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ON RELATIVE ORDER ORIENTED RESULTS OF ENTIRE FUNCTIONS OF TWO COMPLEX VARIABLES

SANJIB KUMAR DATTA, TANMAY BISWAS, AND GOLOK KUMAR MONDAL

Abstract. In this paper we intend to find out relative order (relative lower order) of an entire function of two complex variables f with respect to another entire function of two complex variables g when relative order (relative lower order) of f and relative order (relative lower order) of g with respect to another entire function h of two complex variables are given.

1. Introduction,Definitions and Notations

Let f be an entire function of two complex variables holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : \ |z_i| \leq r_i, \ i = 1, 2 \text{ for all } \ r_1 \geq 0, r_2 \geq 0\}$$

and $M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_i| \leq r_i, \ i = 1, 2\}$. Then in view of maximum principal and Hartogs's theorem [2, p.21, p.51], $M_f(r_1, r_2)$ is increasing function of r_1, r_2. We do not explain the standard definitions and notations of the theory of entire functions as those are available in [2].

The following definition is well known:

Definition 1 ([2], p.339 (see also [1])). *The order* ${}_{v_2}\rho_f$ *and the lower order* ${}_{v_2}\lambda_f$ *of an entire function* f *of two complex variables are defined as*

$${}_{v_2}\rho_f = \limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_f(r_1, r_2)}{\log(r_1 r_2)} \ \textit{and} \ {}_{v_2}\lambda_f = \liminf_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_f(r_1, r_2)}{\log(r_1 r_2)} .$$

An entire function of two complex variables for which order and lower order are the same is said to be of regular growth. Functions which are not of regular growth are said to be of irregular growth.

If we consider the above definition for single variable, then the definition coincides with the classical definition of order (see [18]) which is generally used in computational purpose. Generalizing this notion, Bernal (see [3, 4]) introduced the definition of relative order between two entire functions of single variable. During the past decades, several authors (see [13, 14, 15, 16]) made close investigations on the properties of relative order of entire functions of single variable. In fact, some works relating to the growth estimates of composite entire functions of single variable on the basis of relative order of entire functions have been explored in (see [6, 7, 8, 10, 11, 9]). In the case of relative order, it was then natural for Banerjee and Datta [5] to define the relative order of entire functions of two complex variables as follows.

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Definition 2 ([5])**.** *The relative order between two entire functions of two complex variables denoted by ${}_{v_2}\rho_g(f)$ is define as:*

$$\begin{aligned} {}_{v_2}\rho_g(f) &= \inf\left\{\mu > 0 : M_f(r_1, r_2) < M_g\left(r_1^{\mu}, r_2^{\mu}\right); r_1 \geq R(\mu), r_2 \geq R(\mu)\right\} \\ &= \limsup_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)} \end{aligned}$$

where g is an entire function holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : |z_i| \leq r_i, \ i = 1, 2 \textit{ for all } r_1 \geq 0, r_2 \geq 0\}$$

and the definition coincides with Definition 1 see [5] *if $g(z) = \exp(z_1 z_2)$.*

Likewise, one can define the relative lower order of f with respect to g denoted by ${}_{v_2}\lambda_g(f)$ as follows:

$${}_{v_2}\lambda_g(f) = \liminf_{r \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)}.$$

An entire function of two complex variables for which relative order and relative lower order with respect to another entire function of two complex variables are the same is said to be of regular relative growth with respect to that entire function. Functions which are not of regular relative growth with respect to entire functions are said to be of irregular relative growth with respect to respective entire functions.

Now a question may arise about relative order (relative lower order) of f with respect to another entire function g when relative order (relative lower order) of f and g with respect to another entire function h are respectively given. In this paper we intend to provide this answer. In fact, in this paper we wish to extend some results of [12].

2. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f, g and h be any three entire functions of two complex variables such that relative order (relative lower order) of f with respect to h and relative order (relative lower order) of f with respect to h are ${}_{v_2}\rho_h(f)$ $({}_{v_2}\lambda_h(f))$ and ${}_{v_2}\rho_h(g)$ $({}_{v_2}\lambda_h(g))$ respectively. Then*

$$\begin{aligned} \frac{{}_{v_2}\lambda_h(f)}{{}_{v_2}\rho_h(g)} &\leq {}_{v_2}\lambda_g(f) \leq \min\left\{\frac{{}_{v_2}\lambda_h(f)}{{}_{v_2}\lambda_h(g)}, \frac{{}_{v_2}\rho_h(f)}{{}_{v_2}\rho_h(g)}\right\} \\ &\leq \max\left\{\frac{{}_{v_2}\lambda_h(f)}{{}_{v}\lambda_h(g)}, \frac{{}_{v_2}\rho_h(f)}{{}_{v_2}\rho_h(g)}\right\} \leq_{v_2} \rho_g(f) \leq \frac{{}_{v_2}\rho_h(f)}{{}_{v_2}\lambda_h(g)} . \end{aligned}$$

Proof. From the definitions of ${}_{v_2}\rho_h(f)$ and ${}_{v_2}\lambda_h(f)$ we have for all sufficiently large values of r_1, r_2 that

$$\begin{aligned} & M_h^{-1} M_f(r_1, r_2) \leq \exp\left\{\left({}_{v_2}\rho_h(f) + \varepsilon\right)\log(r_1 r_2)\right\} \\ & \textit{i.e.,}\ M_f(r_1, r_2) \leq M_h\left[\exp\left\{\left({}_{v_2}\rho_h(f) + \varepsilon\right)\log(r_1 r_2)\right\}\right] , \end{aligned} \tag{1}$$

$$\begin{aligned} & M_h^{-1} M_f(r_1, r_2) \geq \exp\left\{\left({}_{v_2}\lambda_h(f) - \varepsilon\right)\log(r_1 r_2)\right\} \\ & \textit{i.e.,}\ M_f(r_1, r_2) \geq M_h\left[\exp\left\{\left({}_{v_2}\lambda_h(f) - \varepsilon\right)\log(r_1 r_2)\right\}\right] . \end{aligned} \tag{2}$$

and also for a sequence of values of r_1, r_2 tending to infinity we get that

$$M_h^{-1} M_f(r_1, r_2) \geq \exp\{({}_{v_2}\rho_h(f) - \varepsilon)\log(r_1 r_2)\}$$
$$i.e., \ M_f(r_1, r_2) \geq M_h[\exp\{({}_{v_2}\rho_h(f) - \varepsilon)\log(r_1 r_2)\}] \ . \tag{3}$$
$$M_h^{-1} M_f(r_1, r_2) \leq \exp\{({}_{v_2}\lambda_h(f) + \varepsilon)\log(r_1 r_2)\}$$
$$i.e., \ M_f(r_1, r_2) \leq M_h[\exp\{({}_{v_2}\lambda_h(f) + \varepsilon)\log(r_1 r_2)\}] \ . \tag{4}$$

Similarly from the definitions of ${}_{v_2}\rho_h(g)$ and ${}_{v_2}\lambda_h(g)$ it follows for all sufficiently large values of r_1, r_2 that

$$M_h^{-1} M_g(r_1, r_2) \leq \exp\{({}_{v_2}\rho_h(g) + \varepsilon)\log(r_1 r_2)\}$$
$$i.e., \ M_g(r_1, r_2) \leq M_h[\exp\{({}_{v_2}\rho_h(g) + \varepsilon)\log(r_1 r_2)\}]$$
$$i.e., \ M_h(r_1, r_2) \geq M_g\left[\exp\left[\frac{\log(r_1 r_2)}{({}_{v_2}\rho_h(g) + \varepsilon)}\right]\right] \ . \tag{5}$$

$$M_h^{-1} M_g(r_1, r_2) \geq \exp\{({}_{v_2}\lambda_h(g) - \varepsilon)\log(r_1 r_2)\}$$
$$i.e., \ M_g(r_1, r_2) \geq M_h[\exp\{({}_{v_2}\lambda_h(g) - \varepsilon)\log(r_1 r_2)\}]$$
$$i.e., \ M_h(r_1, r_2) \leq M_g\left[\exp\left[\frac{\log(r_1 r_2)}{({}_{v_2}\lambda_h(g) - \varepsilon)}\right]\right] \tag{6}$$

and for a sequence of values of r_1, r_2 tending to infinity we obtain that

$$M_h^{-1} M_g(r_1, r_2) \geq \exp\{({}_{v_2}\rho_h(g) - \varepsilon)\log(r_1 r_2)\}$$
$$i.e., \ M_g(r_1, r_2) \geq M_h[\exp\{({}_{v_2}\rho_h(g) - \varepsilon)\log(r_1 r_2)\}]$$
$$i.e., \ M_h(r_1, r_2) \leq M_g\left[\exp\left[\frac{\log(r_1 r_2)}{({}_{v_2}\rho_h(g) - \varepsilon)}\right]\right] \ . \tag{7}$$

$$M_h^{-1} M_g(r_1, r_2) \leq \exp\{({}_{v_2}\lambda_h(g) + \varepsilon)\log(r_1 r_2)\}$$
$$i.e., \ M_g(r_1, r_2) \leq M_h[\exp\{({}_{v_2}\lambda_h(g) + \varepsilon)\log(r_1 r_2)\}]$$
$$i.e., \ M_h(r_1, r_2) \geq M_g\left[\exp\left[\frac{\log(r_1 r_2)}{({}_{v_2}\lambda_h(g) + \varepsilon)}\right]\right] \ . \tag{8}$$

Now from (3) and in view of (5), we get for a sequence of values of r_1, r_2 tending to infinity that

$$\log M_g^{-1} M_f(r_1, r_2) \geq \log M_g^{-1} M_h[\exp\{({}_{v_2}\rho_h(f) - \varepsilon)\log(r_1 r_2)\}]$$

$$i.e., \ \log M_g^{-1} M_f(r_1, r_2)$$
$$\geq \log M_g^{-1} M_g\left[\exp\left[\frac{\log\exp\{({}_{v_2}\rho_h(f) - \varepsilon)\log(r_1 r_2)\}}{({}_{v_2}\rho_h(g) + \varepsilon)}\right]\right]$$

$$i.e., \ \log M_g^{-1} M_f(r_1, r_2) \geq \frac{({}_{v_2}\rho_h(f) - \varepsilon)}{({}_{v_2}\rho_h(g) + \varepsilon)}\log(r_1 r_2)$$
$$i.e., \ \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)} \geq \frac{({}_{v_2}\rho_h(f) - \varepsilon)}{({}_{v_2}\rho_h(g) + \varepsilon)} \ .$$

As $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r_1,r_2\to\infty}\frac{\log M_g^{-1}M_f\,(r_1,r_2)}{\log\,(r_1r_2)}\geq\frac{{}_{v_2}\rho_h\,(f)}{{}_{v_2}\rho_h\,(g)}$$

$$i.e.,\ {}_{v_2}\rho_g\,(f)\geq\frac{{}_{v_2}\rho_h\,(f)}{{}_{v_2}\rho_h\,(g)}\ . \tag{9}$$

Analogously, from (2) and in view of (8) it follows for a sequence of values of r_1, r_2 tending to infinity that

$$\log M_g^{-1}M_f\,(r_1,r_2)\geq\log M_g^{-1}M_h\left[\exp\left\{\left({}_{v_2}\lambda_h\,(f)-\varepsilon\right)\log\,(r_1r_2)\right\}\right]$$

$$i.e.,\ \log M_g^{-1}M_f\,(r_1,r_2)$$
$$\geq\log M_g^{-1}M_g\left[\exp\left[\frac{\log\exp\left\{\left({}_{v_2}\lambda_h\,(f)-\varepsilon\right)\log\,(r_1r_2)\right\}}{\left({}_{v_2}\lambda_h\,(g)+\varepsilon\right)}\right]\right]$$

$$i.e.,\ \log M_g^{-1}M_f\,(r_1,r_2)\geq\frac{\left({}_{v_2}\lambda_h\,(f)-\varepsilon\right)}{\left({}_{v_2}\lambda_h\,(g)+\varepsilon\right)}\log\,(r_1r_2)$$

$$i.e.,\ \frac{\log M_g^{-1}M_f\,(r_1,r_2)}{\log\,(r_1r_2)}\geq\frac{\left({}_{v_2}\lambda_h\,(f)-\varepsilon\right)}{\left({}_{v_2}\lambda_h\,(g)+\varepsilon\right)}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r_1,r_2\to\infty}\frac{\log M_g^{-1}M_f\,(r_1,r_2)}{\log\,(r_1r_2)}\geq\frac{{}_{v_2}\lambda_h\,(f)}{{}_{v_2}\lambda_h\,(g)}$$

$$i.e.,\ {}_{v_2}\rho_g\,(f)\geq\frac{{}_{v_2}\lambda_h\,(f)}{{}_{v_2}\lambda_h\,(g)}\ . \tag{10}$$

Again in view of (6), we have from (1) for all sufficiently large values of r_1, r_2 that

$$\log M_g^{-1}M_f\,(r_1,r_2)\leq\log M_g^{-1}M_h\left[\exp\left\{\left({}_{v_2}\rho_h\,(f)+\varepsilon\right)\log\,(r_1r_2)\right\}\right]$$

$$i.e.,\ \log M_g^{-1}M_f\,(r_1,r_2)$$
$$\leq\log M_g^{-1}M_g\left[\exp\left[\frac{\log\exp\left\{\left({}_{v_2}\rho_h\,(f)+\varepsilon\right)\log\,(r_1r_2)\right\}}{\left({}_{v_2}\lambda_h\,(g)-\varepsilon\right)}\right]\right]$$

$$i.e.,\ \log M_g^{-1}M_f\,(r_1,r_2)\leq\frac{\left({}_{v_2}\rho_h\,(f)+\varepsilon\right)}{\left({}_{v_2}\lambda_h\,(g)-\varepsilon\right)}\log\,(r_1r_2)$$

$$i.e.,\ \frac{\log M_g^{-1}M_f\,(r_1,r_2)}{\log\,(r_1r_2)}\leq\frac{\left({}_{v_2}\rho_h\,(f)+\varepsilon\right)}{\left({}_{v_2}\lambda_h\,(g)-\varepsilon\right)}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r_1,r_2\to\infty}\frac{\log M_g^{-1}M_f\,(r_1,r_2)}{\log\,(r_1r_2)}\leq\frac{{}_{v_2}\rho_h\,(f)}{{}_{v_2}\lambda_h\,(g)}$$

$$i.e.,\ {}_{v_2}\rho_g\,(f)\leq\frac{{}_{v_2}\rho_h\,(f)}{{}_{v_2}\lambda_h\,(g)}\ . \tag{11}$$

Again from (2) and in view of (5), we get for all sufficiently large values of r_1, r_2 that

$$\log M_g^{-1}M_f\,(r_1,r_2)\geq\log M_g^{-1}M_h\left[\exp\left\{\left({}_{v_2}\lambda_h\,(f)-\varepsilon\right)\log\,(r_1r_2)\right\}\right]$$

$$i.e., \ \log M_g^{-1} M_f(r_1, r_2)$$
$$\geq \log M_g^{-1} M_g \left[\exp\left[\frac{\log \exp\{({}_{v_2}\lambda_h(f) - \varepsilon)\log(r_1 r_2)\}}{({}_{v_2}\rho_h(g) + \varepsilon)}\right]\right]$$

$$i.e., \ \log M_g^{-1} M_f(r_1, r_2) \geq \frac{({}_{v_2}\lambda_h(f) - \varepsilon)}{({}_{v_2}\rho_h(g) + \varepsilon)} \log(r_1 r_2)$$
$$i.e., \ \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)} \geq \frac{({}_{v_2}\lambda_h(f) - \varepsilon)}{({}_{v_2}\rho_h(g) + \varepsilon)} .$$

As $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\liminf_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)} \geq \frac{{}_{v_2}\lambda_h(f)}{{}_{v_2}\rho_h(g)}$$
$$i.e., \ {}_{v_2}\lambda_g(f) \geq \frac{{}_{v_2}\lambda_h(f)}{{}_{v_2}\rho_h(g)} . \tag{12}$$

Also in view of (7), we get from (1) for a sequence of values of r_1, r_2 tending to infinity that

$$\log M_g^{-1} M_f(r_1, r_2) \leq \log M_g^{-1} M_h\left[\exp\{({}_{v_2}\rho_h(f) + \varepsilon)\log(r_1 r_2)\}\right]$$

$$i.e., \ \log M_g^{-1} M_f(r_1, r_2)$$
$$\leq \log M_g^{-1} M_g \left[\exp\left[\frac{\log \exp\{({}_{v_2}\rho_h(f) + \varepsilon)\log(r_1 r_2)\}}{({}_{v_2}\rho_h(g) - \varepsilon)}\right]\right]$$

$$i.e., \ \log M_g^{-1} M_f(r_1, r_2) \leq \frac{({}_{v_2}\rho_h(f) + \varepsilon)}{({}_{v_2}\rho_h(g) - \varepsilon)} \log(r_1 r_2)$$
$$i.e., \ \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)} \leq \frac{({}_{v_2}\rho_h(f) + \varepsilon)}{({}_{v_2}\rho_h(g) - \varepsilon)} .$$

Since $\varepsilon (> 0)$ is arbitrary, we get from above that

$$\liminf_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)} \leq \frac{{}_{v_2}\rho_h(f)}{{}_{v_2}\rho_h(g)}$$
$$i.e., \ {}_{v_2}\lambda_g(f) \leq \frac{{}_{v_2}\rho_h(f)}{{}_{v_2}\rho_h(g)} . \tag{13}$$

Similarly from (4) and in view of (6), it follows for a sequence of values of r_1, r_2 tending to infinity that

$$\log M_g^{-1} M_f(r_1, r_2) \leq \log M_g^{-1} M_h\left[\exp\{({}_{v_2}\lambda_h(f) + \varepsilon)\log(r_1 r_2)\}\right]$$

$$i.e., \ \log M_g^{-1} M_f(r_1, r_2)$$
$$\leq \log M_g^{-1} M_g \left[\exp\left[\frac{\log \exp\{({}_{v_2}\lambda_h(f) + \varepsilon)\log(r_1 r_2)\}}{({}_{v_2}\lambda_h(g) - \varepsilon)}\right]\right]$$

$$i.e., \ \log M_g^{-1} M_f(r_1, r_2) \leq \frac{({}_{v_2}\lambda_h(f) + \varepsilon)}{({}_{v_2}\lambda_h(g) - \varepsilon)} \log(r_1 r_2)$$
$$i.e., \ \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)} \leq \frac{({}_{v_2}\lambda_h(f) + \varepsilon)}{({}_{v_2}\lambda_h(g) - \varepsilon)} .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\liminf_{r_1,r_2\to\infty}\frac{\log M_g^{-1}M_f\,(r_1,r_2)}{\log\,(r_1r_2)}\le\frac{{}_{v_2}\lambda_h\,(f)}{{}_{v_2}\lambda_h\,(g)}$$

$$i.e.,\ {}_{v_2}\lambda_g\,(f)\le\frac{{}_{v_2}\lambda_h\,(f)}{{}_{v_2}\lambda_h\,(g)}\ . \tag{14}$$

Thus the theorem follows from (9), (10), (11), (12), (13) and (14). □

In view of Theorem 1, one can easily verify the following corollaries:

Corollary 1. *Let f be an entire function of two complex variables with regular relative growth with respect to an entire function h of two complex variables and g be entire another entire function of two complex variables. Then*

$${}_{v_2}\lambda_g\,(f)=\frac{{}_{v_2}\rho_h\,(f)}{{}_{v_2}\rho_h\,(g)}\quad and\quad {}_{v_2}\rho_g\,(f)=\frac{{}_{v_2}\rho_h\,(f)}{{}_{v_2}\lambda_h\,(g)}\ .$$

In addition, if ${}_{v_2}\rho_h\,(f)={}_{v_2}\rho_h\,(g)$, then

$${}_{v_2}\lambda_g\,(f)={}_{v_2}\rho_f\,(g)=1\ .$$

Corollary 2. *Let f, g, h be an three entire functions of two complex variables such that g is of regular relative growth with respect to an entire function h. Then*

$${}_{v_2}\lambda_g\,(f)=\frac{{}_{v_2}\lambda_h\,(f)}{{}_{v_2}\rho_h\,(g)}\quad and\quad {}_{v_2}\rho_g\,(f)=\frac{{}_{v_2}\rho_h\,(f)}{{}_{v_2}\rho_h\,(g)}\ .$$

In addition, if ${}_{v_2}\rho_h\,(f)={}_{v_2}\rho_h\,(g)$ then

$${}_{v_2}\rho_g^{(p,q)}\,(f)={}_{v_2}\lambda_f^{(q,p)}\,(g)=1\ .$$

Corollary 3. *Let f and g be any two entire functions of two complex variables with regular relative growth with respect to another entire function h of two complex variables respectively. Then*

$${}_{v_2}\lambda_g\,(f)={}_{v_2}\rho_g\,(f)=\frac{{}_{v_2}\rho_h\,(f)}{{}_{v_2}\rho_h\,(g)}\ .$$

Corollary 4. *Let f and g be any two entire functions of two complex variables with regular relative growth and regular relative growth with respect to another entire function h of two complex variables respectively. Also suppose that ${}_{v_2}\rho_h\,(f)={}_{v_2}\rho_h\,(g)$. Then*

$${}_{v_2}\lambda_g\,(f)={}_{v_2}\rho_g\,(f)={}_{v_2}\lambda_f\,(g)={}_{v_2}\rho_f\,(g)=1\ .$$

Corollary 5. *Let f, g and h be any three entire functions of two complex variables such that either f is not of regular relative growth or g is not of regular relative growth with respect to h. Then*

$${}_{v_2}\rho_g\,(f)\,.{}_{v_2}\rho_f\,(g)\ge 1\ .$$

when f and g are both of regular relative growth with respect to h respectively, then

$${}_{v_2}\rho_g\,(f)\,.{}_{v_2}\rho_f\,(g)=1\ .$$

Corollary 6. *Let f, g and h be any three entire functions of two complex variables such that either f is not of regular relative growth or g is not of regular relative growth with respect to h. Then*

$${}_{v_2}\lambda_g\,(f)\,.{}_{v_2}\lambda_f\,(g)\le 1\ .$$

when f and g are both of regular relative growth with respect to h respectively, then

$${}_{v_2}\lambda_g\,(f)\,.{}_{v_2}\lambda_f\,(g)=1\ .$$

Corollary 7. *Let f and g be any two entire functions of two complex variables. Then*

$$\begin{aligned} (i)\ {}_{v_2}\lambda_g(f) &= \infty \text{ when } {}_{v_2}\rho_h(g)=0\ , \\ (ii)\ {}_{v_2}\rho_g(f) &= \infty \text{ when } {}_{v_2}\lambda_h(g)=0\ , \\ (iii)\ {}_{v_2}\lambda_g(f) &= 0 \text{ when } {}_{v_2}\rho_h(g)=\infty \end{aligned}$$

and

$$(iv)\ {}_{v_2}\rho_g(f)=0 \text{ when } {}_{v_2}\lambda_h(g)=\infty\ .$$

Corollary 8. *Let f and g be any two entire functions of two complex variables. Then*

$$\begin{aligned} (i)\ {}_{v_2}\rho_g(f) &= 0 \text{ when } {}_{v_2}\rho_h(f)=0\ , \\ (ii)\ {}_{v_2}\lambda_g(f) &= 0 \text{ when } {}_{v_2}\lambda_h(f)=0\ , \\ (iii)\ {}_{v_2}\rho_g(f) &= \infty \text{ when } {}_{v_2}\rho_h(f)=\infty \end{aligned}$$

and

$$(iv)\ {}_{v_2}\lambda_g(f)=\infty \text{ when } {}_{v_2}\lambda_h(f)=\infty\ .$$

Example 1. *Let us consider any three given natural numbers m, n and p. Also suppose that a and b any two positive real numbers so that*

$$f(z_1,z_2)=\exp\left(z_1^m z_2^m\right),\ g(z_1,z_2)=\exp\left(az_1^n z_2^n\right) \text{ and } h(z_1,z_2)=\exp\left(bz_1^p z_2^p\right).$$

Then

$${}_{v_2}\rho_h(f)= {}_{v_2}\lambda_h(f)=\frac{m}{p} \text{ and } {}_{v_2}\rho_h(g)= {}_{v_2}\lambda_h(g)=\frac{n}{p}.$$

Now

$${}_{v_2}\rho_g(f)= {}_{v_2}\lambda_g(f)=\frac{{}_{v_2}\rho_h(f)}{{}_{v_2}\rho_h(g)}=\frac{\frac{m}{p}}{\frac{n}{p}}=\frac{m}{n}.$$

Example 2. *Let n be any natural number and a any positive real number and consider*

$$f(z_1,z_2)=h(z_1,z_2)=\exp\left(z_1^n z_2^n\right) \text{ and } g(z_1,z_2)=\exp\left(az_1^n z_2^n\right).$$

In this case f and g are two entire functions with regular relative growth with respect to another entire function h, thus

$${}_{v_2}\rho_g(f)= {}_{v_2}\lambda_g(f)=\frac{{}_{v_2}\rho_h(f)}{{}_{v_2}\rho_h(g)}=\frac{n}{n}=1.$$

Clearly

$${}_{v_2}\rho_f(g)= {}_{v_2}\lambda_f(g)=1.$$

Example 3. *Let m, n and p be any three natural numbers and consider*

$$f(z_1,z_2)=\exp\left(z_1^m z_2^m\right),\ g(z_1,z_2)=z_1^n z_2^n \text{ and } h(z_1,z_2)=\exp\left(z_1^p z_2^p\right).$$

Then

$${}_{v_2}\rho_h(f)= {}_{v_2}\lambda_h(f)=\frac{m}{p} \text{ and } {}_{v_2}\rho_h(g)= {}_{v_2}\lambda_h(g)=0.$$

Now

$${}_{v_2}\rho_g(f)= {}_{v_2}\lambda_g(f)=\infty$$

and

$${}_{v_2}\rho_f(g)= {}_{v_2}\lambda_f(g)=0.$$

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Department of Mathematics, University of Kalyani
P.O.- Kalyani, Dist-Nadia, PIN- 741235
West Bengal, India

Rajbari, Rabindrapalli, R. N. Tagore Road
P.O.- Krishnagar, Dist-Nadia, PIN- 741101
West Bengal, India.

Dhulauri Rabindra Vidyaniketan (H.S.)
Vill +P.O.- Dhulauri , P.S.- Domkal
Dist-Murshidabad , PIN- 742308, West Bengal, India
E-mail address: golok.mondal13@rediffmail.com