TJMM
7 (2015), No. 1, 53-57

THE ORDER OF CONVEXITY FOR AN INTEGRAL OPERATOR

ADRIANA OPREA

ABSTRACT. For analytic functions f_i, g_i for $i = 1, 2, .., n$, $n \in \mathbb{N}^*$, in the open unit disk $\mathcal{U}$, we study some convexity properties for a new general integral operator.

1. INTRODUCTION AND PRELIMINARIES

Let $U = \{z : |z| < 1\}$ be the unit disk and A be the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathcal{U} \tag{1}$$

which are analytic in $\mathcal{U}$ and satisfy the conditions

$$f(0) = f'(0) - 1 = 0.$$

We denote by S the class of univalent and regular functions.

A function $f \in \mathcal{A}$ is a starlike function of order β, $0 \leq \beta < 1$ and we denote this class by $S^*(\beta)$ if it satisfies

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, z \in \mathcal{U}$$

We denote by $K(\beta)$ the class of convex functions of order β, $0 \leq \beta < 1$ that satisfies the inequality

$$\text{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \beta, z \in \mathcal{U}$$

A function $f \in \mathcal{A}$ belongs to class $R(\beta)$, $0 \leq \beta < 1$, if

$$\text{Re}(f'(z)) > \beta, z \in \mathcal{U}$$

The family $B(\mu, \beta)$, $\mu \geq 0$, $0 \leq \beta < 1$, which contains the functions f that satisfy the condition:

$$\left| f'(z) \left(\frac{z}{f(z)}\right)^{\mu} - 1 \right| < 1 - \beta, z \in \mathcal{U} \tag{2}$$

was studied by Frasin and Jahangiri in [3].

This family is a comprehensive class of analytic functions that contains other new classes of analytic univalent functions such as $B(1, \beta) = S^*_\beta$ and $B(0, \beta) = R_\beta$. Frasin and Darus in [2] have introduced another interesting subclass $B(2, \beta) = B(\beta)$.

In order to derive our main results, we have to recall here the following General Schwarz Lemma.

2010 *Mathematics Subject Classification.* Primary 30C45; Secondary 30C75.

Key words and phrases. Analytic functions; Integral Operators; General Schwarz Lemma;Convexity Order.

This work was supported by the strategic project PERFORM, POSDRU 159/1.5/S/138963, inside POSDRU Romania 2014, co-financed by the European Social Fund-Investing in People.

Lemma 1 (General Schwarz-Lemma [4]). *Let f the function regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$, M fixed. If f has at $z = 0$ one zero with multiplicity $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in \mathcal{U}_R. \tag{3}$$

the equality (in the inequality (3)) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

2. Main results

In this paper, we define a new general integral operator:

$$G_n(z) = \int_0^z \prod_{i=1}^n \left(f_i'(t) e^{g_i(t)} \right)^{\alpha_i} dt,$$

in the open unit disc $\mathcal{U}$, when $f_i, g_i \in \mathcal{A}$, $\alpha_i \in \mathbb{C}$ and we study the convexity order for this integral operator.

Theorem 1. *Let $f_i \in \mathcal{A}$, $g_i \in \mathcal{B}(\mu_i, \beta_i)$, $\mu_i \geq 1$, $0 \leq \beta_i < 1$ and $\alpha_i \in \mathbb{C}$, for $i = 1, 2, ..., n$. If*

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq M_i, M_i \geq 1, |g_i(z)| \leq N_i, N_i \geq 1, \ z \in \mathcal{U} \tag{4}$$

for $i = 1, 2, ..., n$ and $0 < \sum_{i=1}^n |\alpha_i|(M_i + (2 - \beta_i) N_i^{\mu_i}) \leq 1$, then the integral operator

$$G_n(z) = \int_0^z \prod_{i=1}^n \left(f_i'(t) e^{g_i(t)} \right)^{\alpha_i} dt \tag{5}$$

is in the class $\mathcal{K}(\delta)$, where $\delta = 1 - \sum_{i=1}^n |\alpha_i|(M_i + (2 - \beta_i) N_i^{\mu_i})$.

Proof. Let $g_i \in \mathcal{B}(\mu_i, \beta_i)$, $\mu_i \geq 1$, $0 \leq \beta_i < 1$, for $i = 1, 2, ..., n$.

From relation (5), we have:

$$\frac{G_n''(z)}{G_n'(z)} = \sum_{i=1}^n \alpha_i \left(\frac{f_i''(z)}{f_i'(z)} + g_i'(z) \right) \tag{6}$$

It follows that:

$$\left| \frac{zG_n''(z)}{G_n'(z)} \right| \leq \sum_{i=1}^n |\alpha_i||z| \left| \frac{f_i''(z)}{f_i'(z)} + g_i'(z) \right| \leq \tag{7}$$

$$\leq \sum_{i=1}^n |\alpha_i| \left(\left| \frac{f_i''(z)}{f_i'(z)} \right| + \left| g_i'(z) \left(\frac{z}{g_i(z)} \right)^{\mu_i} \right| \left| \left(\frac{g_i(z)}{z} \right)^{\mu_i} \right| \right)$$

Applying the General Schwarz Lemma, we have: $|g_i(z)| \leq |z| N_i, z \in \mathcal{U}$ for $i = 1, 2, ..., n$. So, from (7), we obtain:

$$\left| \frac{zG_n''(z)}{G_n'(z)} \right| \leq \sum_{i=1}^n |\alpha_i| \left(M_i + \left| g_i'(z) \left(\frac{z}{g_i(z)} \right)^{\mu_i} \right| N_i^{\mu_i} \right), z \in \mathcal{U} \tag{8}$$

From (8), we have:

$$\left| \frac{zG_n''(z)}{G_n'(z)} \right| \leq \sum_{i=1}^n |\alpha_i| \left(M_i + \left(\left| g_i'(z) \left(\frac{z}{g_i(z)} \right)^{\mu_i} - 1 \right| + 1 \right) N_i^{\mu_i} \right), z \in \mathcal{U} \tag{9}$$

From (2) and (9), we have:

$$\left|\frac{zG_n''(z)}{G_n'(z)}\right| \le \sum_{i=1}^{n} |\alpha_i| \left(M_i + (2-\beta_i)N_i^{\mu_i}\right) = 1 - \delta.$$

So, $G_n \in \mathcal{K}(\delta)$. □

If we consider $\beta_i = \beta$, $\mu_i = \mu$ for $i = 1, 2, ..., n$ in Theorem 1, we get:

Corollary 1. *Let $f_i \in \mathcal{A}$, $g_i \in \mathcal{B}(\mu, \beta)$, $\mu \ge 1$, $0 \le \beta < 1$ and $\alpha_i \in \mathbb{C}$, for $i = 1, 2, ..., n$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \le M_i, M_i \ge 1, |g_i(z)| \le N_i, N_i \ge 1,\ z \in \mathcal{U} \tag{10}$$

for $i = 1, 2, ..., n$ and $0 < \sum_{i=1}^{n} |\alpha_i|(M_i + (2-\beta)N_i^{\mu}) \le 1$, then the integral operator

$$G_n(z) = \int_0^z \prod_{i=1}^{n} \left(f_i'(t)e^{g_i(t)}\right)^{\alpha_i} dt \tag{11}$$

is in the class $\mathcal{K}(\delta)$, where $\delta = 1 - \sum_{i=1}^{n} |\alpha_i|(M_i + (2-\beta)N_i^{\mu})$.

If we consider $M_i = M$ and $N_i = N$, for $i = 1, 2, ..., n$ in Theorem 1, we get:

Corollary 2. *Let $f_i \in \mathcal{A}$, $g_i \in \mathcal{B}(\mu_i, \beta_i)$, $\mu_i \ge 1$, $0 \le \beta_i < 1$ and $\alpha_i \in \mathbb{C}$, for $i = 1, 2, ..., n$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \le M, M \ge 1, |g_i(z)| \le N, N \ge 1,\ z \in \mathcal{U} \tag{12}$$

for $i = 1, 2, ..., n$ and $0 < \sum_{i=1}^{n} |\alpha_i|(M + (2-\beta_i)N^{\mu_i}) \le 1$, then the integral operator

$$G_n(z) = \int_0^z \prod_{i=1}^{n} \left(f_i'(t)e^{g_i(t)}\right)^{\alpha_i} dt \tag{13}$$

is in the class $\mathcal{K}(\delta)$, where $\delta = 1 - \sum_{i=1}^{n} |\alpha_i|(M + (2-\beta_i)N^{\mu_i})$.

If we consider $\beta_i = \beta$, $\mu_i = \mu$ for $i = 1, 2, ..., n$ in Corollary 2, we have:

Corollary 3. *Let $f_i \in \mathcal{A}$, $g_i \in \mathcal{B}(\mu, \beta)$, $\mu \ge 1$, $0 \le \beta < 1$ and $\alpha_i \in \mathbb{C}$, for $i = 1, 2, ..., n$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \le M, M \ge 1, |g_i(z)| \le N, N \ge 1,\ z \in \mathcal{U} \tag{14}$$

for $i = 1, 2, ..., n$ and $0 < \sum_{i=1}^{n} |\alpha_i|(M + (2-\beta)N^{\mu}) \ge 1$, then the integral operator

$$G_n(z) = \int_0^z \prod_{i=1}^{n} \left(f_i'(t)e^{g_i(t)}\right)^{\alpha_i} dt \tag{15}$$

is in the class $\mathcal{K}(\delta)$, where $\delta = 1 - \sum_{i=1}^{n} |\alpha_i|(M + (2-\beta)N^{\mu})$.

If we consider $\delta = 0$ in Corollary 2, we have:

Corollary 4. *Let $f_i \in \mathcal{A}$, $g_i \in \mathcal{B}(\mu_i, \beta_i)$, $\mu_i \ge 1$, $0 \le \beta_i < 1$ and $\alpha_i \in \mathbb{C}$, for $i = 1, 2, ..., n$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \le M, M \ge 1, |g_i(z)| \le N, N \ge 1,\ z \in \mathcal{U} \tag{16}$$

for $i = 1, 2, ..., n$ and $\sum_{i=1}^{n} |\alpha_i|(M + (2-\beta_i)N^{\mu_i}) = 1$, then the integral operator G_n is convex in $\mathcal{U}$.

If we consider $\beta_i = \beta$ and $\mu_i = \mu$ for $i = 1, 2, ...n$ in Corollary 4, we have:

Corollary 5. *Let* $f_i \in \mathcal{A}$*,* $g_i \in \mathcal{B}(\mu, \beta)$*,* $\mu \geq 1$*,* $0 \leq \beta < 1$ *and* $\alpha_i \in \mathbb{C}$*, for* $i = 1, 2, ..., n$*. If*

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq M, M \geq 1, |g_i(z)| \leq N, N \geq 1, \ z \in \mathcal{U} \tag{17}$$

for $i = 1, 2, ..., n$ *and* $\sum_{i=1}^{n} |\alpha_i|(M + (2-\beta)N^{\mu}) = 1$*, then the integral operator* G_n *is convex in* $\mathcal{U}$*.*

Considering $\mu_i = 0$ for $i = 1, 2, ..n$ in Corollary 2, we have:

Corollary 6. *Let* $f_i \in \mathcal{A}$*,* $g_i \in \mathcal{R}(\beta_i)$*,* $0 \leq \beta_i < 1$ *and* $\alpha_i \in \mathbb{C}$*, for* $i = 1, 2, ..., n$*. If*

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq M, M \geq 1, \ z \in \mathcal{U} \tag{18}$$

for $i = 1, 2, ..., n$ *and* $0 < \sum_{i=1}^{n} |\alpha_i|(M + 2 - \beta_i) \leq 1$*, then the integral operator*

$$G_n(z) = \int_0^z \prod_{i=1}^{n} \left(f_i'(t) e^{g_i(t)} \right)^{\alpha_i} dt \tag{19}$$

is in the class $\mathcal{K}(\delta)$*, where* $\delta = 1 - \sum_{i=1}^{n} |\alpha_i|(M + 2 - \beta_i)$*.*

If we consider $\beta_i = \beta$ for $i = 1, 2, ...n$ in Corollary 6, we have:

Corollary 7. *Let* $f_i \in \mathcal{A}$*,* $g_i \in \mathcal{R}(\beta)$*,* $0 \leq \beta < 1$ *and* $\alpha_i \in \mathbb{C}$*, for* $i = 1, 2, ..., n$*. If*

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq M, M \geq 1, \ z \in \mathcal{U} \tag{20}$$

for $i = 1, 2, ..., n$ *and* $0 < \sum_{i=1}^{n} |\alpha_i|(M + 2 - \beta) \leq 1$*, then the integral operator*

$$G_n(z) = \int_0^z \prod_{i=1}^{n} \left(f_i'(t) e^{g_i(t)} \right)^{\alpha_i} dt \tag{21}$$

is in the class $\mathcal{K}(\delta)$*, where* $\delta = 1 - \sum_{i=1}^{n} |\alpha_i|(M + 2 - \beta)$*.*

Considering $\mu_i = 1$ for $i = 1, 2, ..n$ in Corollary 2, we have:

Corollary 8. *Let* $f_i \in \mathcal{A}$*,* $g_i \in S^*(\beta_i)$*,* $0 \leq \beta_i < 1$ *and* $\alpha_i \in \mathbb{C}$*, for* $i = 1, 2, ..., n$*. If*

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq M, M \geq 1, |g_i(z)| \leq N, N \geq 1, \ z \in \mathcal{U} \tag{22}$$

for $i = 1, 2, ..., n$ *and* $0 < \sum_{i=1}^{n} |\alpha_i|(M + (2 - \beta_i)N) \leq 1$*, then the integral operator ,*G_n *is in the class* $\mathcal{K}(\delta)$*, where* $\delta = 1 - \sum_{i=1}^{n} |\alpha_i|(M + (2 - \beta_i)N)$*.*

If we consider $\beta_i = \beta$ for $i = 1, 2, ..., n$ in Corollary 5 we have:

Corollary 9. *Let* $f_i \in \mathcal{A}$*,* $g_i \in S^*(\beta)$*,* $0 \leq \beta < 1$ *and* $\alpha_i \in \mathbb{C}$*, for* $i = 1, 2, ..., n$*. If*

$$\left| \frac{f_i''(z)}{f_i'(z)} \right| \leq M, M \geq 1, |g_i(z)| \leq N, N \geq 1, \ z \in \mathcal{U} \tag{23}$$

for $i = 1, 2, ..., n$ *and* $0 < \sum_{i=1}^{n} |\alpha_i|(M + (2 - \beta)N) \leq 1$*, then the integral operator ,*G_n *is in the class* $\mathcal{K}(\delta)$*, where* $\delta = 1 - \sum_{i=1}^{n} |\alpha_i|(M + (2 - \beta)N)$*.*

Considering $\mu_i = 2$ for $i = 1, 2, ..n$ in Corollary 2, we have:

Corollary 10. *Let $f_i \in \mathcal{A}$, $g_i \in \mathcal{B}(\beta_i)$, $0 \le \beta_i < 1$ and $\alpha_i \in \mathbb{C}$, for $i = 1, 2, ..., n$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \le M, M \ge 1, |g_i(z)| \le N, N \ge 1, \ z \in \mathcal{U} \tag{24}$$

for $i = 1, 2, ..., n$ and $0 < \sum_{i=1}^{n} |\alpha_i|(M + (2 - \beta_i)N^2) \le 1$, then the integral operator , G_n is in the class $\mathcal{K}(\delta)$, where $\delta = 1 - \sum_{i=1}^{n} |\alpha_i|(M + (2 - \beta_i)N^2)$.

If we consider $\beta_i = \beta$ for $i = 1, 2, ..., n$ in Corollary 10 we have:

Corollary 11. *Let $f_i \in \mathcal{A}$, $g_i \in \mathcal{B}(\beta)$, $0 \le \beta < 1$ and $\alpha_i \in \mathbb{C}$, for $i = 1, 2, ..., n$. If*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \le M, M \ge 1, |g_i(z)| \le N, N \ge 1, \ z \in \mathcal{U} \tag{25}$$

for $i = 1, 2, ..., n$ and $0 < \sum_{i=1}^{n} |\alpha_i|(M + (2 - \beta)N^2) \le 1$, then the integral operator , G_n is in the class $\mathcal{K}(\delta)$, where $\delta = 1 - \sum_{i=1}^{n} |\alpha_i|(M + (2 - \beta)N^2)$.

If we consider $n = 1$, $\beta_i = \delta = 0$ for $i = 1, 2, ..., n$ in Corollary 5, we have:

Corollary 12. *Let $f \in \mathcal{A}$, $g \in S^*$, and $\alpha \in \mathbb{C}$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| \le M, M \ge 1, |g(z)| \le N, N \ge 1, \ z \in \mathcal{U} \tag{26}$$

and $|\alpha| = \frac{1}{M+2N}$, then the integral operator $G_1(z) = \int_0^z \left(f'(t)e^{g(t)}\right)^{\alpha} dt$ is convex in $\mathcal{U}$.

Remark 1. *For $n = 1$ and $\alpha_i = 1$ in relation (5), we obtain the integral operator $I(f, g)(z) = \int_0^z \left(f'(t)e^{g(t)}\right) dt$, defined and studied by N. Ularu and D. Breaz in* [6].

Remark 2. *For $n = 1$ in relation (5), we obtain the integral operator $I_1(f, g)(z) = \int_0^z \left(f'(t)e^{g(t)}\right)^{\alpha} dt$, defined and studied by N. Ularu and D. Breaz in* [6] *and* [5].

References

[1] Breaz, D., Breaz, N., *Two integral operators*, Studia Universitatis Babeș Bolyai, Mathematica **3** (2002), Cluj Napoca, 13–21.

[2] B.A. Frasin and M. Darus, *On certain analytic univalent functions*, Internat. J. Math. and Math. Sci. **25** (5) (2001), 305–310.

[3] B.A. Frasin, J. Jahangiri, *A new and comprehensive class of analytic functions*, Anal. Univ. Oradea Fasc. Math. **XV** (2008), 59–62.

[4] O. Mayer, *The Function Theory of One Variable Complex*, București, 1981.

[5] N. Ularu, D. Breaz, *Univalence criterion and convexity for an integral opertor*, Applied Mathematics Letter **25** (2012), 658–661.

[6] N. Ularu, D. Breaz, *Univalence condition and properties for two integral opertors*, Applied Sciences **15** (2013), 112–117.

University of Pitești
Department of Mathematics
Târgul din Vale Str., No.1, 110040, Pitești, Argeș, România
E-mail address: adriana_oprea@yahoo.com