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AN INTEGRAL EQUATION FROM PHYSICS - A SYNTHESIS SURVEY - PART I

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Abstract. In this synthesis survey we will present in several parts, the results obtained in the study of the integral equation from physics:

$$x(t) = \int_a^b K(t, s, x(s), x(a), x(b))ds + f(t), \quad t \in [a, b].$$

that appeared in the 70s and was a mathematical model in the operation of turbojets. This is the first part of the synthesis survey and it contains the results concerning the existence and respectively the existence and uniqueness of the solution of this integral equation. Also it contains some properties of the solution of this integral equation, formulated as integral inequalities. This part ends with three examples.

1. Introduction

We formulate the following problem:

"We consider the functional-integral equation

$$x(t) = \int_a^b K(t, s, x(s), \min_{a \le \zeta \le s} x(\zeta), \max_{s \le \zeta \le b} x(\zeta))ds + f(t), \quad t \in [a, b],$$

and it observe that if is searching for ascending solutions, then we obtain a Fredholm-type integral equation with modified argument, having the form:

$$x(t) = \int_a^b K(t, s, x(s), x(a), x(b))ds + f(t), \quad t \in [a, b] \tag{1}$$

where $K : [a, b] \times [a, b] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ or $K : [a, b] \times [a, b] \times J^3 \longrightarrow \mathbb{R}$, $J \subset \mathbb{R}$ closed interval, and $f : [a, b] \longrightarrow \mathbb{R}$."

This integral equation appeared in the 70s and based on some issues relating to the operation of turbojets, was proposed as mathematical model.

The following main issues regarding the solution of this integral equation was studied: existence and respectively existence and uniqueness of the solution; Gronwall lemmas and comparison theorems; data dependence (continuous data dependence of the solution, differentiability of the solution with respect to a parameter, differentiability of the solution with respect to a and b) and the approximation of the solution.

In what follows, we will present a synthesis survey, in several parts, on the analysis of the solution of this integral equation.

This is the first part of survey and it contains the results that we have obtained regarding the existence and uniqueness and some properties of the solution of the integral equation (1); these results were published in several papers.

In the section 2 we present the basic results and notations that were used in order to obtaining the results presented in the sections 3 and 4.

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The section 3 contains the results regarding the existence and uniqueness of the solution of the integral equation (1), obtained by applying *the Schauder's Theorem*, *the Banach's Theorem* and *the Perov's Theorem*.

In the section 4 we present some properties of the solution of the integral equation (1), as integral inequalities, obtained by applying *the Abstract Gronwall Lemma* and *the Abstract Comparison Lemma*.

2. Basic notations and results

Let X be a nonempty set, d a metric on X and $A : X \to X$ an operator. In this part of survey we shall use the following notations:

$P(X) := \{Y \subset X \ / \ Y \neq \varnothing\}$ – the set of all nonempty subsets of X

$I(A) := \{Y \in P(X) \ / \ A(Y) \subset Y\}$ – the family of the nonempty subsets of X, invariant for A

$F_A := \{x \in X | A(x) = x\}$ – the fixed points set of A

$A^0 := 1_X,\ A^1 := A,\ A^{n+1} := A \circ A^n,\ n \in N$ – the iterate operators of A.

We consider the Banach space $X = C([a,b], \mathbf{B})$:

$$C([a,b], \mathbf{B}) = \{x : [a,b] \longrightarrow \mathbf{B} | x - \text{continuous function}\},$$

endowed with the Chebyshev norm

$$\|x\|_C := \max_{t \in [a,b]} |x(t)|\,, \text{ for all } x \in C([a,b], \mathbf{B}),$$

where $(\mathbf{B}, +, \mathbb{R}, |\cdot|)$ is a Banach space.

The following definitions and theorems were used to study the existence and uniqueness of the solution of integral equation (1) (see [17, 18, 19, 20, 21, 22, 23, 24, 25]). In order to obtain the results presented in this survey also were used some results from [16] and [26].

Let $\{x(t)\}$ be a set of functions $x \in C([a,b], \mathbf{B})$.

Definition 1. *The functions $x(t)$ are called equal bounded functions on the interval $[a,b]$, if there exists $M > 0$ such that*

$$|x(t)| \leq M \quad \textit{for all} \quad t \in [a,b] \quad \textit{and} \quad x(t) \in \{x(t)\},\ x \in C([a,b], \mathbf{B}).$$

Definition 2. *The functions $x(t)$ are called equal continuous functions on the interval $[a,b]$, if $\forall \varepsilon > 0$, $\exists \eta > 0$ such that for each function $x(t) \in \{x(t)\}$, $x \in C([a,b], \mathbf{B})$, we have*

$$|x(t'') - x(t')| \leq \varepsilon \quad \textit{for all} \quad t'', t' \in [a,b] \quad \textit{and} \quad |t'' - t'| < \eta.$$

Theorem 1 (Ascoli-Arzela)**.** *A subset of the functions from $C([a,b], \mathbf{B})$ is compact if and only if this subset is equal bounded and equal continuous.*

Theorem 2 (Schauder)**.** *Let X be a Banach space and $Y \subset X$ be a nonempty, bounded, convex and closed set. If $A : Y \to Y$ is a completely continuous operator, then A has at least one fixed point.*

Definition 3 (Rus I.A., [18] or [19])**.** *Let (X,d) be a metric space. An operator $A : X \to X$ is Picard operator (PO) if there exists $x^* \in X$ such that:*

(a) $F_A = \{x^*\}$;

(b) the sequence $(A^n(x_0))_{n \in \mathbb{N}}$ converges to x^*, for all $x_0 \in X$.

Definition 4 (Rus I.A., [18] or [19])**.** *Let (X,d) be a metric space. An operator $A : X \to X$ is weakly Picard operator (WPO) if the sequence $(A^n(x_0))_{n \in \mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.*

If A is a weakly Picard operator, then we consider the following operator

$$A^{\infty} : X \to X, \quad A^{\infty}(x) = \lim_{n \longrightarrow \infty} A^n(x)$$

and we observe that $A^{\infty}(X) = F_A$.

Theorem 3 (Banach, Contraction Principle)**.** *Let (X, d) be a complete metric space and $A : X \to X$ an α-contraction ($\alpha < 1$). Under these conditions we have:*

(i) $F_A = \{x^*\}$;

(ii) $x^* = \lim_{n \longrightarrow \infty} A^n(x_0)$, *for all* $x_0 \in X$;

(iii) $d(x^*, A^n(x_0)) \leq \frac{\alpha^n}{1-\alpha} d(x_0, A(x_0))$.

Also, we will use the Banach space $C([a,b], \mathbb{R}^m)$

$$C([a,b], \mathbb{R}^m) := \{x : [a,b] \to \mathbb{R}^m / x \text{ continuous function}\}$$

endowed with the generalized Chebyshev norm defined by the relation:

$$\|x\| := \begin{pmatrix} \|x_1\|_C \\ \dots \\ \|x_m\|_C \end{pmatrix}, \text{ for all } x = \begin{pmatrix} x_1 \\ \dots \\ x_m \end{pmatrix} \in C([a,b], \mathbb{R}^m) \tag{2}$$

where

$$\|x_k\|_C := \max_{t \in [a,b]} |x_k(t)|, \ k = \overline{1,m}.$$

This space is a complete generalized Banach space, where for an element $w \in \mathbb{R}^m$, we denoted $\|w\| = (|w_1|, \dots, |w_m|)$.

Theorem 4 (Perov)**.** *Let (X, d), with $d(x,y) \in \mathbb{R}^m$ be a complete generalized metric space and $A : X \longrightarrow X$ an operator. We suppose that there exists a matrix $Q \in M_{mm}(\mathbb{R}_+)$ such that*

(i) $d(A(x), A(y)) \leq Q d(x,y)$, *for all* $x, y \in X$;

(ii) $Q^n \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Then

(a) $F_A = \{x^*\}$;

(b) $A^n(x) \longrightarrow x^*$ *as* $n \longrightarrow \infty$ *and*

$$d(A^n(x), x^*) \leq Q^n (I - Q)^{-1} d(x_0, A(x_0)).$$

Definition 5 (see [17])**.** *A matrix $Q \in M_{mm}(\mathbb{R})$ converges to zero if Q^k converges to the zero matrix as $k \longrightarrow \infty$.*

The following theorem presents two conditions that are equivalents with the convergence to zero of a matrix $Q \in M_{nn}(\mathbb{R}_+)$. This theorem is useful for applying *Perov's Theorem* in order to study the existence and uniqueness of the solution of an integral equations system.

Theorem 5 (see [17])**.** *Let $Q \in M_{mm}(\mathbb{R}_+)$ be a matrix. The following conditions are equivalents:*

(i) $Q^k \to \infty$ *as* $k \to \infty$;

(ii) The eigenvalues λ_k, $k = \overline{1,m}$ *of the matrix* Q *satisfy the condition* $|\lambda_k| < 1$, $k = \overline{1,m}$;

(iii) The matrix $I - Q$ *is non-singular and*

$$(I - Q)^{-1} = I + Q + \dots + Q^m + \dots$$

Now, we present three lemmas that were used in establishing two integral inequalities as properties of the solution of the integral equation (1).

Let $\leq$ be an order relation on X.

Lemma 1 (Rus I.A., [21]). *Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator, such that:*

(i) the operator A is increasing;
(ii) A is WPO.

Then, the operator A^∞ is increasing.

Lemma 2 (Abstract Comparison Lemma). *Let $(X, d, \leq)$ be an ordered metric space and $A, B, C : X \to X$ three operators, such that:*

(i) $A \leq B \leq C$;
(ii) A, B, C are WPOs;
(iii) the operator B is increasing.

Then

$$x \leq y \leq z \Rightarrow A(x) \leq B(y) \leq C(z).$$

Remark 1. *Let A, B, C be the operators defined in the Abstract Comparison Lemma. In addition, we suppose that $F_B = \{x_B^*\}$, i.e. B is a PO. Then we have*

$$A^\infty(x) \leq x_B^* \leq C^\infty(x), \text{ for all } x \in X.$$

But $A^\infty(X) = F_A$, and $C^\infty(X) = F_C$, and therefore we obtain $F_A \leq x_B^* \leq F_C$.

Lemma 3 (Abstract Gronwall Lemma, see [19]). *Let$(X, d, \leq)$ be an ordered metric space and $A : X \to X$ an operator. We suppose that:*

(i) A is a PO;
(ii) the operator A is increasing.

If we denote with x_A^* the unique fixed point of the operator A, then

(a) $x \leq A(x) \Rightarrow x \leq x_A^*$;
(b) $x \geq A(x) \Rightarrow x \geq x_A^*$.

3. The existence and uniqueness of the solution

Let $(\mathbf{B}, +, \mathbb{R}, |\cdot|)$ be a Banach space.

We consider the nonlinear Fredholm integral equation with modified argument (1)

$$x(t) = \int_a^b K(t, s, x(s), x(a), x(b))ds + f(t), \quad t \in [a, b]$$

where $K : [a, b] \times [a, b] \times \mathbf{B}^3 \longrightarrow \mathbf{B}$, $f : [a, b] \longrightarrow \mathbf{B}$.

We present below, the results of existence and respectively of existence and uniqueness of the solution for this equation.

3.1. Existence of the solution.

Theorem 6 (Dobrițoiu M., [13]). *Suppose that*

(i) $K \in C([a, b] \times [a, b] \times \mathbf{B}^3, \mathbf{B})$;
(ii) $f \in C([a, b], \mathbf{B})$;
(iii) there exists $M_K > 0$ such that

$$|K(t, s, u_1, u_2, u_3)| \leq M_K, \text{ for all } t \in [a, b],\ u_1, u_2, u_3 \in \mathbf{B}.$$

Then the integral equation (1) has at least one solution $x^* \in C([a, b], \mathbf{B})$.

Theorem 7 (Dobrițoiu M., [13]). *Suppose that*

(i) $K \in C([a,b] \times [a,b] \times J^3, \mathbf{B})$, $J \subset \mathbf{B}$ *is a compact subset;*

(ii) $f \in C([a,b], \mathbf{B})$;

(iii) $M_K(b-a) \le r$,

where we denoted with M_K *a positive constant such that, for the restriction* $K|_{[a,b]\times[a,b]\times J^3}$, $J \subset \mathbf{B}$ *compact, we have*

$$|K(t,s,u_1,u_2,u_3)| \le M_K, \text{ for all } t \in [a,b],\ u_1,u_2,u_3 \in J.$$

Then the integral equation (1) *has at least one solution* $x^* \in \overline{B}(f;r) \subset C([a,b], \mathbf{B})$.

In the proofs of these theorems, *the Ascoli-Arzela's Theorem* and *the Schauder's Theorem* have been used. These results have been published in [13].

3.2. **Existence and uniqueness of the solution.**

Theorem 8 (Dobrițoiu M., [13]). *Suppose that*

(i) $K \in C([a,b] \times [a,b] \times \mathbf{B}^3, \mathbf{B})$;

(ii) $f \in C([a,b], \mathbf{B})$;

(iii) there exists $L > 0$ *such that*

$$|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)| \le L(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|),$$

for all $t, s \in [a,b]$, $u_i, v_i \in \mathbf{B}$, $i = 1,2,3$;

(iv) $3L(b-a) < 1$.

Then the integral equation (1) *has a unique solution* $x^* \in C([a,b], \mathbf{B})$.

Theorem 9 (Dobrițoiu M., [13]). *Suppose that*

(i) $K \in C([a,b] \times [a,b] \times J^3, \mathbf{B})$, $J \subset \mathbf{B}$ *is a compact subset;*

(ii) $f \in C([a,b], \mathbf{B})$;

(iii) $M_K(b-a) \le r$ *(invariance condition of the sphere* $\overline{B}(f;r)$*),*
where $M_K > 0$, *such that* $|K(t,s,u,v,w)| \le M_K$, *for all* $t \in [a,b]$, $u,v,w \in J \subset \mathbf{B}$, J *compact subset;*

(iv) there exists $L > 0$ *such that*

$$|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)| \le L(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|),$$

for all $t, s \in [a,b]$, $u_i, v_i \in J$, $i = 1,2,3$;

(v) $3L(b-a) < 1$.

Then the integral equation (1) *has a unique solution* $x^* \in \overline{B}(f;r) \subset C([a,b], \mathbf{B})$.

In the proofs of these theorems, *the Banach's Theorem* has been used. These results have been published in [13].

Remark 2. *The theorem 8 holds true in particular cases* $\mathbf{B} = \mathbb{R}$, $\mathbf{B} = \mathbb{R}^m$, $\mathbf{B} = l^2(\mathbb{R})$, *if one replaces the conditions (i), (ii) and (iii) with conditions which one assures the existence and uniqueness of the solution of the integral equation* (1) *in the space* $C[a,b]$, *in the space* $C([a,b], \mathbb{R}^m)$ *and respectively in the space* $C([a,b], l^2(\mathbb{R}))$.

In what follows we present the result from theorem 11 in the particular cases $\mathbf{B} = \mathbb{R}$, $\mathbf{B} = \mathbb{R}^m$, $\mathbf{B} = l^2(\mathbb{R})$.

In the case $\mathbf{B} = \mathbb{R}$, we present below two theorems of existence and uniqueness of the solution of this integral equation, that was published in [1].

We consider the integral equation (1), where $K : [a,b] \times [a,b] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ or $K : [a,b] \times [a,b] \times J^3 \longrightarrow \mathbb{R}$, $J \subset \mathbb{R}$ closed interval, and $f : [a,b] \subset \mathbb{R}$.

Theorem 10 (Ambro M., [1]). *If*

(i) $K \in C([a,b] \times [a,b] \times \mathbb{R}^3)$, $f \in C[a,b]$;

(ii) *there exists* $L > 0$ *such that*

$$|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)| \le L(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|),$$

for all $t, s \in [a,b]$, $u_i, v_i \in \mathbb{R}$, $i = 1,2,3$;

(iii) $3L(b-a) < 1$ *(contraction condition),*

then the integral equation (1) *has a unique solution* $x^* \in C[a,b]$, *that can be obtained by the successive approximations method, starting at any element* $x_0 \in C[a,b]$. *In addition, if* x_n *is the n-th successive approximation, then the following estimation is hold:*

$$\|x^* - x_n\|_{C[a,b]} \le \frac{[3L(b-a)]^n}{1 - 3L(b-a)} \|x_0 - x_1\|_{C[a,b]} . \tag{3}$$

Theorem 11 (Ambro M., [1]). *We suppose that*

(i) $K \in C([a,b] \times [a,b] \times J^3)$, $J \subset \mathbb{R}$ *compact interval,* $f \in C[a,b]$;

(ii) *there exists* $L > 0$ *such that*

$$|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)| \le L(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3|),$$

for all $t, s \in [a,b]$, $u_i, v_i \in J$, $i = 1,2,3$;

(iii) $3L(b-a) < 1$ *(contraction condition).*
If $r > 0$ *is such that*

$$\left[x \in \overline{B}(f;r)\right] \Longrightarrow [x(t) \in J \subset \mathbb{R}]$$

and

(iv) $M(b-a) \le r$ *(invariance condition of the sphere* $\overline{B}(f;r)$*),*
where $M > 0$, *such that* $|K(t,s,u,v,w)| \le M$, *for all* $t \in [a,b]$, $u,v,w \in J \subset \mathbb{R}$, *J compact,*

then the integral equation (1) *has a unique solution* $x^* \in \overline{B}(f;r) \subset C[a,b]$, *that can be obtained by the successive approximations method, starting at any element* $x_0 \in \overline{B}(f;r)$. *In addition, if* x_n *is the n-th successive approximation, then the estimation* (3) *takes place.*

In the proofs of these theorems, *the Banach's Fixed Point Theorem* (*Contraction Principle*) was used. Also, were used some results from [13].

In the particular case $\mathbf{B} = \mathbb{R}^m$, we have the system of nonlinear integral equations

$$x(t) = \int_a^b K(t,s,x(s),x(a),x(b))ds + f(t), \tag{1'}$$

where $t \in [a,b]$, $K : [a,b] \times [a,b] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ and $f : [a,b] \longrightarrow \mathbb{R}^m$, that have the form:

$$\begin{cases} x_1(t) = \int_a^b K_1(t,s,x(s),x(a),x(b))ds + f_1(t), \\ x_2(t) = \int_a^b K_2(t,s,x(s),x(a),x(b))ds + f_2(t), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_m(t) = \int_a^b K_m(t,s,x(s),x(a),x(b))ds + f_m(t), \end{cases} \tag{1''}$$

and we present below some theorems of existence and uniqueness of the solution for this integral equations system.

Theorem 12 (Dobriţoiu M., [11], [12]). *We suppose that*

(i) $K \in C([a,b] \times [a,b] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$;

(ii) $f \in C([a,b], \mathbb{R}^m)$;

(iii) *there exists* $L > 0$*, such that*

$$|K_i(t,s,u_1,u_2,u_3) - K_i(t,s,v_1,v_2,v_3)| \le$$
$$\le L(\|u_1 - v_1\|_{\mathbb{R}^m} + \|u_2 - v_2\|_{\mathbb{R}^m} + \|u_3 - v_3\|_{\mathbb{R}^m})$$

for all $t, s \in [a,b]$, $u_1,u_2,u_3,v_1,v_2,v_3 \in \mathbb{R}^m$, $i = \overline{1,m}$;

(iv) $3L(b-a) < 1$.

Under these conditions, the system of integral equations (1') *has a unique solution* $x^* \in C([a,b],\mathbb{R}^m)$*, that can be obtained by the successive approximations method, starting at any element from the space* $C([a,b],\mathbb{R}^m)$*. If* $x_0 = (x_{01}, x_{02}, ..., x_{0m})$ *is the starting function and* $x_k = (x_{k1}, x_{k2}, ..., x_{km})$ *is the k-th successive approximation, then the following estimation is hold:*

$$\|x^* - x_k\|_{C([a,b],\mathbb{R}^m)} \le \frac{[3L(b-a)]^k}{1 - 3L(b-a)} \|x_0 - x_1\|_{C([a,b],\mathbb{R}^m)} \tag{4}$$

In the proof of this theorem, *the Banach's Theorem*, has been used.

Remark 3. *If, in particular case, we consider the Euclidean norm,* $\|\cdot\|_E$*, the Minkowski's norm,* $\|\cdot\|_M$*, and respectively the Chebyshev's norm,* $\|\cdot\|_C$ *on the space* $\mathbb{R}^m$*, then in accordance with these norms, the hypotheses (iii) and (iv) and the estimation* (4)*, in the theorem 12 are modified. We have:*

a) Euclidean norm

(iii_1) *there exists* $L > 0$ *, such that*

$$|K_i(t,s,u_1,u_2,u_3) - K_i(t,s,v_1,v_2,v_3)| \le$$
$$\le L\left[\left(\sum_{j=1}^m |u_{1j} - v_{1j}|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^m |u_{2j} - v_{2j}|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^m |u_{3j} - v_{3j}|^2\right)^{\frac{1}{2}}\right]$$

for all $t, s \in [a,b]$, $u_1,u_2,u_3,v_1,v_2,v_3 \in \mathbb{R}^m$, $i = \overline{1,m}$;

(iv_1) $3\sqrt{m}L(b-a) < 1$ *and*

$$\|x^* - x_k\|_{C([a,b],\mathbb{R}^m)} \le \frac{[3\sqrt{m}L(b-a)]^k}{1 - 3\sqrt{m}L(b-a)} \|x_0 - x_1\|_{C([a,b],\mathbb{R}^m)}. \tag{4_1}$$

b) Minkowski's norm

(iii_2) *there exists* $L > 0$ *, such that*

$$|K_i(t,s,u_1,u_2,u_3) - K_i(t,s,v_1,v_2,v_3)| \le$$
$$\le L\left(\sum_{j=1}^m |u_{1j} - v_{1j}| + \sum_{j=1}^m |u_{2j} - v_{2j}| + \sum_{j=1}^m |u_{3j} - v_{3j}|\right)$$

for all $t, s \in [a,b]$, $u_1,u_2,u_3,v_1,v_2,v_3 \in \mathbb{R}^m$, $i = \overline{1,m}$;

(iv_2) $3mL(b-a) < 1$ *and*

$$\|x^* - x_k\|_{C([a,b],\mathbb{R}^m)} \le \frac{[3mL(b-a)]^k}{1 - 3mL(b-a)} \|x_0 - x_1\|_{C([a,b],\mathbb{R}^m)}. \tag{4_2}$$

c) Chebyshev's norm

(iii_3) *there exists* $L > 0$*, such that*

$$\max_{1\le i\le m} |K_i(t,s,u_1,u_2,u_3) - K_i(t,s,v_1,v_2,v_3)| \le$$
$$\le L\left(\max_{1\le j\le m} |u_{1j} - v_{1j}| + \max_{1\le j\le m} |u_{2j} - v_{2j}| + \max_{1\le j\le m} |u_{3j} - v_{3j}|\right)$$

for all $t, s \in [a,b]$, $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}^m$*;*
(iv₃) $3L(b-a) < 1$ *and*

$$\|x^* - x_k\|_{C([a,b],\mathbb{R}^m)} \leq \frac{[3L(b-a)]^k}{1-3L(b-a)} \|x_0 - x_1\|_{C([a,b],\mathbb{R}^m)}. \quad (4_3)$$

We present below, another result of existence and uniqueness for the solution of the system (1"), in the space $C([a,b],\mathbb{R}^m)$, obtained by applying *the Perov's Theorem* and using *the generalized Chebyshev's norm*, defined by relation (2) in the section 1.

Theorem 13 (Dobriţoiu M., [12])**.** *We suppose that*

(i) $K \in C([a,b] \times [a,b] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$*;*
(ii) $f \in C([a,b],\mathbb{R}^m)$*;*
(iii) there exists $Q \in M_{mm}(\mathbb{R}_+)$ *such that*

$$\|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)\|_C \leq$$
$$\leq Q\left(\|u_1 - v_1)\|_C + \|u_2 - v_2)\|_C + \|u_3 - v_3)\|_C\right)$$

for all $t, s \in [a,b]$, $ui, vi \in \mathbb{R}^m$, $i = 1,2,3$*;*
(iv) $[3(b-a)Q]^n \to 0$ *as* $n \to \infty$*.*

Then the system of integral equations (1") *has a unique solution* $x^* \in C([a,b],\mathbb{R}^m)$*, that can be obtained by applying the successive approximations method, starting at any element* $x_0 \in C([a,b],\mathbb{R}^m)$*. If* x_n *is the n-th successive approximation, then the following estimation is hold:*

$$\|x^* - x_n\|_C \leq [3(b-a)Q]^n \cdot [I - 3(b-a)Q]^{-1} \|x_0 - x_1\|_C. \quad (5)$$

Regarding the existence and uniqueness of the solution of the integral equations system (1") in the sphere $\overline{B}(f;r) \in C([a,b],\mathbb{R}^m)$

$$\overline{B}(f;r) = \{x \in C([a,b],\mathbb{R}^m) \mid \|x - f\|_C \leq r,\ r \in M_{m1}(\mathbb{R}_+)\} \subset C([a,b],\mathbb{R}^m)$$

we obtained the following result:

Theorem 14 (Dobriţoiu M., [12])**.** *We suppose that*

(i) $K \in C([a,b] \times [a,b] \times J^3, \mathbb{R}^m)$*,* $J \subset \mathbb{R}^m$ *compact subset;*
(ii) $f \in C([a,b],\mathbb{R}^m)$*;*
(iii) there exists $L > 0$ *, such that*

$$|K_i(t,s,u_1,u_2,u_3) - K_i(t,s,v_1,v_2,v_3)| \leq$$
$$\leq L\left(\|u_1 - v_1\|_{\mathbb{R}^m} + \|u_2 - v_2\|_{\mathbb{R}^m} + \|u_3 - v_3\|_{\mathbb{R}^m}\right)$$

for all $t, s \in [a,b]$, $u_1, u_2, u_3, v_1, v_2, v_3 \in J \subset \mathbb{R}^m$, $i = \overline{1,m}$*;*
(iv) $3L(b-a) < 1$.
In addition, we suppose that if $r > 0$ *is such that*

$$\left[x \in \overline{B}(f;r)\right] \Longrightarrow [x(t) \in J \subset \mathbb{R}^m]$$

and the following condition is fulfilled:
(v) $M(b-a) \leq r$ *(invariance condition of the sphere* $\overline{B}(f;r)$*), where we denoted by* M *a positive constant, such that for the restriction* $K\ |_{[a,b]\times[a,b]\times J^3}$*,* $J \subset \mathbb{R}^m$ *compact, we have*

$$|K(t,s,u_1,u_2,u_3)| \leq M,$$

for all $t, s \in [a,b]$, $u_1, u_2, u_3 \in J \subset \mathbb{R}^m$.

Under these conditions, the system of integral equations (1") *has a unique solution* $x^* \in \overline{B}(f;r) \subset C([a,b],\mathbb{R}^m)$*, that can be obtained by the successive approximations method, starting at any element from* $\overline{B}(f;r) \subset C([a,b],\mathbb{R}^m)$*. If* x_0 *is the starting function and* x_k *is the k-th successive approximation, then the estimation* (4) *is hold.*

Another result of existence and uniqueness of the solution of the integral equations system (1") in the sphere $\overline{B}(f;r) \subset C([a,b],\mathbb{R}^m)$ is obtained by applying *the Perov's Theorem.*

Theorem 15 (Dobrițoiu M., [12]). *We suppose that*

(i) $K \in C([a,b]\times[a,b]\times J^3,\mathbb{R}^m)$*,* $J \subset \mathbb{R}^m$ *compact subset;*

(ii) $f \in C([a,b],\mathbb{R}^m)$*;*

(iii) there exists $Q \in M_{mm}(\mathbb{R}_+)$ *such that*

$$\|K(t,s,u_1,u_2,u_3) - K(t,s,v_1,v_2,v_3)\|_C \leq$$
$$\leq Q\cdot(\|u_1-v_1\|_C + \|u_2-v_2\|_C + \|u_3-v_3\|_C)$$

for all $t,s \in [a,b]$*,* $u_i,v_i \in J \subset \mathbb{R}^m$*,* $i=\overline{1,3}$*.*
If $r \in M_{m1}(\mathbb{R}_+)$ *is a matrix, such that*

$$\left[x \in \overline{B}(f;r)\right] \Longrightarrow [x(t) \in J \subset \mathbb{R}^m]$$

and the following condition is fulfilled:

(iv) $M_K(b-a) \leq r$ *(invariance condition of the sphere* $\overline{B}(f;r)$*),*
where we denoted with $M_K = \begin{pmatrix} M_K^1 \\ \cdots \\ M_K^m \end{pmatrix} \in M_{m1}(\mathbb{R}_+)$ *a matrix having positive constants as elements, such that for the restriction* $K\mid_{[a,b]\times[a,b]\times J^3}$*,* $J \subset \mathbb{R}^m$ *compact subset, we have*

$$\|K(t,s,u_1,u_2,u_3)\|_C \leq M_K,$$

for all $t,s \in [a,b]$*,* $u_1,u_2,u_3 \in J \subset \mathbb{R}^m$*,*
and

(v) $[3(b-a)Q]^n \to 0$ *as* $n \to \infty$*,*
then the system of integral equations (1") *has a unique solution* $x^* \in \overline{B}(f;r) \subset C([a,b],\mathbb{R}^m))$*, that can be obtained by the successive approximations method, starting at any element* $x_0 \in \overline{B}(f;r)$*, and if the n-th successive approximation is* x_n*, then the estimation* (5) *is hold.*

The results obtained in the particular case $\mathbf{B} = l^2(\mathbb{R})$, have been published in (Dobrițoiu M., [12]).

4. Gronwall-type lemmas and comparison results

The theorems presented in this section contain some integral inequalities that are the properties of the solution of the integral equation (1). These results are obtained by applying *the Picard operators technique*, *the Abstract Gronwall Lemma* and *the Abstract Comparison Lemma.*

Let $(\mathbf{B},+,\mathbb{R},|\cdot|)$ be an ordered Banach space.

We consider the integral equation (1), where $K : [a,b]\times[a,b]\times\mathbf{B}^3 \longrightarrow \mathbf{B}$ and $f : [a,b] \longrightarrow \mathbf{B}$.

Theorem 16 (Dobrițoiu M., [14]). *We suppose that:*

(i) $K \in C([a,b]\times[a,b]\times\mathbf{B}^3,\mathbf{B})$*,* $f \in C([a,b],\mathbf{B})$*;*

(ii) $K(t,s,\cdot,\cdot,\cdot)$ *is increasing for all* $t,s \in [a,b]$*;*

(iii) *there exists* $L_K > 0$*, such that*

$$|K(t,s,u_1,v_1,w_1) - K(t,s,u_2,v_2,w_2)| \leq L_K(|u_1-u_2| + |v_1-v_2| + |w_1-w_2|),$$

for all $t,s \in [a,b]$*,* $u_1,u_2,v_1,v_2,w_1,w_2 \in \mathbf{B}$*;*

(iv) $3L_K(b-a) < 1$*,*

and let $x^* \in C([a,b],\mathbf{B})$ *be the unique solution of the integral equation* (1)*.*
Under these conditions

(a) *if* $x \in C([a,b],\mathbf{B})$ *is a lower-solution of the integral equation* (1) *then* $x \leq x^*$*;*
(b) *if* $x \in C([a,b],\mathbf{B})$ *is an upper-solution of the integral equation* (1) *then* $x \geq x^*$*.*

In the proof of this theorem, *the Picard operators technique* and *the Abstract Gronwall Lemma*, have been used.

Remark 4. *The theorem 16 holds true in particular cases* $\mathbf{B} = \mathbb{R}$*,* $\mathbf{B} = \mathbb{R}^m$*,* $\mathbf{B} = l^2(\mathbb{R})$*, if one replaces the conditions (i) and (iii) with conditions which one assures the existence and uniqueness of the solution of the integral equation* (1) *in the space* $C[a,b]$*, in the space* $C([a,b],\mathbb{R}^m)$ *and respectively in the space* $C([a,b],l^2(\mathbb{R}))$*.*

Now, we consider the integral equation (1) corresponding to the functions Ki, fi, $i = 1,2,3$, i.e.

$$x(t) = \int_a^b K_1(t,s,x(s),x(a),x(b))ds + f_1(t) \tag{6}$$

$$x(t) = \int_a^b K_2(t,s,x(s),x(a),x(b))ds + f_2(t) \tag{7}$$

$$x(t) = \int_a^b K_3(t,s,x(s),x(a),x(b))ds + f_3(t) \tag{8}$$

where $K_i : [a,b] \times [a,b] \times \mathbf{B}^3 \longrightarrow B$, $f_i : [a,b] \longrightarrow \mathbf{B}$, $i = 1,2,3$.

Theorem 17 (Dobrițoiu M., [14])**.** *We suppose that the functions* K_i *and* f_i*,* $i = 1,2,3$ *satisfies the following conditions:*

(1) $K_i \in C([a,b] \times [a,b] \times \mathbf{B}^3, \mathbf{B})$*,* $f_i \in C([a,b],\mathbf{B})$*,* $i = 1,2,3$*;*
(2) $K_2(t,s,\cdot,\cdot,\cdot)$ *is increasing for all* $t,s \in [a,b]$*;*
(3) $K_1 \leq K_2 \leq K_3$ *and* $f_1 \leq f_2 \leq f_3$*;*
(4) *there exists* $L_i > 0$*,* $i = 1,2,3$ *such that*

$$|K_i(t,s,u_1,v_1,w_1) - K_i(t,s,u_2,v_2,w_2)| \leq L_i(|u_1-u_2| + |v_1-v_2| + |w_1-w_2|),$$

for all $t,s \in [a,b]$*,* $u_1,u_2,v_1,v_2,w_1,w_2 \in \mathbf{B}$*;*
(5) $3L_i(b-a) < 1$ *,* $i = 1,2,3$*.*

If we denote by x_1^**,* x_2^**, and respectively* x_3^* *the unique solution of the integral equation* (6)*,* (7) *and respectively* (8)*, then*

$$x_1^* \leq x_2^* \leq x_3^*.$$

In the proof of this theorem, *the Picard operators technique* and *the Abstract Comparison Lemma*, have been used.

5. Examples

Example 1 (Dobrițoiu M., [12]). *We consider the integral equation with modified argument*

$$x(t) = \int_0^1 \left(\frac{1}{7} \sin(x(s)) + \frac{x(0)+x(1)}{5} \right) ds + \cos t, \quad t \in [0,1] \tag{9}$$

where
$K \in C([0,1]\times[0,1]\times\mathbb{R}^3)$, $K(t,s,u_1,u_2,u_3) = \frac{1}{7}\sin u_1 + \frac{u_2+u_3}{5}$,
$f \in C[0,1]$, $f(t) = \cos t$ *and*
$x \in C[0,1]$.

We will check the conditions of theorem 10 of existence and uniqueness for the solution of the integral equation (9) in the space $C[0,1]$.
In order to study the existence and uniqueness of the solution of the integral equation (9) in the space $C[0,1]$, we attach to this equation, the operator $A : C[0,1] \to C[0,1]$, defined by the relation:

$$A(x)(t) = \int_0^1 \left(\frac{1}{7} \sin(x(s)) + \frac{x(0)+x(1)}{5} \right) ds + \cos t, \quad t \in [0,1].$$

The set of the solutions of the integral equation (9), in the space $C[0,1]$, coincides with the fixed points set of the operator A, i.e. with F_A.
The function K satisfies the Lipschitz condition with the constant $\frac{1}{7}$ with respect to the third argument, and with the constant $\frac{1}{5}$ with respect to the last two arguments.
Also we obtain that

$$\|A(x) - A(y)\|_{C[0,1]} \le \left(\frac{1}{7} + \frac{1}{5} + \frac{1}{5} \right) \cdot \|x-y\|_{C[0,1]} \cdot \int_0^1 ds = \frac{19}{35} \|x-y\|_{C[0,1]}$$

and it results that the operator A is a contraction with the coefficient $\frac{19}{35}$.
The conditions of the theorem 10 being fulfilled, it results that the integral equation (9) has a unique solution $x^* \in C[0,1]$, that can be obtained by the successive approximations method, starting at any element $x_0 \in C[0,1]$, and if the n-th successive approximation is x_n , then the following estimation is hold:

$$d(x^*, x_n) \le \frac{19^n}{35^{n-1} \cdot 16} d(x_0, x_1). \tag{10}$$

By an analogous reason and applying the theorem 11 it results that the integral equation (9) has a unique solution in the sphere

$$\overline{B}(\cos t; r) = \{x \in C[0,1] \mid \|x - \cos t\|_{C[0,1]} \le r,\ r \in \mathbb{R}_+\},$$

$x^* \in \overline{B}(\cos t; r) \subset C[0,1]$. This solution can be obtained by the successive approximations method, starting at any element $x_0 \in \overline{B}(\cos t; r) \subset C[0,1]$, and if the n-th successive approximation is x_n, then the estimation (10) is hold.

Example 2 (Dobrițoiu M., [12]). *Now, we consider the system of integral equations with modified argument*

$$\begin{cases} x_1(t) = \int_0^1 \left[\frac{2t+1}{15} x_1(s) + \frac{1}{5} x_1(0) + \frac{1}{5} x_1(1)\right] ds + 2t + 1 \\ x_2(t) = \int_0^1 \left[\frac{2t+1}{21} x_2(s) + \frac{1}{7} x_2(0) + \frac{1}{7} x_2(1)\right] ds + \sin t \end{cases}, \quad t \in [0,1] \tag{11}$$

where
$K \in C\left([0,1]\times[0,1]\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2, \mathbb{R}^2\right)$,
$K(t,s,u_1,u_2,u_3) = (K_1(t,s,u_1,u_2,u_3),\ K_2(t,s,u_1,u_2,u_3))$,
$K_1(t,s,u_1,u_2,u_3) = \frac{2t+1}{15} u_{11} + \frac{1}{5} u_{21} + \frac{1}{5} u_{31}$,

$K_2\left(t,s,u_1,u_2,u_3\right)=\frac{2t+1}{21}u_{12}+\frac{1}{7}u_{22}+\frac{1}{7}u_{32},$
$t,s\in[0,1],$
$u_1,u_2,u_3\in\mathbb{R}^2,\ u_1=(u_{11},u_{12}),\ u_2=(u_{21},u_{22}),\ u_3=(u_{31},u_{32}),$
$f\in C([0,1],\mathbb{R}^2),\ f(t)=(f_1(t),f_2(t)),\ f_1(t)=2t+1,\ f_2(t)=\sin t,$ *and*
$x\in C([0,1],\mathbb{R}^2),\ x(t)=(x_1(t),x_2(t)).$

In order to establish the existence and uniqueness conditions of the solution for the system (11), we will apply Theorem 13.
The operator $A: C([0,1],\mathbb{R}^2)\longrightarrow C([0,1],\mathbb{R}^2)$, $A=(A_1,A_2)$, attached to the system (11), defined by the relations:

$$\begin{cases} A_1(x)(t):=\int_0^1\left[\frac{2t+1}{15}x_1(s)+\frac{1}{5}x_1(0)+\frac{1}{5}x_1(1)\right]ds+2t+1 \\ A_2(x)(t):=\int_0^1\left[\frac{2t+1}{21}x_2(s)+\frac{1}{7}x_2(0)+\frac{1}{7}x_2(1)\right]ds+\sin t \end{cases}$$

satisfies the Lipschitz condition with the matrix $Q=\begin{pmatrix}1/5 & 0\\ 0 & 1/7\end{pmatrix}$. From the Theorem 5 it results that the matrix $3(b-a)Q=3Q=\begin{pmatrix}3/5 & 0\\ 0 & 3/7\end{pmatrix}$ converges to zero and therefore, the operator A is a contraction with this matrix.
The conditions of the Theorem 13 being satisfied, it results that the system of integral equations (11) has a unique solution $x^*\in C([0,1],\mathbb{R}^2)$. If x_n is the n-th successive approximation, then the following estimation is hold:

$$\|x^*-x_n\|\le\begin{pmatrix}3/5 & 0\\ 0 & 3/7\end{pmatrix}^n\cdot\begin{pmatrix}2/5 & 0\\ 0 & 4/7\end{pmatrix}^{-1}\cdot\|x_0-x_1\|.$$

Example 3 (Dobriţoiu M., [14])**.** *In the case* $\mathbf{B}=\mathbb{R}$, *we consider the integral equation with modified argument*

$$x(t)=\int_0^1\left(\frac{t}{7}x(s)+\frac{1}{7}x(0)+\frac{1}{5}x(1)\right)ds+\frac{13}{14}t-\frac{1}{5},\quad t\in[0,1] \tag{12}$$

where
$K\in C([0,1]\times[0,1]\times\mathbb{R}^3), K(t,s,u_1,u_2,u_3)=\frac{t}{7}u_1+\frac{1}{7}u_2+\frac{1}{5}u_3,$
$f\in C[0,1],\ f(t)=\frac{13}{14}t-\frac{1}{5},$
$x\in C[0,1],$
and one verifies the conditions of the theorem 16.

The solution of the integral equation (12) is $x^*(t)=t$, $t\in[0,1]$.
We attach to this integral equation the operator $A: C[0,1]\to C[0,1]$, defined by the relation:

$$A(x)(t)=\int_0^1\left(\frac{t}{7}x(s)+\frac{1}{7}x(0)+\frac{1}{5}x(1)\right)ds+\frac{13}{14}t-\frac{1}{5},\quad t\in[0,1].$$

The solutions set of the integral equation (12) in the $C[0,1]$ space, coincides with the fixed points set of the operator A, defined above.
Since the function K satisfies the Lipschitz condition with the constant $\frac{1}{7}$ with respect to the third and fourth arguments, and with the constant $\frac{1}{5}$ with respect to the last argument, it results that the operator A is contraction with the coefficient $\frac{17}{35}$. Therefore A is PO.
According to *Banach's Theorem* it results that the integral equation (12) has a unique solution $x^*\in C[0,1]$. This unique solution is $x^*(t)=t,\ t\in[0,1]$.
Since the function $K(t,s,\cdot,\cdot,\cdot)$ is increasing for all $t,s\in[0,1]$, it results that the conditions

of the theorem 16 are satisfied ($\mathbf{B} = \mathbb{R}$) and therefore, the solution of the integral equation (12) has two properties presented as the following integral inequalities:
– if $x \in C[0,1]$ is a lower-solution of the integral equation (12) then

$$x(t) \leq \int_0^1 \left(\frac{t}{7}x^*(s) + \frac{1}{7}x^*(0) + \frac{1}{5}x^*(1) \right) ds + \frac{13}{14}t - \frac{1}{5}$$

- if $x \in C[0,1]$ is an upper-solution of the integral equation (12) then

$$x(t) \geq \int_0^1 \left(\frac{t}{7}x^*(s) + \frac{1}{7}x^*(0) + \frac{1}{5}x^*(1) \right) ds + \frac{13}{14}t - \frac{1}{5}.$$

These results and examples complete our presentation in the first part of this synthesis survey.

References

[1] Ambro, M., *Aproximarea soluțiilor unei ecuații integrale cu argument modificat*, Studia Univ. Babeș-Bolyai, Cluj-Napoca, Mathematica, **2** (1978), 26–32 (in Romanian).

[2] András, Sz., *Ecuații integrale Fredholm–Volterra*, Ed. Didactică și Pedagogică, București, 2005 (in Romanian).

[3] Berinde, V., *Generalized contractions and applications*, Cub Press 22, Baia Mare, 1997.

[4] Dobrițoiu, M., *Existence and continuous dependence on data of the solution of an integral equation*, *Bulletins for Applied&Computer Mathematics*, Budapest, BAM – CVI / 2005.

[5] Dobrițoiu, M., *The generalization of an integral equation*, Proceedings of the 9th National Conference of the Romanian Mathematical Society, Ed. Univ. de Vest Timișoara, 2005, Seria Alef, ISBN 973-7608-37-2, 392–396.

[6] Dobrițoiu, M., *On an integral equation with modified argument*, Acta Universitatis Apulensis, Alba-Iulia, Mathematics–Informatics, **11** (2006), ISSN 1582-5329, 387–391.

[7] Dobrițoiu, M., *Analysis of an integral equation with modified argument*, Studia Univ. Babeș-Bolyai, Cluj-Napoca, Mathematica, **51** (1) (2006), 81–94.

[8] Dobrițoiu, M., *A Fredholm-Volterra integral equation with modified argument*, Analele Univ. din Oradea, fasc. Matematica, tom XIII, 2006, ISSN 1221-1265, 133–138.

[9] Dobrițoiu, M., *Properties of the solution of an integral equation with modified argument*, Carpathian J. Math., **23** (2007), No. 1-2, 77–80.

[10] Dobrițoiu, M., *Gronwall-type lemmas for an integral equation with modified argument*, Proceedings of ICTAMI 2007, Alba Iulia.

[11] Dobrițoiu, M., *System of integral equations with modified argument*, Carpathian J. Math., **24** (2008), No. 2, 26–36.

[12] Dobrițoiu, M., *Ecuații integrale cu argument modificat*, Presa Universitară Clujeană, Cluj-Napoca, 2009 (in Romanian).

[13] Dobrițoiu, M., A Nonlinear Fredholm Integral Equation, Transylv. J. Math. Mech., **1** 2009, No. 1-2, Edyro Press Publisher, ISSN 2067-239X, 25–32.

[14] Dobrițoiu, M., *Gronwall Type Integral Inequalities via Picard Operators*, Proceedings of the 5th International Conference "Dynamical Systems and Applications ", Ovidius University Annals Series: Civil Engineering, **1** (2009), Special Issue 11, Ovidius University Press, 291–298.

[15] Dobrițoiu, M., *The solution of a system of nonlinear integral*, Transylv. J. Math. Mech., **2** (2010), No. 1, Edyro Press Publisher, ISSN 2067-239X, 89–95.

[16] Precup, R., *Ecuații integrale neliniare*, Univ. Babeș-Bolyai Cluj-Napoca, 1993 (in Romanian).

[17] Rus, I.A., *Principii și aplicații ale teoriei punctului fix*, Ed. Dacia, Cluj-Napoca, 1979 (in Romanian).

[18] Rus, I.A., *Weakly Picard mappings*, Comment. Math. Univ. Caroline, **34** (3) (1993), 769–773.

[19] Rus, I.A., *Picard operators and applications*, Babeș-Bolyai University of Cluj-Napoca, **3** 1996.

[20] Rus, I.A., *Generalized contractions and applications*, Presa Universitară Clujeană, Cluj-Napoca, 2001.

[21] Rus, I.A., *Weakly Picard operators and applications*, Seminar on Fixed Point Theory, Cluj-Napoca, **2** (2001), 41–58.

[22] Rus, I.A., *Picard operators and applications*, Scientiae Mathematicae Japonicae, **58** (2003), 191–219.

[23] Rus, I.A., *Fixed points, upper and lower fixed points: abstract Gronwall lemmas*, Carpathian J. Math., **20** (2004), No. 1, 125–134.

[24] Rus, I.A., *Metric space with fixed point property with respect to contractions*, Studia Univ. Babeș-Bolyai, Cluj-Napoca, Mathematica, **51** (2006), no. 3, 115–121.

[25] Rus, I A., *The theory of a metrical fixed point theorem: theoretical and applicative relevances*, Fixed Point Theory, **9** (2008), No. 2, 541–559.

[26] Șerban, M.A., *Existence and uniqueness theorems for the Chandrasekhar's equation*, Académie Roumaine Filiale de Cluj-Napoca, Mathematica, Tome **41** (64), Nr. 1, 1999, 91–103.

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