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NEW INTEGRAL SOLUTIONS FOR A THERMOELASTIC QUARTER-PLANE

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Abstract. In this paper new influence functions on the thermal displacements and stresses of a unit point heat source for a boundary value problems of thermoelasticity for a quarter-plane were obtained. Also, new integral solutions of Green's for a boundary value problems were derived, thermal stresses caused by the temperature gradient acting on a segment of the boundary line were calculated. All these results are presented in terms of elementary functions for canonical domains of Cartesian system coordinates. Using the computer program Maple 18, the graphical presentations of thermal stresses caused by a unit point of heat source and of thermal stresses for one boundary value problems caused by the temperature gradient acting on a segment of the boundary line were constructed.

1. Introduction

Green's function plays a very important role in finding integral solutions for boundary value problems. Using the Green's functions, the integral solutions for various problems can be determined, but the most difficult in this method is the construction of these functions. If the integral solutions are determined, then thermoelastic displacements and thermal stresses can be derived. Thermoelastic displacements can be calculated using the Maysel's integral formula [1]. In this case the solution of boundary value problems is not represented directly via the known values, but via the temperature field, which must be found. Then a volume integral is calculated. To avoid having to determine the temperature field in [2, 3, 4, 5, 6] the author V. Șeremet has proposed the generalization of the Maysel's and Green's integral formulas in thermoelasticity:

$$u_i(\xi) = a^{-1} \int\limits_V F(x) U_i(x,\xi) dV(x) - \int\limits_{\Gamma_D} T(y) \frac{\partial U_i(y,\xi)}{\partial n_y} d\Gamma_D(y)$$

$$+ \int\limits_{\Gamma_N} \frac{\partial T(y)}{\partial n_y} U_i(y,\xi) d\Gamma_N(y) + a^{-1} \int\limits_{\Gamma_M} \left[\alpha T(y) + a \frac{\partial T(y)}{\partial n_y} \right] U_i(y,\xi) d\Gamma_M(y); i = 1,2,3, \quad (1)$$

where:

Γ_D, Γ_N și Γ_M are the parts of the body surface Γ ($\Gamma = \Gamma_D \cup \Gamma_N \cup \Gamma_M$), which the Dirichlet's boundary conditions (temperature $T(y)$), the Neumann's boundary conditions (heat flux $a\frac{\partial T(y)}{\partial n_y}\Big)$ and mixed boundary conditions (heat exchange between exterior medium and surface of the body represented by law $\left[\alpha T(y) + a\frac{\partial T(y)}{\partial n_y}\right]\Big)$ are prescribed; a is thermal conductivity; $F(x)$ is the internal heat source; α is the coefficient of convective heat conductivity; $\gamma = \alpha_t(2\mu + 3\lambda)$ is the thermoelastic constant; α_t is the coefficient of the linear thermal expansion, but λ, μ are Lame's constants of elasticity.

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If the thermoelastic displacements are determined using formula (1), the temperature field need not to be determined. The thermoelastic displacements can be derived from directly via the prescribed internal heat source, temperature, heat flux or a heat exchange between exterior medium and surface of the body.

The thermal stresses for three-dimensional canonical domains of Cartesian system of coordinates will be calculated by using the following type of Green's integral formula [2]:

$$\sigma_{ij}(\xi) = a^{-1}\int_V F(x)\Sigma_{ij}(x,\xi)dV(x) - \int_{\Gamma_D} T(y)\frac{\partial \Sigma_{ij}(y,\xi)}{\partial n_y}d\Gamma_D(y) + \int_{\Gamma_N} \frac{\partial T(y)}{\partial n_y}\Sigma_{ij}(y,\xi)d\Gamma_N(y)$$
$$+ a^{-1}\int_{\Gamma_M}\left[\alpha T(y) + a\frac{\partial T(y)}{\partial n_y}\right]\Sigma_{ij}(y,\xi)d\Gamma_M(y); i,j = 1,2,3. \quad (2)$$

The matrix $\sigma_{ij}(\xi)$ and $\Sigma_{ij}(x,\xi)$ in Eq. (2) are defined by the components:

$$\sigma_{ij}(\xi) = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}; \quad \Sigma_{ij}(x,\xi) = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{pmatrix}, \quad (3)$$

where:

Σ_{ij} represents the influence functions for thermal stresses of a unit point heat source and σ_{ij} represent the influence functions for thermal stresses caused by internal heat source, temperature, heat flux or a heat exchange between exterior medium and surface of the body.

The thermal stresses Σ_{ij} and σ_{ij} can be determined of Duhamel-Neumann law [7]:

$$\Sigma_{ij} = \mu(U_{i,j} + U_{j.i}) + \delta_{ij}(\lambda\Theta - \gamma G_T); \Theta = U_{k,k}(x,\xi); i,j,k = 1,2,3; \quad (4)$$

$$\sigma_{ij} = \mu(u_{i,j} + u_{j.i}) + \delta_{ij}(\lambda\theta - \gamma T); \theta = u_{k,k}; i,j,k = 1,2,3, \quad (5)$$

where:

δ_{ij} is Kronecker's symbol, if $i = j \longrightarrow 1$ and if $i \neq j \longrightarrow 0$.

2. Thermal stresses Σ_{ij} within quarter-plane of a unit point heat source

It is required to determine thermal stresses $\sigma_{ij}(\xi); i,j = 1,2$ of a particular boundary value problems in the quarter-plane $P(0 \le x_1, x_2 < \infty)$ with boundary thermal conditions Dirichlet. In this body acting the temperature gradient $T = T(y_1, 0)$ on a segment from boundary straight lines $\Gamma_{20}(0 \le y_1 < \infty; y_2 = 0)$:

$$T(y) = \begin{cases} T_{10}(0, y_2) = 0, y \in \Gamma_{10}; \\ T_{20}(y_1, 0) = T_0 = const, y \in (a \le y_1 \le b; y_2 = 0), y \in \Gamma_{20}; 0 \le a < b; \\ T_{20}(y_1, 0) = 0, y \in (0 \le y_1 < a; y_2 = 0) \cup (b < y_1 < \infty; y_2 = 0), y \in \Gamma_{20}. \end{cases} \quad (6)$$

The mechanical boundary conditions:

– on the marginal line $\Gamma_{10}(y_1 = 0; 0 \le y_2 < \infty)$:

$$\Gamma_{10} \longrightarrow \sigma_{11} = \sigma_{12} = 0; \quad (7)$$

– on the marginal line $\Gamma_{20}(0 \le y_1 < \infty; y_2 = 0)$:

$$\Gamma_{20} \longrightarrow u_1 = 0; \sigma_{22} = 0. \quad (8)$$

The mechanical and thermal boundary conditions of a particular boundary value problems within quarter-plane are showed in the Figure 1.

To solve the problem it is necessary to determine thermal stresses $\Sigma_{ij}(x,\xi); i,j = 1,2$ of a unit point heat source, which are obtained if thermoelastic displacements $U_i(x,\xi)$; $i = 1,2$ of a unit point heat source are known.

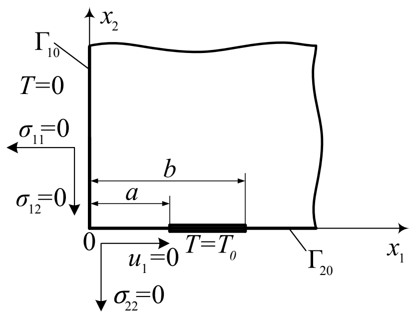

FIGURE 1. The scheme of the quarter-plane with the mechanical boundary conditions σ_{11}, σ_{12}, σ_{22}, u_1 and the thermal boundary conditions T applied on the boundary straight lines Γ_{10} and Γ_{20} for a boundary value problems.

In this article for the first time the thermoelastic displacements $U_i(x,\xi)$; $i=1,2$ and thermal stresses $\Sigma_{ij}(x,\xi); i,j=1,2$ of a unit point heat source using the structural formulas obtained by the method of harmonic integral representations [2] has been proposed. Thermal stresses $\sigma_{ij}(x,\xi); i,j=1,2$ in the quarter-plane P with thermal boundary conditions (6) and mechanical boundary conditions (7), (8) were obtained by the temperature gradient acting on a segment of the boundary line using of a particular integral formula (2). In the field literature [8], [9] boundary value problems within qurter-plane are solved by using ΘG - convolution method, but other mechanical and thermal boundary conditions .

2.1. **Determination of thermoelastic displacements U_i.** In the quarter-plane $P(0 \leq x_1; x_2 < \infty)$ with thermal boundary conditions Dirichlet thermoelastic displacements $U_i(x,\xi); i=1,2$ must be calculated for mechanical and thermal boundary conditions:

– on the marginal line $\Gamma_{10}(y_1=0; 0 \leq y_2 < \infty)$:

$$\Sigma_{11}(x,y) = \Sigma_{12}(x,y) = 0; x \in P; G_T(y,\xi) = 0; y \equiv (0,y_2) \in \Gamma_{10}; \tag{9}$$

– on the marginal line $\Gamma_{20}(0 \leq y_1 < \infty; y_2 = 0)$:

$$\Sigma_{22}(x,y) = 0; U_1(x,y) = 0; x \in P; G_T(y,\xi) = 0; y \equiv (y_1,0) \in \Gamma_{20}. \tag{10}$$

All the mechanical and thermal boundary conditions are showed in Figure 2.

To determine the thermal displacement using the structural formulas $U_i(x,\xi)$ and $\Theta(x,\xi)$ which has been demonstrated in 16 theorem of the monograph [2] for three-dimensional canonical domains of Cartesian system of coordinates with boundary conditions (9) and (10). These structural formulas are valid for a two-dimensional problem. In this case thermoelastic displacements take the following form:

$$U_i(x,\xi) = \frac{\gamma}{2(\lambda+2\mu)}\left[\xi_i G_T(x,\xi) - x_i G_i(x,\xi) - 2\left(x_1\xi_1\frac{\partial}{\partial\xi_i} + \xi_i\right)W_T(x,\xi)\right.$$
$$\left. +2\frac{\partial}{\partial\xi_i}\left(\int \xi_1 W_T(x,\xi)d\xi_1 - \frac{\mu}{\lambda+\mu}(\delta_{2i}-\delta_{1i})x_1\int W_T(x,\xi)d\xi_1\right)\right]; i=1,2, \tag{11}$$

where:

$W_T(x,\xi)$ is regular part of the Green's functions $G_T(x,\xi)$;

$\delta_{1i}; \delta_{2i}$ are Kronecker's symbols;

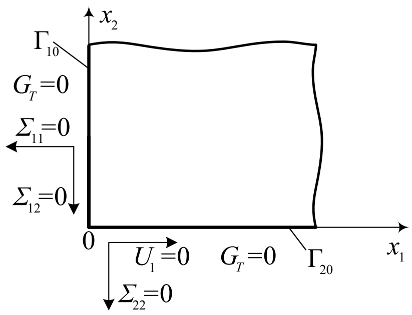

FIGURE 2. The scheme of the quarter-plane with the mechanical boundary conditions Σ_{11}, Σ_{12}, Σ_{22}, U_1 and the thermal boundary conditions G_T applied on the boundary straight lines Γ_{10} and Γ_{20}.

and volume dilatation:

$$\Theta(x,\xi) = \frac{\gamma}{\lambda+2\mu}\left(G_T(x,\xi) + \frac{2\mu}{\lambda+\mu}x_1\frac{\partial}{\partial x_1}W_T(x,\xi)\right). \tag{12}$$

Green's functions G_T, G_Θ and $G_i; i=1,2$ are connected with the boundary conditions (9) and (10) as follows: if on the marginal line a quarter-plane P thermal stresses are known then derivatives of Green's functions are equal to zero, and if on the marginal line thermoelastic displacements are known then Green's functions are equal to zero:

$$\Sigma_{11} = \Sigma_{12} = 0; G_T = 0 \Rightarrow G_{1,1} = G_{2,1} = G_{\Theta,1} = 0, \tag{13}$$

on the marginal line $\Gamma_{10}(y_1 = 0; 0 \le y_2 < \infty)$ and

$$\Sigma_{22} = 0; U_1 = 0; G_T = 0 \Rightarrow U_{1,1} = U_{2,2} = 0 \Rightarrow \Theta = 0; G_1 = G_{2,2} = G_\Theta = 0, \tag{14}$$

on the marginal line $\Gamma_{20}(0 \le y_1 < \infty; y_2 = 0)$.

Green's functions $G_T; G_\Theta; G_1$ and G_2 for quarter-plane P are extracted from handbook [3] or encyclopedia [10] and these functions will be calculated by the following expressions:

$$G_T(x,\xi) = G^{(1)}(x,\xi) = \frac{1}{4\pi}\ln\frac{r_1r_2}{rr_{12}}; \tag{15}$$

$$G_\Theta(x,\xi) = G_1(x,\xi) = G^{(4)}(x,\xi) = \frac{1}{4\pi}\ln\frac{r_2r_{12}}{rr_1}; \tag{16}$$

$$G_2(x,\xi) = G^{(2)}(x,\xi) = -\frac{1}{4\pi}\ln r_2r_{12}rr_1 + c, \tag{17}$$

where:
$r = (x_1-\xi_1)^2 + (x_2-\xi_2)^2$; $r_1 = (x_1+\xi_1)^2 + (x_2-\xi_2)^2$;
$r_2 = (x_1-\xi_1)^2 + (x_2+\xi_2)^2$; $r_{12} = (x_1+\xi_1)^2 + (x_2+\xi_2)^2$.

Green's function $G_2(x,\xi) = G^{(2)}(x,\xi)$ contains an undetermined constant c, because the solution to this boundary value problems is characterized by the indetermination, but the result of this problem is obtained with precision of a constant.

The regular part of the Green's function $G_T(x,\xi)$ (15) is that part which contains inferior index 1 (that part of the $G_T(x,\xi)$ which are reflected via marginal line Γ_{10}. So, $W_T(x,\xi)$ of the formulas (11) and (12) is calculated with the following equation:

$$W_T(x,\xi) = \frac{1}{4\pi}\ln\frac{r_1}{r_{12}}. \tag{18}$$

Substituting expressions (15),(16), (17), (18) in the formula (11) and undetermined constant c of Green's function $G_2(x,\xi) = G^{(2)}(x,\xi)$ has been taken equal to zero $c = 0$. The final expressions for thermoelastic displacements $U_i(x,\xi); i = 1,2$ of a unit point heat source for quarter-plane P are presented in the following formulas:

$$U_1(x,\xi) = \frac{\gamma}{8\pi(\lambda+2\mu)}\left[(x_1+\xi_1)\ln\frac{r_1}{r_{12}} + (x_1-\xi_1)\ln\frac{r}{r_2} + 2x_1\left(\frac{\mu}{\lambda+\mu} + \xi_1\frac{\partial}{\partial x_1}\right)\ln\frac{r_1}{r_{12}}\right] \tag{19}$$

$$U_2(x,\xi) = \frac{\gamma}{8\pi(\lambda+2\mu)}\left[(x_2+\xi_2)\ln(r_1r_2) + (x_2-\xi_2)\ln(rr_{12}) - 2\left(x_1\xi_1\frac{\partial}{\partial\xi_2} + \xi_2\right)\ln\frac{r_1}{r_{12}}\right.$$
$$\left. + 2\frac{\partial}{\partial\xi_2}\left(\int\xi_1\ln\frac{r_1}{r_{12}}d\xi_1 - \frac{\mu}{\lambda+\mu}x_1\int\ln\frac{r_1}{r_{12}}d\xi_1\right)\right]. \tag{20}$$

2.2. **Determination of thermal stresses** Σ_{ij}**.** The thermal stresses $\Sigma_{ij}(x,\xi)$ are calculated using Duhamel-Neumann law (4), which for two-dimensional problems can be rewritten in the following form:

$$\Sigma_{ij} = \mu(U_{i,j} + U_{j.i}) + \delta_{ij}(\lambda\Theta - \gamma G_T); \Theta = U_{k,k}(x,\xi); i,j,k = 1,2, \tag{21}$$

where: $\Theta(x,\xi)$ - thermoelastic volume dilatation determined by the formula (12):

$$\Theta(x,\xi) = \frac{\gamma}{4\pi(\lambda+2\mu)}\left(\ln\frac{r_1r_2}{rr_{12}} + \frac{2\mu}{\lambda+\mu}x_1\frac{\partial}{\partial x_1}\ln\frac{r_1}{r_{12}}\right). \tag{22}$$

Substituting Green's function $G_T(x,\xi)$ (15), thermoelastic volume dilatation $\Theta(x,\xi)$ (22) and expressions for thermoelastic displacements $U_i(x,\xi); i = 1,2$ of a unit point heat source (19)-(20) in the Duhamel-Neumann law (21), we obtain the expressions for main thermoelastic functions for thermal stresses $\Sigma_{ij}(x,\xi)$:

$$\Sigma_{11}(x,\xi) = \frac{\gamma\mu}{4\pi(\lambda+2\mu)}\left(\ln\frac{rr_{12}}{r_1r_2} + (x_1+\xi_1)\frac{\partial}{\partial\xi_1}\ln\frac{r_1}{r_{12}}\right.$$
$$\left. +(x_1-\xi_1)\frac{\partial}{\partial\xi_1}\ln\frac{r}{r_2} - 2x_1\xi_1\frac{\partial^2}{\partial\xi_1^2}\ln\frac{r_1}{r_{12}}\right); \tag{23}$$

$$\Sigma_{22}(x,\xi) = \frac{\gamma\mu}{4\pi(\lambda+2\mu)}\left(\ln\frac{rr_{12}}{r_1r_2} + (x_2+\xi_2)\frac{\partial}{\partial\xi_2}\ln(r_{12}r_2)\right.$$
$$\left. +(x_2-\xi_2)\frac{\partial}{\partial\xi_2}\ln(rr_1) - 2x_1\xi_1\frac{\partial^2}{\partial\xi_2^2}\ln\frac{r_1}{r_{12}} + 2(x_1-\xi_1)\frac{\partial}{\partial\xi_1}\ln\frac{r_1}{r_{12}}\right); \tag{24}$$

$$\Sigma_{12}(x,\xi) = \frac{\gamma\mu}{4\pi(\lambda+2\mu)}\left((x_1-\xi_1)\frac{\partial}{\partial\xi_2}\ln\frac{rr_{12}}{r_2r_1} + (x_2+\xi_2)\frac{\partial}{\partial\xi_2}\ln(r_{12}r_2)\right.$$
$$\left. +(x_2-\xi_2)\frac{\partial}{\partial\xi_2}\ln(rr_1) + 2\xi_1\frac{\partial}{\partial\xi_2}\ln\frac{r_1}{r_{12}} - 4x_1\xi_1\frac{\partial}{\partial\xi_1}\frac{\partial}{\partial\xi_2}\ln\frac{r_1}{r_{12}}\right). \tag{25}$$

The graphics of normal thermal stresses $\Sigma_{11}(x,\xi)$, $\Sigma_{22}(x,\xi)$ and tangential thermal stresses $\Sigma_{12}(x,\xi)$ constructed by using computer program Maple 18 are presented in the Figures 3,*a*, 4,*a* și 5,*a* of the Appendix.

3. Explicit thermal stresses σ_{ij} of a particular boundary value problem within quarter-plane

According to thermal boundary conditions (6): heat flux $a\frac{\partial T(y)}{\partial n_y} = 0$, heat exchange between exterior medium and surface of the body $\left[\alpha T(y) + a\frac{\partial T(y)}{\partial n_y}\right] = 0$. Quarter-plane P is acting the temperature gradient $T = T(y_1, 0)$ on a segment from boundary straight lines $\Gamma_{20}(0 \leq y_1 < \infty; y_2 = 0)$, so, the internal heat source $F(x) = 0$. In this case the type of Green's integral formula (2) has the following form:

$$\sigma_{ij}(\xi) = -\int_0^\infty T_{20}(y_1, 0) Q_{ij}(y_1, 0; \xi) dy_1, \tag{26}$$

where:

$$Q_{ij}(y_1, 0; \xi) = (\partial/\partial n_{y_2})\Sigma_{ij}(y, \xi). \tag{27}$$

The matrix of thermal stresses $\sigma(\xi)$ and $\Sigma(x, \xi)$ are defined by the components:

$$\sigma_{ij}(\xi) = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}; \quad \Sigma_{ij}(\xi) = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}. \tag{28}$$

One by one substituting expressions (23) - (25) in the formula (26), we obtain the formulas for thermal stresses:

$$\sigma_{11}(\xi) = \frac{\gamma\mu}{2\pi(\lambda + 2\mu)} T_0 \int_a^b \frac{\partial}{\partial \xi_2} \left[\ln\frac{r_{10}}{r_0} - (y_1 + \xi_1)\frac{\partial}{\partial \xi_1}\ln r_{10}\right.$$
$$\left. -(y_1 - \xi_1)\frac{\partial}{\partial \xi_1}\ln r_0 + 2y_1\xi_1\frac{\partial^2}{\partial \xi_1^2}\ln r_{10}\right] dy_1, \tag{29}$$

$$\sigma_{22}(\xi) = \frac{\gamma\mu}{2\pi(\lambda + 2\mu)} T_0 \int_a^b \left[2\frac{\partial}{\partial \xi_2}\ln r_{10} - \xi_2\frac{\partial^2}{\partial \xi_1^2}\ln(r_{10}r_0)\right.$$
$$\left. -2y_1\xi_1\frac{\partial^2}{\partial \xi_1^2}\frac{\partial}{\partial \xi_2}\ln r_{10} - 2(y_1 - \xi_1)\frac{\partial}{\partial \xi_1}\frac{\partial}{\partial \xi_2}\ln r_{10}\right] dy_1, \tag{30}$$

$$\sigma_{12}(\xi) = \frac{\gamma\mu}{2\pi(\lambda + 2\mu)} T_0 \int_a^b \left[2\xi_1\frac{\partial^2}{\partial \xi_1^2}\ln r_{10} - (y_1 - \xi_1)\frac{\partial^2}{\partial \xi_1^2}\ln\frac{r_{10}}{r_0}\right.$$
$$\left. +\frac{\partial}{\partial \xi_1}\ln(r_{10}r_0) + \xi_2\frac{\partial}{\partial \xi_1}\frac{\partial}{\partial \xi_2}\ln(r_{10}r_0) + 4y_1\xi_1\frac{\partial}{\partial \xi_1}\frac{\partial^2}{\partial \xi_2^2}\ln r_{10}\right] dy_1, \tag{31}$$

where:

$$r_0 = r(y_1, 0; \xi) = (y_1 - \xi_1)^2 + \xi_2^2; r_{10} = r_1(y_1, 0; \xi) = (y_1 + \xi_1)^2 + \xi_2^2. \tag{32}$$

After solving the integrals from the formulas (29) - (31) we obtain the following final analytical expressions for thermal stresses $\sigma_{ij}(\xi)$ in the quarter-plane P caused by the temperature gradient T_0 acting on a segment $a \leq y_1 \leq b$ of the boundary line Γ_{20}:

$$\sigma_{11}(\xi) = \frac{\gamma\mu T_0}{2\pi(\lambda + 2\mu)}\frac{\partial}{\partial \xi_2}\left[(y_1 - \xi_1)\ln\frac{r_{10}}{r_0}\right.$$
$$\left.\left. +4\xi_2\left(\arctan\frac{y_1 + \xi_1}{\xi_2} - \arctan\frac{y_1 - \xi_1}{\xi_2}\right) + 2\xi_1\left(y_1\frac{\partial}{\partial \xi_1}\ln r_{10} - 4\right)\right]\right|_a^b, \tag{33}$$

$$\sigma_{22}(\xi) = \frac{\gamma\mu T_0}{2\pi(\lambda+2\mu)}\left\{8\arctan\frac{y_1+\xi_1}{\xi_2} - \xi_2\frac{\partial}{\partial\xi_1}\ln\frac{r_{10}}{r_0}\right.$$
$$\left.-2\left[y_1 - \xi_1\left(2 - y_1\frac{\partial}{\partial\xi_1}\right)\right]\frac{\partial}{\partial\xi_2}\ln r_{10}\right\}\Bigg|_a^b, \quad (34)$$

$$\sigma_{12}(\xi) = \frac{\gamma\mu T_0}{2\pi(\lambda+2\mu)}\left[6\xi_1\frac{\partial}{\partial\xi_1}\ln r_{10} - (y_1-\xi_1)\frac{\partial}{\partial\xi_1}\ln(r_{10}r_0)\right.$$
$$\left.+\left(2+\xi_2\frac{\partial}{\partial\xi_2}\right)\ln\frac{r_{10}}{r_0} + 4y_1\xi_1\frac{\partial^2}{\partial\xi_2^2}\ln r_{10}\right]\Bigg|_a^b. \quad (35)$$

The graphics of normal thermal stresses $\sigma_{11}(\xi)$, $\sigma_{22}(\xi)$ and tangential thermal stresses $\sigma_{12}(\xi)$ constructed by using computer program Maple 18 are presented in the Figures 3,*b*, 4,*b* și 5,*b* of the Appendix.

4. Conclusions

The expressions for thermoelastic displacements $U_i(x,\xi); i = 1,2$ (19) and (20), thermal stresses $\Sigma_{ij}(x,\xi); i,j = 1,2$ (23) - (25) and $\sigma_{ij}(\xi); i,j = 1,2$ (33) - (35) in the quarter-plane for the boundary conditions (6), (7) and (8) were obtained for the firts time. All expressions are presented in terms of elementary functions. In determining the thermal stresses $\sigma_{ij}(\xi); i,j = 1,2$ for the particular problem it was not necessary to establish the temperature field and use Maysel's formula, and then to calculate the integral volume, but the thermal stresses $\Sigma_{ij}(x,\xi); i,j = 1,2$ of a unit point heat source and the temperature gradient T_0 acting of the boundary line, then the integral surface is solved. All thermal stresses $\Sigma_{ij}(x,\xi); i,j = 1,2$ and $\sigma_{ij}(\xi); i,j = 1,2$ are presented graphically using computer program Maple 18 and they are included into the Appendix. Using the thermal stresses $\Sigma_{ij}(x,\xi); i,j = 1,2$ (23) - (25) of a unit point heat source, the thermal stresses for a particular boundary value problems can be determined caused by the internal heat source and/or the temperature gradient applied to one or both line in the quarter-plane of the boundary conditions indicated. Using the thermal stresses $\sigma_{ij}(\xi); i,j = 1,2$ (33) - (35), it is easy to determine the thermal stresses by the temperature gradient of any value, applied within any segment of the line Γ_{20}.

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Appendix. Graphics of normal thermal stresses $\Sigma_{11}, \Sigma_{22}, \sigma_{11}, \sigma_{22}$ and tangential thermal stresses Σ_{12}, σ_{12} in the thermoelastic quarter-plane, caused by the unit point heat source and by the temperature gradient

Graphics of the thermal stresses $\Sigma_{11}(x,\xi), \Sigma_{22}(x,\xi)$ and $\Sigma_{12}(x,\xi)$ caused by a unit point heat source applied in the point $x_1 = 5m, x_2 = 5m$ within quarter-plane P were constructed using computer program Maple 18. Graphics of the thermal stresses $\sigma_{11}(\xi), \sigma_{22}(\xi)$ and $\sigma_{12}(\xi)$ caused by the temperature gradient $T_0 = 50K$ acting on a segment $a \le y_1 \le b, (a = 4m, b = 6m)$ of the boundary line $\Gamma_{20}(0 \le y_1 < \infty; y_2 = 0)$ were constructed using computer program Maple 18. The value of elastic and thermal constants are: the Poisson ration $\nu = 0,3$; modulus of elasticity $E = 2,1 \cdot 10^5 MPa$ and coefficient of linear thermal expansion $\alpha_t = 1,2 \cdot 10^{-5}(K^{-1})$.

Normal thermal stresses $\Sigma_{11}(x,\xi)$ of a unit heat source applied in the point $x_1 = 5m, x_2 = 5m$ calculated by the formula (23) are presented in the Figure 3, *a*. Normal thermal stresses $\sigma_{11}(\xi)$ caused by the temperature gradient $T_0 = 50K$ calculated by the formula (33) are presented in the Figure 3, *b*.

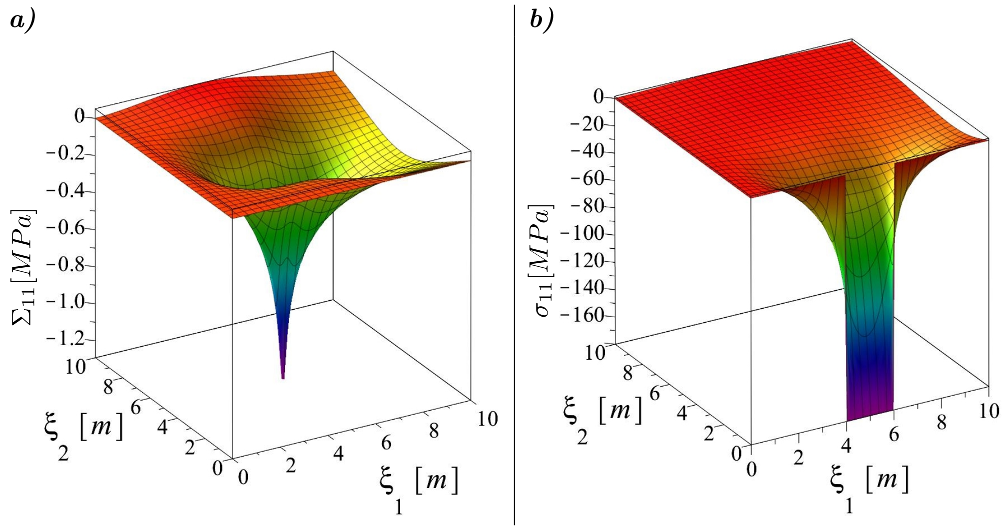

Figure 3. Graphics of normal thermal stresses $\Sigma_{11}(x,\xi)$ and $\sigma_{11}(\xi)$ in the quarter-plane P in dependence of $0 \le \xi_1, \xi_2 \le 10$, caused by a unit heat source applied in the point $x_1 = 5m, x_2 = 5m$ - Figure 3, *a*; and by the temperature gradient $T_0 = 50K$ on the segment $4m \le y_1 \le 6m$ of the boundary line Γ_{20} - Figure 3, *b*.

Analyzing the Figure 3 graphics one can observe the following:

- the boundary conditions are respected: on the marginal line $\Gamma_{10}(\xi_1 = 0; 0 \le \xi_2 < \infty) \to \Sigma_{11}(x,\xi) = 0$ (Figure 3, *a*); $\sigma_{11}(\xi) = 0$ (Figure 3, *b*);
- graphics have a local maximum: in the point of application of the unit heat source $x_1 = 5m, x_2 = 5m$ (Figure 3, *a*) and on the segment $4m \le y_1 \le 6m$ on the marginal line Γ_{20} by the temperature gradient (Figure 3, *b*).

Normal thermal stresses $\Sigma_{22}(x,\xi)$ of a unit heat source applied in the point $x_1 = 5m, x_2 = 5m$ calculated by the formula (24) are presented in the Figure 4, *a*. Normal thermal stresses $\sigma_{22}(\xi)$ caused by the temperature gradient $T_0 = 50K$ calculated by the formula (34) are presented in the Figure 4, *b*.

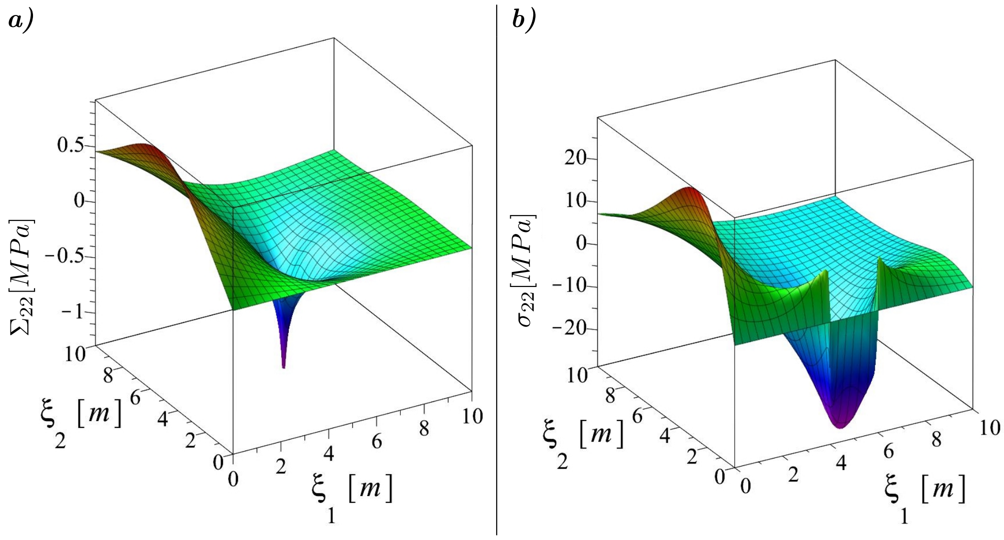

FIGURE 4. Graphics of normal thermal stresses $\Sigma_{22}(x,\xi)$ and $\sigma_{22}(\xi)$ in the quarter-plane P in dependence of $0 \leq \xi_1, \xi_2 \leq 10$, caused by a unit heat source applied in the point $x_1 = 5m, x_2 = 5m$ - Figure 4, *a*; and by the temperature gradient $T_0 = 50K$ on the segment $4m \leq y_1 \leq 6m$ of the boundary line Γ_{20} - Figure 4, *b*.

Analyzing the Figure 4 graphics one can observe the following:

- the boundary conditions are respected: on the marginal line $\Gamma_{20}(0 \leq \xi_1 < \infty; \xi_2 = 0) \to \Sigma_{22}(x,\xi) = 0$ (Figure 4, *a*); $\sigma_{22}(\xi) = 0$ (Figure 4, *b*);
- the graphic (Figure 4, *a*) has a local maximum in the point of application of the unit heat source $x_1 = 5m, x_2 = 5m$; and the graphic (Figure 4, *b*) has a discontinuity near the points $a = 4m, b = 6m$ on the segment $4m \leq y_1 \leq 6m$ of the boundary line Γ_{20}.

Tangential thermal stresses $\Sigma_{12}(x,\xi)$ of a unit heat source applied in the point $x_1 = 5m, x_2 = 5m$ calculated by the formula (25) are presented in the Figure 5, *a*. Normal thermal stresses $\sigma_{12}(\xi)$ caused by the temperature gradient $T_0 = 50K$ calculated by the formula (35) are presented in the Figure 5, *b*.

Analyzing the Figure 5 graphics one can observe the following:

- the boundary conditions are respected: on the marginal line $\Gamma_{10}(\xi_1 = 0; 0 \leq \xi_2 < \infty) \to \Sigma_{12}(x,\xi) = 0$ (Figure 5, *a*); $\sigma_{12}(\xi) = 0$ (Figure 5, *b*);
- the graphic (Figure 5, *a*) has a discontinuity in the point of application of the unit heat source $x_1 = 5m, x_2 = 5m$; and the graphic (Figure 5, *b*) has a local maximum at the points $a = 4m, b = 6m$ on the segment $4m \leq y_1 \leq 6m$ of the boundary line Γ_{20} .

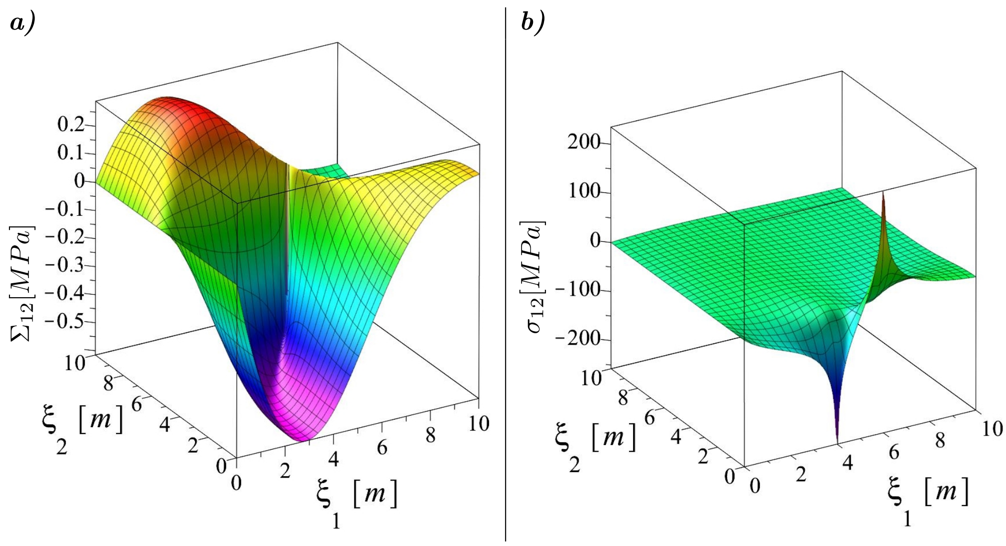

FIGURE 5. Graphics of tangential thermal stresses $\Sigma_{12}(x, \xi)$ and $\sigma_{12}(\xi)$ in the quarter-plane P in dependence of $0 \leq \xi_1, \xi_2 \leq 10$, caused by a unit heat source applied in the point $x_1 = 5m, x_2 = 5m$ - Figure 5, a; and by the temperature gradient $T_0 = 50K$ on the segment $4m \leq y_1 \leq 6m$ of the boundary line Γ_{20} - Figure 5, b.

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