TJMM
7 (2015), No. 1, 31-43

SOME PERTURBED OSTROWSKI TYPE INEQUALITIES FOR ABSOLUTELY CONTINUOUS FUNCTIONS (III)

S.S. DRAGOMIR

Abstract. In this paper, some new perturbed Ostrowski type inequalities for absolutely continuous functions are established.

1. Introduction

In order to obtain various perturbed Ostrowski type inequalities, in the earlier paper [26] we established the following equality:

Lemma 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$ and $x\in[a,b]$. Then for any $\lambda_1(x)$ and $\lambda_2(x)$ complex numbers, we have*

$$f(x)+\frac{1}{2(b-a)}\left[(b-x)^2\lambda_2(x)-(x-a)^2\lambda_1(x)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{1}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda_1(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda_2(x)\right]dt,$$

where the integrals in the right hand side are taken in the Lebesgue sense.

The following equality in terms of one parameter holds:

Corollary 1. *With the assumption in Lemma 1, we have for any $\lambda(x)\in\mathbb{C}$ that*

$$f(x)+\left(\frac{a+b}{2}-x\right)\lambda(x)-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2}$$
$$=\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-\lambda(x)\right]dt+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-\lambda(x)\right]dt.$$

Remark 1. *If we take $\lambda(x)=0$ in (2), then we get Montgomery's identity for absolutely continuous functions, namely*

$$f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^x (t-a)f'(t)\,dt+\frac{1}{b-a}\int_x^b (t-b)f'(t)\,dt, \tag{3}$$

for $x\in[a,b]$.

We have the following midpoint representation:

Corollary 2. *With the assumption in Lemma 1, we have for any $\lambda_1,\lambda_2\in\mathbb{C}$ that*

$$f\left(\frac{a+b}{2}\right)+\frac{1}{8}(b-a)(\lambda_2-\lambda_1)-\frac{1}{b-a}\int_a^b f(t)\,dt \tag{4}$$

2010 *Mathematics Subject Classification.* 25D15, 25D10.
Key words and phrases. Ostrowski's inequality, integral inequalities, perturbed Ostrowski type inequalities.

$$= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t)-\lambda_1\right]dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\lambda_2\right]dt.$$

In particular, if $\lambda_1 = \lambda_2 = \lambda$, *then we have the equality*

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \tag{5}$$
$$= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t)-\lambda\right]dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-\lambda\right]d.$$

The identity (1) has many particular cases of interest.

If $x \in (a,b)$ is a point of differentiability for the absolutely continuous function $f : [a,b] \to \mathbb{C}$, then we have the equality:

$$f(x) + \left(\frac{a+b}{2} - x\right)f'(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \tag{6}$$
$$= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'(x)\right]dt + \frac{1}{b-a}\int_x^b (t-b)\left[f'(t)-f'(x)\right]dt.$$

In particular we have

$$f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt \tag{7}$$
$$+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t)-f'\left(\frac{a+b}{2}\right)\right]dt$$

provided $f'\left(\frac{a+b}{2}\right)$ exists and is finite.

For $x \in (a,b)$, if we take in (1)

$$\lambda_1(x) = \frac{f(x)-f(a)}{x-a} \text{ and } \lambda_2(x) = \frac{f(b)-f(x)}{b-x},$$

then we get, after some elementary calculations,

$$\frac{1}{2}\left[f(x) + \frac{(b-x)f(b)+(x-a)f(a)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \tag{8}$$
$$= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - \frac{f(x)-f(a)}{x-a}\right]dt$$
$$+\frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - \frac{f(b)-f(x)}{b-x}\right]dt.$$

In particular, we have

$$\frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(b)+f(a)}{2}\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \tag{9}$$
$$= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - \frac{f\left(\frac{a+b}{2}\right)-f(a)}{\frac{b-a}{2}}\right]dt$$
$$+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - \frac{f(b)-f\left(\frac{a+b}{2}\right)}{\frac{b-a}{2}}\right]dt.$$

If we assume that the lateral derivatives $f'_+(a)$ and $f'_-(b)$ exist and are finite, then we have from (1) for $\lambda_1(x) = f'_+(a)$ and $\lambda_2(x) = f'_-(b)$

$$f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'_-(b) - (x-a)^2 f'_+(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \tag{10}$$
$$= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - f'_+(a)\right]dt + \frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - f'_-(b)\right]dt,$$

for all $x \in [a,b]$.

In particular, we have

$$f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)\left[f'_-(b) - f'_+(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \tag{11}$$
$$= \frac{1}{b-a}\int_a^{\frac{a+b}{2}} (t-a)\left[f'(t) - f'_+(a)\right]dt + \frac{1}{b-a}\int_{\frac{a+b}{2}}^b (t-b)\left[f'(t) - f'_-(b)\right]dt.$$

If we take in (1) $\lambda_2(x) = \lambda_2(x) = f'\left(\frac{a+b}{2}\right)$, provided this derivative exists and is finite, then we get

$$f(x) + \left(\frac{a+b}{2} - x\right) f'\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)\,dt \tag{12}$$
$$= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - f'\left(\frac{a+b}{2}\right)\right]dt + \frac{1}{b-a}\int_x^b (t-b)\left[f'(t) - f'\left(\frac{a+b}{2}\right)\right]dt,$$

for all $x \in [a,b]$.

In [26] we obtained the following perturbed Ostrowski type inequality:

Theorem 1. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \right. \tag{13}$$
$$\left. + \frac{1}{4(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \right|$$
$$\le \frac{1}{4}(b-a)\left[\left(\frac{x-a}{b-a}\right)^2 \bigvee_a^x (f') + \left(\frac{b-x}{b-a}\right)^2 \bigvee_x^b (f')\right]$$
$$\le \frac{1}{4}(b-a)\begin{cases} \left[\frac{1}{4} + \left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{2}\bigvee_a^b (f') + \frac{1}{2}\left|\bigvee_a^x (f') - \bigvee_x^b (f')\right|\right], \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{2p} + \left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[\left[\bigvee_a^x (f')\right]^q + \left[\bigvee_x^b (f')\right]^q\right]^{1/q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left[\frac{1}{2} + \left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b (f') \end{cases}$$

for any $x \in [a,b]$.

Another perturbed Ostrowski type inequality obtained in [27] is as follows:

Theorem 2. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\left| f(x) + \left(\frac{a+b}{2} - x \right) f'(x) - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{14}$$
$$\le \frac{1}{b-a} \left[\int_a^x (t-a) \bigvee_t^x (f')\,dt + \int_x^b (b-t) \bigvee_x^t (f')\,dt \right]$$
$$\le \frac{1}{2}(b-a) \left[\left(\frac{x-a}{b-a} \right)^2 \bigvee_a^x (f')\,dt + \left(\frac{b-x}{b-a} \right)^2 \bigvee_x^b (f') \right]$$
$$\le \frac{1}{2}(b-a) \begin{cases} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \left[\frac{1}{2} \bigvee_a^b (f') + \frac{1}{2} \left| \bigvee_a^x (f') - \bigvee_x^b (f') \right| \right], \\[2ex] \left[\left(\frac{x-a}{b-a} \right)^{2p} + \left(\frac{b-x}{b-a} \right)^{2p} \right]^{1/p} \left[\left[\bigvee_a^x (f') \right]^q + \left[\bigvee_x^b (f') \right]^q \right]^{1/q} \\ p > 1, \frac{1}{p} + \frac{1}{q} = 1, \\[2ex] \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b (f'), \end{cases}$$

for any $x \in [a,b]$.

For other Ostrowski type inequalities see [1]-[19] and [23]-[46].

Motivated by the above results, we establish in this paper other perturbed Ostrowski type inequalities for complex valued differentiable functions.

2. Inequalities for Derivatives of Bounded Variation

Assume that the function $f : I \to \mathbb{C}$ is differentiable on the interior of I, denoted $\mathring{I}$, and $[a,b] \subset \mathring{I}$. Then, from (10) we have the equality

$$f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \tag{15}$$
$$= \frac{1}{b-a} \int_a^x (t-a) [f'(t) - f'(a)]\,dt + \frac{1}{b-a} \int_x^b (b-t) [f'(b) - f'(t)]\,dt,$$

for any $x \in [a,b]$.

In particular, for $x = \frac{a+b}{2}$, we have

$$f\left(\frac{a+b}{2} \right) + \frac{1}{8}(b-a) [f'(b) - f'(a)] - \frac{1}{b-a} \int_a^b f(t)\,dt \tag{16}$$
$$= \frac{1}{b-a} \int_a^{\frac{a+b}{2}} (t-a) [f'(t) - f'(a)]\,dt + \frac{1}{b-a} \int_{\frac{a+b}{2}}^b (b-t) [f'(b) - f'(t)]\,dt.$$

Theorem 3. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then for any $x \in [a,b]$*

$$\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{17}$$
$$\leq \frac{1}{b-a} \left[\int_a^x (t-a) \bigvee_a^t (f')\,dt + \int_x^b (b-t) \bigvee_t^b (f')\,dt \right]$$
$$\leq \frac{1}{b-a} \begin{cases} \frac{1}{2}(x-a)^2 \bigvee_a^x (f'), \\ \frac{1}{(q+1)^{1/q}} (x-a)^{1+1/q} \left(\int_a^x \left(\bigvee_a^t (f') \right)^p dt \right)^{1/p}, \\ (x-a) \int_a^x \left(\bigvee_a^t (f') \right) dt \end{cases}$$
$$+ \frac{1}{b-a} \begin{cases} \frac{1}{2}(b-x)^2 \bigvee_x^b (f'), \\ \frac{1}{(q+1)^{1/q}} (b-x)^{1+1/q} \left(\int_x^b \left(\bigvee_t^b (f') \right)^p dt \right)^{1/p}, \\ (b-x) \int_x^b \left(\bigvee_t^b (f') \right) dt. \end{cases}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Taking the modulus in (15) we have

$$\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{18}$$
$$\leq \frac{1}{b-a} \int_a^x (t-a) \left| f'(t) - f'(a) \right| dt + \frac{1}{b-a} \int_x^b (b-t) \left| f'(b) - f'(t) \right| dt,$$

for any $x \in [a,b]$.

Since the derivative $f' : \mathring{I} \to \mathbb{C}$ is of bounded variation on $[a,b]$, then

$$\left| f'(t) - f'(a) \right| \leq \bigvee_a^t (f') \text{ for any } t \in [a,x]$$

and

$$\left| f'(b) - f'(t) \right| \leq \bigvee_t^b (f') \text{ for any } t \in [x,b].$$

Therefore

$$\int_a^x (t-a) \left| f'(t) - f'(a) \right| dt \leq \int_a^x (t-a) \bigvee_a^t (f')\,dt$$

and

$$\int_x^b (b-t)\,|f'(b)-f'(t)|\,dt \le \int_x^b (b-t) \bigvee_t^b (f')\,dt$$

for any $x \in [a,b]$.

Adding these two inequalities and dividing by $b-a$ we get the first inequality in (17).

Using Hölder's integral inequality we have

$$\int_a^x (t-a) \bigvee_a^t (f')\,dt \le \begin{cases} \bigvee_a^x (f') \int_a^x (t-a)\,dt, \\ \left(\int_a^x (t-a)^q\,dt\right)^{1/q} \left(\int_a^x \left(\bigvee_a^t (f')\right)^p dt\right)^{1/p}, \\ (x-a) \int_a^x \left(\bigvee_a^t (f')\right) dt, \end{cases}$$

$$= \begin{cases} \frac{1}{2}(x-a)^2 \bigvee_a^x (f'), \\ \frac{1}{(q+1)^{1/q}} (x-a)^{1+1/q} \left(\int_a^x \left(\bigvee_a^t (f')\right)^p dt\right)^{1/p}, \\ (x-a) \int_a^x \left(\bigvee_a^t (f')\right) dt \end{cases}$$

and

$$\int_x^b (b-t) \bigvee_t^b (f')\,dt \le \begin{cases} \frac{1}{2}(b-x)^2 \bigvee_x^b (f'), \\ \frac{1}{(q+1)^{1/q}} (b-x)^{1+1/q} \left(\int_x^b \left(\bigvee_x^b (f')\right)^p dt\right)^{1/p}, \\ (b-x) \int_x^b \left(\bigvee_x^b (f')\right) dt. \end{cases}$$

□

Remark 2. *From the first branch in* (17) *we have the sequence of inequalities*

$$\left| f(x) + \frac{1}{2(b-a)} \left[(b-x)^2 f'(b) - (x-a)^2 f'(a) \right] - \frac{1}{b-a} \int_a^b f(t)\,dt \right| \tag{19}$$

$$\le \frac{1}{b-a} \left[\int_a^x (t-a) \bigvee_a^t (f')\,dt + \int_x^b (b-t) \bigvee_t^b (f')\,dt \right]$$

$$\le \frac{1}{2}(b-a) \left[\left(\frac{x-a}{b-a}\right)^2 \bigvee_a^x (f') + \left(\frac{b-x}{b-a}\right)^2 \bigvee_x^b (f') \right]$$

$$\leq \frac{1}{2}(b-a)\begin{cases} \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\left[\frac{1}{2}\bigvee_a^b(f')+\frac{1}{2}\left|\bigvee_a^x(f')-\bigvee_x^b(f')\right|\right], \\ \\ \left[\left(\frac{x-a}{b-a}\right)^{2p}+\left(\frac{b-x}{b-a}\right)^{2p}\right]^{1/p}\left[\left[\bigvee_a^x(f')\right]^q+\left[\bigvee_x^b(f')\right]^q\right]^{1/q} \\ p>1, \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\bigvee_a^b(f'), \end{cases}$$

for any $x \in [a,b]$.

From the second branch in (17) *we have*

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{20}$$
$$\leq \frac{1}{b-a}\left[\int_a^x (t-a)\bigvee_a^t(f')\,dt+\int_x^b (b-t)\bigvee_t^b(f')\,dt\right]$$
$$\leq \frac{1}{(q+1)^{1/q}}\left\{\left(\frac{x-a}{b-a}\right)^{1+1/q}\left(\int_a^x\left(\bigvee_a^t(f')\right)^p dt\right)^{1/p}\right.$$
$$\left.+\left(\frac{b-x}{b-a}\right)^{1+1/q}\left(\int_x^b\left(\bigvee_t^b(f')\right)^p dt\right)^{1/p}\right\}(b-a)^{1/q}$$
$$\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/p}$$
$$\times\left[\int_a^x\left(\bigvee_a^t(f')\right)^p dt+\int_x^b\left(\bigvee_t^b(f')\right)^p dt\right]^{1/p}(b-a)^{1/q}$$
$$\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/p}$$
$$\times\left[(x-a)\left(\bigvee_a^x(f')\right)^p+(b-x)\left(\bigvee_x^b(f')\right)^p\right]^{1/p}(b-a)^{1/q}$$

for any $x \in [a,b]$ *and* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

From the third branch in (17) *we have*

$$\left|f(x)+\frac{1}{2(b-a)}\left[(b-x)^2 f'(b)-(x-a)^2 f'(a)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{21}$$
$$\leq \frac{1}{b-a}\left[\int_a^x (t-a)\bigvee_a^t(f')\,dt+\int_x^b (b-t)\bigvee_t^b(f')\,dt\right]$$
$$\leq \left(\frac{x-a}{b-a}\right)\int_a^x\left(\bigvee_a^t(f')\right)dt+\left(\frac{b-x}{b-a}\right)\int_x^b\left(\bigvee_t^b(f')\right)dt$$

$$\leq \begin{cases} \left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]\left[\int_a^x \left(\bigvee_a^t (f')\right) dt + \int_x^b \left(\bigvee_t^b (f')\right) dt\right] \\ \left[\left(\frac{x-a}{b-a}\right)^q + \left(\frac{b-x}{b-a}\right)^q\right]^{1/q} \\ \times \left[\left[\int_a^x \left(\bigvee_a^t (f')\right) dt\right]^p + \left[\int_x^b \left(\bigvee_t^b (f')\right) dt\right]^p\right]^{1/p} \\ \max\left\{\int_a^x \left(\bigvee_a^t (f')\right) dt, \int_x^b \left(\bigvee_t^b (f')\right) dt\right\} \end{cases}$$

for any $x \in [a, b]$ *and* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 3. *We observe that, if we take* $x = \frac{a+b}{2}$ *in* (19) *then we get the perturbed midpoint inequality*

$$\left| f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{22}$$
$$\leq \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} (t-a) \bigvee_a^t (f')\,dt + \int_{\frac{a+b}{2}}^b (b-t) \bigvee_t^b (f')\,dt\right] \leq \frac{1}{8}(b-a)\bigvee_a^b (f').$$

3. Inequalities for Lipschitzian Derivatives

We start with the following result.

Theorem 4. *Let* $f : I \to \mathbb{C}$ *be a differentiable function on* $\mathring{I}$ *and* $[a, b] \subset \mathring{I}$. *Let* $x \in (a, b)$. *If* $\alpha_i > -1$ *and* $L_{\alpha_i} > 0$ *with* $i = 1, 2$ *are such that*

$$|f'(t) - f'(a)| \leq L_{\alpha_1} (t-a)^{\alpha_1} \text{ for any } t \in [a, x) \tag{23}$$

and

$$|f'(b) - f'(t)| \leq L_{\alpha_2} (b-t)^{\alpha_2} \text{ for any } t \in (x, b], \tag{24}$$

then we have

$$\left| f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \tag{25}$$
$$\leq \frac{1}{b-a}\left[\frac{L_{\alpha_1}}{\alpha_1+2}(x-a)^{\alpha_1+2} + \frac{L_{\alpha_2}}{\alpha_2+2}(b-x)^{\alpha_2+2}\right].$$

Proof. Using the conditions (23) and (24) we have

$$\left| f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right|$$
$$\leq \frac{1}{b-a}\int_a^x (t-a)\,|f'(t) - f'(a)|\,dt + \frac{1}{b-a}\int_x^b (b-t)\,|f'(b) - f'(t)|\,dt$$
$$\leq \frac{1}{b-a}L_{\alpha_1}\int_a^x (t-a)^{\alpha_1+1}\,dt + \frac{1}{b-a}L_{\alpha_2}\int_x^b (b-t)^{\alpha_2+1}\,dt$$
$$= \frac{1}{b-a}L_{\alpha_1}\frac{(x-a)^{\alpha_1+2}}{\alpha_1+2} + \frac{1}{b-a}L_{\alpha_2}\frac{(b-x)^{\alpha_2+2}}{\alpha_2+2}$$

$$= \frac{1}{b-a}\left[\frac{L_{\alpha_1}}{\alpha_1+2}(x-a)^{\alpha_1+2}+\frac{L_{\alpha_2}}{\alpha_2+2}(b-x)^{\alpha_2+2}\right]$$

and the inequality (25) is obtained. □

Corollary 3. *Let $f: I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. If the derivative is f' of r-H-Hölder type on $[a,b]$, i.e. we have the condition*

$$|f'(t) - f'(s)| \leq H|t-s|^r$$

for any $t, s \in [a,b]$, where $r \in (0,1]$ and $H > 0$ are given, then

$$\left|f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{H}{r+2}\left[\left(\frac{x-a}{b-a}\right)^{r+2} + \left(\frac{b-x}{b-a}\right)^{r+2}\right](b-a)^{r+1}, \tag{26}$$

for any $x \in [a,b]$.

In particular, if f' is Lipschitzian with the constant $L > 0$, then

$$\left|f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{3}L\left[\left(\frac{x-a}{b-a}\right)^{3} + \left(\frac{b-x}{b-a}\right)^{3}\right](b-a)^{2}, \tag{27}$$

for any $x \in [a,b]$.

Remark 4. *With the assumptions of Corollary 3 we have the midpoint inequality*

$$\left|f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{H}{2^{r+1}(r+2)}(b-a)^{r+1}. \tag{28}$$

If if f' is Lipschitzian with the constant $L > 0$, then

$$\left|f\left(\frac{a+b}{2}\right) + \frac{1}{8}(b-a)\left[f'(b) - f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{12}L(b-a)^2. \tag{29}$$

4. Inequalities for Differentiable Functions with the Property (S)

Let $f: I \to \mathbb{C}$ be a differentiable convex function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Then f' is monotonic nondecreasing and by the equality (15) we have

$$f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \geq 0 \tag{30}$$

or, equivalently

$$\frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] \geq \frac{1}{b-a}\int_a^b f(t)\,dt - f(x) \tag{31}$$

for any $x \in [a,b]$.

We observe that the inequalities (30) and (31) remain valid for the larger class of differentiable functions f that that satisfy the *property* (S) on the interval $[a,b]$, namely

$$f'(a) \leq f'(t) \leq f'(b) \tag{S}$$

for any $t \in [a,b]$.

Theorem 5. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$.*

(i) Let $x \in [a,b]$. If f satisfies the property (S) on the interval $[a,x]$ and $[x,b]$, then

$$f'(x)\left(\frac{a+b}{2}-x\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt - f(x). \tag{32}$$

(ii) If f satisfies the property (S) on the interval $[a,b]$, then for any $x \in [a,b]$

$$\frac{f(a)(x-a)+f(b)(b-x)}{b-a} - \frac{1}{b-a}\int_a^b f(t)\,dt \tag{33}$$
$$\le \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right].$$

Proof. (i) Since f satisfies the property (S) on the interval $[a,x]$ and $[x,b]$, then

$$\begin{aligned}
&f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \\
&= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'(a)\right]dt + \frac{1}{b-a}\int_x^b (b-t)\left[f'(b)-f'(t)\right]dt \\
&\le \frac{1}{b-a}\int_a^x (t-a)\left[f'(x)-f'(a)\right]dt + \frac{1}{b-a}\int_x^b (b-t)\left[f'(b)-f'(x)\right]dt \\
&= \frac{f'(x)-f'(a)}{b-a}\int_a^x (t-a)\,dt + \frac{f'(b)-f'(x)}{b-a}\int_x^b (b-t)\,dt \\
&= \frac{f'(x)-f'(a)}{b-a}\cdot\frac{(x-a)^2}{2} + \frac{f'(b)-f'(x)}{b-a}\cdot\frac{(b-x)^2}{2} \\
&= \frac{1}{2(b-a)}\left[(f'(x)-f'(a))(x-a)^2 + (f'(b)-f'(x))(b-x)^2\right] \\
&= \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - f'(x)\left(\frac{a+b}{2}-x\right),
\end{aligned}$$

which proves the inequality (32).

(ii) If f satisfies the property (S) on the interval $[a,b]$, then for any $x \in [a,b]$

$$\begin{aligned}
&\frac{1}{b-a}\int_a^x (t-a)\left[f'(t)-f'(a)\right]dt + \frac{1}{b-a}\int_x^b (b-t)\left[f'(b)-f'(t)\right]dt \\
&\le \frac{x-a}{b-a}\int_a^x \left[f'(t)-f'(a)\right]dt + \frac{b-x}{b-a}\int_x^b \left[f'(b)-f'(t)\right]dt \\
&= \frac{1}{b-a}(x-a)\left[f(x)-f(a)-f'(a)(x-a)\right] \\
&+ \frac{1}{b-a}(b-x)\left[f'(b)(b-x)-f(b)+f(x)\right] \\
&= \frac{1}{b-a}\left[f(x)(x-a)-f(a)(x-a)-f'(a)(x-a)^2\right] \\
&+ \frac{1}{b-a}\left[f'(b)(b-x)^2-f(b)(b-x)+f(x)(b-x)\right] \\
&= \frac{1}{b-a}\left\{f'(b)(b-x)^2-f'(a)(x-a)^2-f(a)(x-a)-f(b)(b-x)+f(x)(b-a)\right\} \\
&= \frac{f'(b)(b-x)^2-f'(a)(x-a)^2}{b-a} + f(x) - \frac{f(a)(x-a)+f(b)(b-x)}{b-a},
\end{aligned}$$

which proves the inequality (33). □

Remark 5. *The inequality* (32) *was obtained for the case of convex functions in* [20] *while* (33) *was established for convex functions in* [21] *with different proofs.*

Further, we use the Čebyšev inequality for synchronous functions (functions with same monotonicity), namely

$$\frac{1}{d-c}\int_c^d g(t)\,h(t)\,dt \geq \frac{1}{d-c}\int_c^d g(t)\,dt \cdot \frac{1}{d-c}\int_c^d h(t)\,dt. \tag{34}$$

Theorem 6. *Let $f : I \to \mathbb{C}$ be a differentiable function on $\mathring{I}$ and $[a,b] \subset \mathring{I}$. Let $x \in [a,b]$. If f is convex on the interval $[a,x]$ and $[x,b]$, then*

$$\frac{1}{2}\left[f(x) + \frac{f(a)(x-a) + f(b)(b-x)}{b-a}\right] \geq \frac{1}{b-a}\int_a^b f(t)\,dt. \tag{35}$$

Proof. We have

$$f(x) + \frac{1}{2(b-a)}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] - \frac{1}{b-a}\int_a^b f(t)\,dt \tag{36}$$
$$= \frac{1}{b-a}\int_a^x (t-a)\left[f'(t) - f'(a)\right]dt + \frac{1}{b-a}\int_x^b (b-t)\left[f'(b) - f'(t)\right]dt$$

for any $x \in [a,b]$.

Since f' is monotonic nondecreasing on $[a,x]$, then by Čebyšev inequality (26) we have

$$\begin{aligned}\int_a^x (t-a)\left[f'(t) - f'(a)\right]dt &\geq \frac{1}{x-a}\int_a^x (t-a)\,dt \cdot \int_a^x \left[f'(t) - f'(a)\right]dt \\ &= \frac{1}{2}(x-a)\left[f(x) - f(a) - f'(a)(x-a)\right] \\ &= \frac{1}{2}\left[f(x)(x-a) - f(a)(x-a) - f'(a)(x-a)^2\right]\end{aligned}$$

and, by the same inequality,

$$\begin{aligned}\int_x^b (b-t)\left[f'(b) - f'(t)\right]dt &\geq \frac{1}{b-x}\int_x^b (b-t)\,dt \cdot \int_x^b \left[f'(b) - f'(t)\right]dt \\ &= \frac{1}{2}(b-x)\left[f'(b)(b-x) - f(b) + f(x)\right] \\ &= \frac{1}{2}\left[f'(b)(b-x)^2 - f(b)(b-x) + f(x)(b-x)\right].\end{aligned}$$

If we add these two inequalities, then we get

$$\begin{aligned}&\int_a^x (t-a)\left[f'(t) - f'(a)\right]dt + \int_x^b (b-t)\left[f'(b) - f'(t)\right]dt \\ &\geq \frac{1}{2}\left[f(x)(x-a) - f(a)(x-a) - f'(a)(x-a)^2\right] \\ &+ \frac{1}{2}\left[f'(b)(b-x)^2 - f(b)(b-x) + f(x)(b-x)\right] \\ &= \frac{1}{2}\left[(b-x)^2 f'(b) - (x-a)^2 f'(a)\right] + \frac{1}{2}f(x)(b-a) \\ &- \frac{1}{2}\left[f(a)(x-a) + f(b)(b-x)\right].\end{aligned}$$

Dividing by $b-a$ and utilizing the equality (36) we deduce the inequality (35). □

Remark 6. *If the function is convex on the whole interval* $[a, b]$*, then, the inequality* (35) *is true for any* $x \in [a, b]$.

References

[1] Acu, A.M., Baboș, A. and Sofonea, F., *The mean value theorems and inequalities of Ostrowski type*, Sci. Stud. Res. Ser. Math. Inform. **21** (2011), no. 1, 5–16.

[2] Acu, A.M. and Sofonea, F., *On an inequality of Ostrowski type*, J. Sci. Arts **3** (16) (2011), 281–287.

[3] Ahmad, F., Barnett, N.S. and Dragomir, S.S., *New weighted Ostrowski and Čebyšev type inequalities*, Nonlinear Anal. **71** (2009), no. 12, e1408–e1412.

[4] Alomari, M.W., *A companion of Ostrowski's inequality with applications*, Transylv. J. Math. Mech. **3** (2011), no. 1, 9–14.

[5] Alomari, M.W., Darus, M., Dragomir, S.S. and Cerone, P., *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense*, Appl. Math. Lett. **23** (2010), no. 9, 1071–1076.

[6] Anastassiou, G.A., *Ostrowski type inequalities*, Proc. Amer. Math. Soc. **123** (1995), No. 12, 3775–3781.

[7] Anastassiou, G.A., *Univariate Ostrowski inequalities, revisited*, Monatsh. Math. **135** (2002), No. 3, 175–189.

[8] Anastassiou, G.A., *Ostrowski inequalities for cosine and sine operator functions*, Mat. Vesnik **64** (2012), no. 4, 336–346.

[9] Anastassiou, G.A., *Multivariate right fractional Ostrowski inequalities*, J. Appl. Math. Inform. **30** (2012), no. 3-4, 445–454.

[10] Anastassiou, G.A., *Univariate right fractional Ostrowski inequalities*, Cubo **14** (2012), no. 1, 1–7.

[11] Barnett, N.S., Dragomir, S.S. and Gomm, I., *A companion for the Ostrowski and the generalised trapezoid inequalities*, Math. Comput. Modelling **50** (2009), no. 1-2, 179–187.

[12] Cerone, P., Cheung, W.-S. and Dragomir, S.S., *On Ostrowski type inequalities for Stieltjes integrals with absolutely continuous integrands and integrators of bounded variation*, Comput. Math. Appl. **54** (2007), No. 2, 183–191.

[13] Cerone, P. and Dragomir, S.S., *Midpoint-type rules from an inequalities point of view*, Handbook of analytic-computational methods in applied mathematics, 135–200, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[14] Cerone, P. and Dragomir, S.S., *Trapezoidal-type rules from an inequalities point of view*, Handbook of analytic-computational methods in applied mathematics, 65–134, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[15] Cerone, P., Dragomir, S.S. and Pearce, C.E.M., *A generalised trapezoid inequality for functions of bounded variation*, Turk. J. Math. **24** (2000), 147–163.

[16] Dragomir, S.S., *The Ostrowski inequality for mappings of bounded variation*, Bull. Austral. Math. Soc. **60** (1999), 495–826.

[17] Dragomir, S.S., *On the mid-point quadrature formula for mappings with bounded variation and applications*, Kragujevac J. Math. **22** (2000), 13–19.

[18] Dragomir, S.S., *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Math. Ineq. & Appl. **4** (1) (2001), 33–40. Preprint, RGMIA Res. Rep. Coll. **2** (1999), No. 1, Article 7. Online: `http://rgmia.vu.edu.au/v2n1.html`.

[19] Dragomir, S.S., *On the trapezoid quadrature formula and applications*, Kragujevac J. Math. **23** (2001), 25–36.

[20] Dragomir, S.S., *An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure Appl. Math. **3** (2002), no. 2, Article 31, 8 pp.

[21] Dragomir, S.S., *An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure Appl. Math. **3** (2002), no. 3, Article 35, 8 pp.

[22] Dragomir, S.S., *Improvements of Ostrowski and generalised trapezoid inequality in terms of the upper and lower bounds of the first derivative*, Tamkang J. Math. **34** (2003), no. 3, 213–222.

[23] Dragomir, S.S., *Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation*, Arch. Math. (Basel) **91** (2008), no. 5, 450–460.

[24] Dragomir, S.S., *Some inequalities for continuous functions of selfadjoint operators in Hilbert spaces*, Acta Math. Vietnamica, to appear. Preprint RGMIA Res. Rep. Coll. **15** (2012), Art. 16. Online: `http://rgmia.org/v15.php`.

[25] Dragomir, S.S., *Refinements of the Ostrowski inequality in terms of the cumulative variation and applications*, Preprint RGMIA Res. Rep. Coll. **16** (2013), Art. 29, pp. 15. Online: `http://rgmia.org/papers/v16/v16a29.pdf`.

[26] Dragomir, S.S., *Some perturbed Ostrowski type inequalities for absolutely continuous functions (I)*, Preprint RGMIA Res. Rep. Coll. **16** (2013), Art. 94.

[27] Dragomir, S.S., *Some perturbed Ostrowski type inequalities for absolutely continuous functions (II)*, Preprint RGMIA Res. Rep. Coll. **16** (2013), Art. 95.

[28] Liu, Z., *Some inequalities of perturbed trapezoid type*, J. Inequal. Pure Appl. Math. **7** (2006), no. 2, Article 47, 9 pp.

[29] Liu, Z., *A note on Ostrowski type inequalities related to some s-convex functions in the second sense*, Bull. Korean Math. Soc. **49** (2012), no. 4, 775–785.

[30] Liu, Z., *A sharp general Ostrowski type inequality*, Bull. Aust. Math. Soc. **83** (2011), no. 2, 189–209.

[31] Liu, Z., *New sharp bound for a general Ostrowski type inequality*, Tamsui Oxf. J. Math. Sci. **26** (2010), no. 1, 53–59.

[32] Liu, Z., *Some Ostrowski type inequalities and applications*, Vietnam J. Math. **37** (2009), no. 1, 15–22.

[33] Liu, Z., *Some companions of an Ostrowski type inequality and applications*, J. Inequal. Pure Appl. Math. **10** (2009), no. 2, Article 52, 12 pp.

[34] Masjed-Jamei, M. and Dragomir, S.S., *A new generalization of the Ostrowski inequality and applications*, Filomat **25** (2011), no. 1, 115–123.

[35] Pachpatte, B.G., *A note on a trapezoid type integral inequality*, Bull. Greek Math. Soc. **49** (2004), 85–90.

[36] Park, J., *On the Ostrowski like type integral inequalities for mappings whose second derivatives are s^*-convex*, Far East J. Math. Sci. (FJMS) **67** (2012), no. 1, 21–35.

[37] Park, J., *Some Ostrowski like type inequalities for differentiable real (α, m)-convex mappings*, Far East J. Math. Sci. (FJMS) **61** (2012), no. 1, 75–91

[38] Sarikaya, M.Z., *On the Ostrowski type integral inequality*, Acta Math. Univ. Comenian. (N.S.) **79** (2010), no. 1, 129–134.

[39] Sulaiman, W.T., *Some new Ostrowski type inequalities*, J. Appl. Funct. Anal. **7** (2012), no. 1-2, 102–107.

[40] Tseng, K.-L., *Improvements of the Ostrowski integral inequality for mappings of bounded variation II*, Appl. Math. Comput. **218** (2012), no. 10, 5841–5847.

[41] Tseng, K.-L., Hwang, S.-R., Yang, G.-S., and Chou, Y.-M., *Improvements of the Ostrowski integral inequality for mappings of bounded variation I*, Appl. Math. Comput. **217** (2010), no. 6, 2348–2355.

[42] Ujević, N., *Error inequalities for a generalized trapezoid rule*, Appl. Math. Lett. **19** (2006), no. 1, 32–37.

[43] Vong, S.W., *A note on some Ostrowski like type inequalities*, Comput. Math. Appl. **62** (2011), no. 1, 532–535.

[44] Wu, Y. and Wang, Y., *On the optimal constants of Ostrowski like inequalities involving n knots*, Appl. Math. Comput. **219** (2013), no. 14, 7789–7794.

[45] WU, Q. and Yang, S., *A note to Ujević's generalization of Ostrowski's inequality*, Appl. Math. Lett. **18** (2005), no. 6, 657–665.

[46] Xiao, Y.-X., *Remarks on Ostrowski like inequalities*, Appl. Math. Comput. **219** (2012), no. 3, 1158–1162.

Mathematics, College of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia.
School of Computational & Applied Mathematics
University of the Witwatersrand
Private Bag 3, Johannesburg 2050, South Africa
E-mail address: `sever.dragomir@vu.edu.au`
URL: `http://rgmia.org/dragomir`