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UNIVALENCE CRITERIA FOR ANALYTIC FUNCTIONS DEFINED BY CERTAIN GENERALIZED DIFFERENTIAL OPERATOR

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ABSTRACT. In this paper, we consider the generalized differential operator $D^n_{\alpha,\mu}(\lambda,\omega)$ $f(z)$ and we determine some sufficient conditions for univalence of analytic functions defined by this operator. Relevant connections of some results obtained with those in earlier works are also given.

1. Introduction

Let A be the class of functions which are analytic in the open unit disk $U=\{z:|z|<1\}$ given by

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j, z\in U. \tag{1}$$

We denote by S the subclass of A consisting of univalent functions.

For the function $f\in A$ on the form (1), we will use the following differential operator:

$$\begin{aligned}
D^0f(z)&=f(z)=z+\sum_{k=2}^{\infty}a_kz^k,\\
D^1_{\alpha,\mu}(\lambda,\omega)f(z)&=(\alpha-\mu\omega^\lambda)f(z)+(\mu\omega^\lambda-\alpha+1)zf^{'}(z)\\
&=z+\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]a_jz^j,\\
D^2_{\alpha,\mu}(\lambda,\omega)f(z)&=D(D^1_{\alpha,\mu}(\lambda,\omega)f(z)),\\
&\vdots\\
D^n_{\alpha,\mu}(\lambda,\omega)f(z)&=D(D^{n-1}_{\alpha,\mu}(\lambda,\omega)f(z))=z+\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^na_jz^j, \qquad (2)
\end{aligned}$$

for $\alpha,\mu,\lambda,\omega\geq 0, n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, with $D^n_{\alpha,\mu}(\lambda,\omega)f(0)=0$.

The operator $D^n_{\alpha,\mu}(\lambda,\omega)f(z)$ was introduced by [3]. We remark that for $\alpha=\mu=0$, we get Sălăgean's differential operator [14], for $\lambda=1,\omega=1$ and $\alpha=1$ we get the operator introduced by F. Al-Oboudi [1], when $\lambda=1,\omega=1$ we obtain the operator introduced by M. Darus and R.W. Ibrahim [5], if $\alpha=1$ we get the operator introduced by M. Darus and I. Faisal [4].

In this paper we derive certain sufficient conditions of univalence for the generalized differential operator $D^n_{\alpha,\mu}(\lambda,\omega)f(z)$. Also, a number of known univalent conditions would

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follow upon specializing the parameters involved. Recently, the problem of univalence of some differential operators have been discussed by many authors (see [6]).

One of the most important and known univalence criteria for analytic functions defined in the open unit disk U was proved by Becker [2].

Lemma 1 (Becker [2]). *Let $f \in A$. If for all $z \in U$*

$$(1-|z|^2)\left|\frac{zf^{''}(z)}{f'(z)}\right| \le 1, \tag{3}$$

then the function f is univalent in U.

An improvement of Becker's univalence criterion was given by Tudor [12] and Pascu [9].

Lemma 2 (Tudor [12]). *Let η be a real number, $\eta > \frac{1}{2}$ and $f \in A$. If for all $z \in U$*

$$\left|(1-|z|^{2\eta})\frac{zf^{''}(z)}{f'(z)} + 1 - \eta\right| \le \eta, \tag{4}$$

then the function f is univalent in U.

Lemma 3 (Pascu [9]). *Let γ be a complex number, $Re\gamma > 0$ and the function $f \in A$. If for all $z \in U$*

$$\frac{(1-|z|^{2Re\gamma})}{Re\gamma}\left|\frac{zf^{''}(z)}{f'(z)}\right| \le 1, \tag{5}$$

then for any complex number δ, $Re\delta \ge Re\gamma$, the function

$$F_\delta(z) = \left(\delta\int_0^z u^{\delta-1}f^{'}(u)du\right)^{\frac{1}{\delta}} \tag{6}$$

is univalent in U.

A new generalization of Ahlfors's and Becker's criterion of univalence was given by Pescar [10].

Lemma 4 (Pescar [10]). *Let γ, c be complex numbers with $Re\gamma > 0$ and $|c| \le 1$, $c \neq -1$. If $f \in A$ satisfies*

$$\left|c|z|^{2\gamma} + (1-|z|^{2\gamma})\frac{zf^{''}(z)}{\gamma f'(z)}\right| \le 1, \tag{7}$$

for all $z \in U$, then the function $H_\gamma(z)$ defined by

$$H_\gamma(z) = \left(\gamma\int_0^z u^{\gamma-1}f^{'}(u)du\right)^{\frac{1}{\gamma}} \tag{8}$$

is univalent in U.

Yet another univalent condition was given by Ozaki and Nunokawa [8].

Lemma 5 (Ozaki and Nunokawa [8]). *Let $f \in A$ satisfy the following inequality:*

$$\left|\frac{z^2f^{'}(z)}{f^2(z)} - 1\right| \le 1\,(z \in U). \tag{9}$$

Then the function f is univalent in U.

Pescar [11], on the other hand, proved another univalent condition.

Lemma 6 (Pescar [11])**.** *Let $f \in A$ satisfy the inequality (9). Also let $c \in \mathbb{C}$ and $\eta \in \mathbb{R}$ such that $1 \le \eta \le \frac{3}{2}$. If*

$$|c| \le \frac{3-2\eta}{\eta} \text{ and } |f(z)| \le 1 \tag{10}$$

then the function

$$T_\eta(z) = \left(\eta \int_0^z (f(u))^{\eta-1} du\right)^{\frac{1}{\eta}} \tag{11}$$

is univalent in U.

Lemma 7. [7] *Let $f \in S$. If for all $z \in U$,*

$$\frac{z}{f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k, \tag{12}$$

then

$$\sum_{k=1}^{\infty} (k-1)|b_k|^2 \le 1. \tag{13}$$

2. Main results

In this section, we give sufficient conditions to obtain a univalence for analytic functions involving the differential operator $D^n_{\alpha,\mu}(\lambda,\omega)$.

Theorem 1. *Let $f \in A$. If for all $z \in U$*

$$\sum_{j=2}^{\infty} [(j-1)(\mu\omega^\lambda - \alpha) + j]^n j(2j-1)|a_j| \le 1, \tag{14}$$

then the operator $D^n_{\alpha,\mu}(\lambda,\omega)f(z)$ is in the class S.

Proof. Assume that $f \in A$ and the inequality (14) take place. Then, for all $z \in U$, we have

$$\begin{aligned}
(1-|z|^2)\left|\frac{z(D^n_{\alpha,\mu}(\lambda,\omega)f(z))''}{(D^n_{\alpha,\mu}(\lambda,\omega)f(z))'}\right| &\le (1+|z|^2)\left|\frac{z(D^n_{\alpha,\mu}(\lambda,\omega)f(z))''}{(D^n_{\alpha,\mu}(\lambda,\omega)f(z))'}\right| \\
&= (1+|z|^2)\left|\frac{z\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j(j-1)a_j z^{j-2}}{1+\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j a_j z^{j-1}}\right| \\
&\le \frac{2\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j(j-1)|a_j|}{1-\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j|a_j|} \le 1.
\end{aligned} \tag{15}$$

Thus, in view of Lemma 1, $D^n_{\alpha,\mu}(\lambda,\omega)f(z)$ is in the class S. □

Theorem 2. *Let $f \in A$ satisfy the following inequality:*

$$\sum_{j=2}^{\infty} j[2(j+\eta-1)-1][(j-1)(\mu\omega^\lambda-\alpha)+j]^n|a_j| \le 2\eta - 1, \tag{16}$$

for all $z \in U$ and $\eta > \frac{1}{2}$ fixed. Then $D^n_{\alpha,\mu}(\lambda,\omega)f(z)$ is in the class S.

Proof. Let suppose that $f \in A$. Then, if (16) take place, for all $z \in U$ and $\eta > \frac{1}{2}$, we have

$$\left|(1-|z|^{2\eta})\frac{z(D^n_{\alpha,\mu}(\lambda,\omega)f(z))''}{(D^n_{\alpha,\mu}(\lambda,\omega)f(z))'}+1-\eta\right| \le (1+|z|^2)\left|\frac{z(D^n_{\alpha,\mu}(\lambda,\omega)f(z))''}{(D^n_{\alpha,\mu}(\lambda,\omega)f(z))'}\right|+|1-\eta|$$

$$= (1+|z|^2)\left|\frac{\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j(j-1)a_j z^{j-1}}{1+\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j a_j z^{j-1}}\right|+|1-\eta|$$

$$\le \frac{2\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j(j-1)|a_j|}{1-\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j|a_j|}+|1-\eta| \le \eta.$$

Therefore, by applying Lemma 2 we obtain the required result. □

Theorem 3. *Let $f \in A$ satisfy the following inequality:*

$$\sum_{j=2}^{\infty} j[2(j-1)+\Re\gamma][(j-1)(\mu\omega^\lambda-\alpha)+j]^n|a_j| \le Re\gamma \tag{17}$$

where $z \in U$ and $Re\gamma > 0$. Then for any complex number δ, $Re\delta \ge Re\gamma$, the integral operator G_δ given by

$$G_\delta(z) = \left(\delta\int_0^z u^{\delta-1}[D^n_{\alpha,\mu}(\lambda,\omega)f(u)]'du\right)^{\frac{1}{\delta}} \tag{18}$$

is in the class S.

Proof. Let suppose that $f \in A$ and the inequality (17) take place. Then, for all $z \in U$, we have

$$\frac{(1-|z|^{2Re\gamma})}{Re\gamma}\left|\frac{z(D^n_{\alpha,\mu}(\lambda,\omega)f(z))''}{(D^n_{\alpha,\mu}(\lambda,\omega)f(z))'}\right| \le \frac{(1+|z|^{2Re\gamma})}{Re\gamma}\left|\frac{z(D^n_{\alpha,\mu}(\lambda,\omega)f(z))''}{(D^n_{\alpha,\mu}(\lambda,\omega)f(z))'}\right|$$

$$= \frac{(1+|z|^{2Re\gamma})}{Re\gamma}\left|\frac{\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j(j-1)a_j z^{j-1}}{1+\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j a_j z^{j-1}}\right|$$

$$\le \frac{2\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j(j-1)|a_j|}{Re\gamma(1-\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j|a_j|)} \le 1. \tag{19}$$

Thus, in view of Lemma 3, G_γ is in the class S. □

For $\gamma = \delta$ we have the following:

Corollary 1. *Let $f \in A$ satisfy the following inequality:*

$$\sum_{j=2}^{\infty} j[2(j-1)+\Re(\gamma)][(j-1)(\mu\omega^\lambda-\alpha)+j]^n|a_j| \le Re(\gamma) \tag{20}$$

where $z \in U$ and $Re(\gamma) > 0$. Then

$$G_\gamma(z) = \left(\gamma\int_0^z u^{\gamma-1}[D^n_{\alpha,\mu}(\lambda,\omega)f(u)]'du\right)^{\frac{1}{\gamma}} \tag{21}$$

is in the class S.

Theorem 4. *Let $f \in A$ satisfy the following equality:*

$$\sum_{j=2}^{\infty} j(j-1)[(j-1)(\mu\omega^\lambda-\alpha)+j]^n a_j z^{j-1} = 0, \tag{22}$$

where $z \in U$. If $Re(\gamma) > 0$ then the integral operator P_γ given by

$$P_\gamma(z) = \left(\gamma \int_0^z u^{\gamma-1}[D^n_{\alpha,\mu}(\lambda,\omega)f(u)]^{'} du\right)^{\frac{1}{\gamma}} \tag{23}$$

is in the class S.

Proof. Let suppose that $f \in A$ and (22) take place. Then, for all $z \in U$, c and γ complex numbers with $|c| \leq 1$, $c \neq -1$ and $Re\gamma > 0$, we have

$$\left|c|z|^{2\gamma} + (1-|z|^{2\gamma})\frac{z(D^n_{\alpha,\mu}(\lambda,\omega)f(z))^{''}}{\gamma(D^n_{\alpha,\mu}(\lambda,\omega)f(z))^{'}}\right| \leq 1 + (1+|z|^{2\gamma})\left|\frac{z(D^n_{\alpha,\mu}(\lambda,\omega)f(z))^{''}}{\gamma(D^n_{\alpha,\mu}(\lambda,\omega)f(z))^{'}}\right| = 1.$$

Thus, in view of Lemma 4, P_γ is in the class S. □

Theorem 5. *Let $f \in A$. If for all $z \in U$*

$$\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda - \alpha) + j]^n |a_j| \leq \frac{1}{\sqrt{7}}, \tag{24}$$

then the operator $D^n_{\alpha,\mu}(\lambda,\omega)f(z)$ is in the class S.

Proof. Let suppose that $f \in A$. For all $z \in U$ we have

$$\left|\frac{z^2(D^n_{\alpha,\mu}(\lambda,\omega)f(z))^{'}}{2(D^n_{\alpha,\mu}(\lambda,\omega)f(z))^2}\right| = \left|\frac{z^2(1+\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j a_j z^{j-1})}{2(z+\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n a_j z^j)^2}\right| =$$

$$\left|\frac{1+\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j a_j z^{j-1}}{2(1+2\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n a_j z^{j-1} + (\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n a_j z^{j-1})^2)}\right|$$

$$\leq \frac{1+\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n j|a_j|}{2(1-2\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n|a_j| - (\sum_{j=2}^{\infty}[(j-1)(\mu\omega^\lambda-\alpha)+j]^n|a_j|)^2)}$$

which is less than 1 if the inequality (24) holds. This implies that

$$\left|\frac{z^2(D^n_{\alpha,\mu}(\lambda,\omega)f(z))^{'}}{(D^n_{\alpha,\mu}(\lambda,\omega)f(z))^2} - 1\right| \leq 1\,(z \in U), \tag{25}$$

and applying Lemma 5 we find that $D^n_{\alpha,\mu}(\lambda,\omega)f(z)$ is in the class S. □

Theorem 6. *Let $f \in A$ satisfy the inequality (24). Also let $c \in \mathbb{C}$ and $\eta \in \mathbb{R}$ such that $1 \leq \eta \leq \frac{3}{2}$. If*

$$|c| \leq \frac{3-2\eta}{\eta} \text{ and } |D^n_{\alpha,\mu}(\lambda,\omega)f(z)| \leq 1 \tag{26}$$

then the integral operator S_η given by

$$S_\eta(z) = \left(\eta \int_0^z (D^n_{\alpha,\mu}(\lambda,\omega)f(u))^{\eta-1} du\right)^{\frac{1}{\eta}} \tag{27}$$

is in the class in S.

Proof. The required result follows in the same way as Theorem 5 using Lemma 6. □

The next theorem follows as an application of the Theorems 1, 2 and 5.

Theorem 7. *Let $f \in A$. If for all $z \in U$*

$$\frac{z}{D^n_{\alpha,\mu}(\lambda,\omega)f(z)} = 1 + \sum_{k=1}^{\infty} b_k z^k \tag{28}$$

and one of the inequalities (14), (16) or (24) holds then

$$\sum_{k=1}^{\infty}(k-1)|b_k|^2 \leq 1. \tag{29}$$

Proof. Letting $f \in A$ and applying one of the Theorems 1, 2 or 5 we get that $f \in S$. Thus, in view of Lemma 7 we obtain the required result. □

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