \left(R - \mathrm{e}^{\mathrm{i}\alpha+\pi+\mathrm{i}\mathrm{k}\pi} - 1\right)$ which lie on the unit circle $\{z : |z| = 1\}$.

It is clear that F is the composition of f and g. According to Lemma 2, any zero ω_k of F will be given by $\beta_k\alpha$. Since, $|\alpha| \geq 1$ and $|\beta_k| \geq 1$, the result follows. $\square$

3. Proofs of theorems and their corollaries

Proof of Theorem 1. Let us first consider the case when $R > 1$. Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial in $\mathcal{P}_n^\vee$. From the definition, $a_\nu = a_{n-\nu}$ for $0 \leq \nu \leq n$. Thus

$$\begin{aligned}\int_{-\pi}^{\pi} |f(Re^{i\theta})|^2 d\theta &= 2\pi \sum_{\nu=0}^{n} |a_\nu|^2 R^{2\nu} = 2\pi \sum_{\nu=0}^{n} \frac{R^{2\nu} + R^{2n-2\nu}}{2} |a_\nu|^2 \\ &\leq 2\pi \frac{R^{2n}+1}{2} \sum_{\nu=0}^{n} |a_\nu|^2 = \frac{R^{2n}+1}{2} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta,\end{aligned}$$

which gives that

$$M_2(f;R) \leq \left(\frac{R^{2n}+1}{2}\right)^{1/2} M_2(f;1). \tag{21}$$

Let us now assume that $R < 1$. For any real number r, $|f(re^{i\theta})| = |r^n e^{in\theta} f(r^{-1}e^{-i\theta})|$. Thus from (21), we have

$$\int_{-\pi}^{\pi} |f(Re^{i\theta})|^2 d\theta = R^{2n} \int_{-\pi}^{\pi} |f(R^{-1}e^{-i\theta})|^2 d\theta \leq \frac{R^{2n}+1}{2} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 d\theta.$$

So (21) is true for $R < 1$ as well. It can be easily verified that the equality holds in (21) for $f(z) = c(1 + z^n)$. $\square$

Proof of Theorem 3. Let us first consider the case when f is a polynomial of degree 2. Let g_ν be a polynomial in $\mathcal{P}_2^\vee$ whose zeroes are in the left half plane. Furthermore, assume that $z_\nu := x_\nu + iy_\nu$ is its zero in the third quadrant such that $|z_\nu| \leq 1$. The polynomial g_ν can be written as $g_\nu(z) = (z - z_\nu)(z - z_\nu^{-1})$. Let $R > 1$ be a real number. Then

$$\begin{aligned}g_\nu(Re^{i\theta}) &= (Re^{i\theta} - z_\nu)(Re^{i\theta} - z_\nu^{-1}) \\ &= R^2 e^{2i\theta} - Re^{i\theta}(z_\nu + z_\nu^{-1}) + 1 \\ &= Re^{i\theta}\{Re^{i\theta} + R^{-1}e^{-i\theta} - (z_\nu + z_\nu^{-1})\}.\end{aligned}$$

It implies that

$$R^{-1} g_\nu(Re^{i\theta}) = e^{i\theta}\{Re^{i\theta} + R^{-1}e^{-i\theta} - (z_\nu + z_\nu^{-1})\}. \tag{22}$$

Similarly, by taking $R \leftrightarrow R^{-1}$, we have

$$R g_\nu(R^{-1}e^{i\theta}) = e^{i\theta}\{Re^{-i\theta} + R^{-1}e^{i\theta} - (z_\nu + z_\nu^{-1})\}. \tag{23}$$

Then

$$\left|\frac{R^{-1}g_\nu(Re^{i\theta})}{Rg_\nu(R^{-1}e^{i\theta})}\right|^2 = \tag{24}$$
$$\frac{\{(R+R^{-1})\cos\theta - \Re(z_\nu + z_\nu^{-1})\}^2 + \{(R-R^{-1})\sin\theta - \Im(z_\nu + z_\nu^{-1})\}^2}{\{(R+R^{-1})\cos\theta - \Re(z_\nu + z_\nu^{-1})\}^2 + \{(R-R^{-1})\sin\theta + \Im(z_\nu + z_\nu^{-1})\}^2}.$$

Since $\Im(z_\nu + z_\nu^{-1}) \geq 0$, we have

$$\left|\frac{(R-R^{-1})\sin\theta - \Im(z_\nu + z_\nu^{-1})}{(R-R^{-1})\sin\theta + \Im(z_\nu + z_\nu^{-1})}\right| \leq 1 \ (0 \leq \theta \leq \pi)$$

and hence from (3), we have

$$\left|g_\nu(Re^{i\theta})\right| \leq \left|g_\nu(Re^{-i\theta})\right| \ (0 \leq \theta \leq \pi). \tag{25}$$

Thus, the result is true for $n = 2$. Next, consider the general case when f belongs to $\mathcal{P}_n^\vee$. It is clear from the definition of self-reciprocal polynomials that if n is odd then it will necessarily have a zero at -1. Thus, f may be written as

$$f(z) = (z+1)^m \prod_{\nu=1}^{l} g_\nu(z),$$

where $m \geq 0$, $n = m + 2l$, and $g_\nu(z) = (z - z_\nu)(z - z_\nu^{-1}) \in \mathcal{P}_2^\vee$ for each $1 \leq \nu \leq l$. Then

$$\left|\frac{f(Re^{i\theta})}{f(R^{-1}e^{i\theta})}\right| = \left|\frac{Re^{i\theta}+1}{R^{-1}e^{i\theta}+1}\right|^m \prod_{\nu=1}^{l}\left|\frac{g_\nu(Re^{i\theta})}{g_\nu(R^{-1}e^{i\theta})}\right| = R^n \prod_{\nu=1}^{l}\left|\frac{g_\nu(Re^{i\theta})}{g_\nu(Re^{-i\theta})}\right| \tag{26}$$

as

$$\left|\frac{Re^{i\theta}+1}{R^{-1}e^{i\theta}+1}\right|^m = R^m\left|\frac{Re^{i\theta}+1}{e^{i\theta}+R}\right|^m = R^m\{\frac{R^2+1+2R\cos\theta}{R^2+1+2R\cos\theta}\}^{\frac{m}{2}} = R^m.$$

Thus from (25), we have

$$\left|f(Re^{i\theta})\right| \leq \left|f(Re^{-i\theta})\right| \ (0 \leq \theta \leq \pi). \tag{27}$$

□

Proof of Corollary 2. Let α be an arbitrary real number, $R > 1$, and $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ belongs to $\mathcal{P}_n^\vee$. Define a linear operator $\Phi : \mathcal{P}_n \to \mathcal{P}_n$ as follows

$$\Phi(f) = \sum_{\nu=0}^{n} a_\nu (R^\nu + e^{i\alpha}R^{n-\nu})z^\nu.$$

From Lemma 3, the operator Φ preserves the class of polynomials whose zeroes lie outside the unit disk $\{z : |z| = 1\}$, and hence is an admissible operator. By Lemma 1, taking $\phi(u) = |u|^p$ where $p > 0$, we have

$$\begin{aligned}\int_{-\pi}^{\pi} |f(Re^{i\theta}) + e^{i\alpha}f(Re^{-i\theta})|^p d\theta &= \int_{-\pi}^{\pi} |f(Re^{i\theta}) + e^{i\alpha}R^n f(R^{-1}e^{i\theta})|^p d\theta \\ &\leq |R^n e^{i\alpha} + 1|^p \int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta.\end{aligned} \tag{28}$$

The left hand side of (28) may be written as

$$\begin{aligned}&\int_{-\pi}^{\pi} |f(Re^{i\theta}) + e^{i\alpha}f(Re^{-i\theta})|^p d\theta = \\ &= \int_0^{\pi} \left|f(Re^{i\theta})\right|^p \left|e^{-i\alpha}\frac{f(Re^{-i\theta})}{f(Re^{i\theta})} + 1\right|^p d\theta + \int_0^{\pi} \left|f(Re^{i\theta})\right|^p \left|e^{i\alpha}\frac{f(Re^{-i\theta})}{f(Re^{i\theta})} + 1\right|^p d\theta \\ &\geq 2\left|1 + e^{i\alpha}\right|^p \int_0^{\pi} \left|f(Re^{i\theta})\right|^p d\theta.\end{aligned} \tag{29}$$

The inequality (29) follows from the fact that $\left|f(Re^{i\theta})\right| \leq \left|f(Re^{-i\theta})\right|$ when $0 \leq \theta \leq \pi$. and $\left|r + e^{i\gamma}\right| \geq \left|1 + e^{i\gamma}\right|$ if $|r| \geq 1$. From (28) and (29), we get

$$\left|1 + e^{i\alpha}\right|^p \int_0^{\pi} \left|f(Re^{i\theta})\right|^p d\theta \leq \left|R^n e^{i\alpha} + 1\right|^p \int_0^{\pi} \left|f(e^{i\theta})\right|^p d\theta. \tag{30}$$

Integrating both sides of (30) with respect to α from 0 to 2π, we get

$$\int_0^{\pi} |f(Re^{i\theta})|^p d\theta \leq \frac{\int_{-\pi}^{\pi} |R^n e^{i\alpha} + 1|^p d\alpha}{\int_{-\pi}^{\pi} |e^{i\alpha} + 1|^p d\alpha} \int_0^{\pi} |f(e^{i\theta})|^p d\theta. \tag{31}$$

□

Proof of Corollary 3. Let f belong to $\mathcal{P}_n^{\vee}$ which satisfies the conditions stated in corollary 3 and $R > 1$ be a real number. Since $|f(\mathrm{e}^{\mathrm{i}\theta})| = |f(\mathrm{e}^{-\mathrm{i}\theta})|$ for $0 \leq \theta \leq \pi$, $\max_{|z|=1} |f(z)| = \max_{0\leq\theta\leq\pi} |f(\mathrm{e}^{\mathrm{i}\theta})|$. From Lemma 3, we have

$$\left|f(R\mathrm{e}^{\mathrm{i}\theta})\right| + \left|f(R\mathrm{e}^{-\mathrm{i}\theta})\right| \leq (R^n + 1) \max_{0\leq\theta\leq\pi} |f(\mathrm{e}^{\mathrm{i}\theta})| \qquad (0 \leq \theta \leq \pi). \tag{32}$$

From Theorem 3, $\left|f(R\mathrm{e}^{\mathrm{i}\theta})\right| \leq \left|f(R\mathrm{e}^{-\mathrm{i}\theta})\right|$ whenever $(0 \leq \theta \leq \pi)$. Thus for $p \geq 1$, we have

$$\left(\frac{1}{\pi}\int_0^\pi |f(R\mathrm{e}^{\mathrm{i}\theta})|^p d\theta\right)^{1/p} \leq$$

$$\leq \left(\frac{1}{\pi}\int_0^\pi (\frac{R^n+1}{2})^p d\theta\right)^{1/p} \max_{0\leq\theta\leq\pi} |f(\mathrm{e}^{\mathrm{i}\theta})| \leq \left(\frac{R^{np}+1}{2}\right)^{1/p} \max_{0\leq\theta\leq\pi} |f(\mathrm{e}^{\mathrm{i}\theta})|.$$

This completes the proof of Corollary 3. □

Proof of Corollary 4. Let f be a polynomial in $\mathcal{P}_n$ such that all its zeroes are in $[-1, 0]$. Define $F(z) = z^n f\left((z + z^{-1})/2\right)$. It can be easily seen that F is a self-reciprocal polynomial of degree $2n$ whose zeroes lie in the left half plane on the unit circle. By Theorem 3, we have

$$\left|F(R\mathrm{e}^{\mathrm{i}\theta})\right| \leq \left|F(R\mathrm{e}^{-\mathrm{i}\theta})\right| \qquad (0 \leq \theta \leq \pi).$$

Which is equivalent to

$$\left|f\left(\frac{R\mathrm{e}^{\mathrm{i}\theta} + R^{-1}\mathrm{e}^{-\mathrm{i}\theta}}{2}\right)\right| \leq \left|f\left(\frac{R\mathrm{e}^{-\mathrm{i}\theta} + R^{-1}\mathrm{e}^{\mathrm{i}\theta}}{2}\right)\right| \tag{33}$$

and the result follows. □

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