TJMM
6 (2014), No. 2, 181-200

A NEW TECHNIQUE TO DERIVE MANY EXPLICIT THERMOELASTIC GREEN'S FUNCTIONS

VICTOR ŞEREMET

Abstract. This study is devoted to a new technique to derive main thermoelastic Green's functions (MTGFs), based on their new integral representations via Green's functions for Poison's equation (GFPE). The derived new integral representations for MTGFs permitted to prove a theorem about their constructive formulas expressed in terms of respective GFPE and some simplest integrals. The influence functions of thermoelastic displacements are generated by a unitary heat source. This source is applied in an arbitrary inner point of a generalized thermoelastic octant at the different homogeneous mechanical and thermal boundary conditions, prescribed on its marginal quadrants. According to the proved theorem many MTGFs for a group of two-and three dimensional BVPs of thermoelasticity for a plane, a half-plane, a quadrant, a space, a quarter-space and an octant may be obtained by changing the respective well-known GFPE and calculating some simplest integrals. Some concrete new MTGFs for octant, quarter-space and half-space are presented. The graphical and numerical computer evaluation of MTGFs for a thermoelastic octant by using Maple 15 software also is included. All MTGFs are obtained in terms of elementary functions that are very important for their numerical implementation. The analytical checking of MTGFs is given for a new BVP for thermoelastic octant. Using the proposed technique it is possible to extend all obtained results for any domain of Cartesian system of coordinates.

1. Introduction

The most difficult problem of Green's function method (GFM) is the derivation of Green's functions (GFs). In monographs [1, 2, 3, 4] are presented the methods to derive GFs for ordinary and partial differential equations. The derivation and application of GFs and Green's matrices for two dimensional (2D) BVPs for elliptic system in the theory of elasticity are presented in the monographs [5, 6]. GFs for advanced materials are derived in the monographs [7, 8]. A large systematic list of GFs for 2D BVPs of Poisson's equation, derived for canonical Cartesian and polar domains is given in the encyclopedia [9]. Respectively, in the handbook [10] is given GFs and Green's matrices (GMs) for two– and 3D BVPs in the theory of elasticity, derived for canonical Cartesian domains. Among theories of thermoelasticity [11, 12, 13, 14, 15, 16, 17, 18] the best developed theory which is widely used in practical calculations is the theory of thermal stresses, i.e. the theory of uncoupled thermoelasticity. This theory presents a synthesis of the theory of heat conduction and of the theory of elasticity. In the theory of uncoupled heat conduction to solve a BVP a Green's integral formula provides the temperature field resulted from a given thermal exposure. The analogous Green's integral formula determines the field of elastic displacements, produced by the known mechanical actions. But in thermoelasticity according to the integral Maysel's formula [13, 15, 16] the solution of a BVP is not

2010 *Mathematics Subject Classification.* 74605, 35J08.
Key words and phrases. Green's functions, thermoelasticity, heat conduction, elasticity, thermoelastic Green's functions, volume dilatation, integral representations.

represented directly in terms of the given data, but in terms of a temperature field, which in the most cases is to be found. This fact introduces certain inconvenience in application of Maysel's formula except the case when the temperature field is known already. To avoid mentioned above inconvenience the author of this work has proposed the following new integral formula in thermoelasticity [10, 19, 20, 21, 22, 23]:

$$\begin{aligned} u_i(\xi) = a^{-1} \int_V F(x)\, U_i(x,\xi)\, dV(x) - \int_{\Gamma_D} T(y)\, \partial U_i(y,\xi)/\partial n_y d\Gamma_D(y) + \\ + \int_{\Gamma_N} \partial T(y)/\partial n_y U_i(y,\xi)\, d\Gamma_N(y) + \\ + a^{-1} \int_{\Gamma_M} [\alpha T(y) + a \partial T(y)/\partial n_y]\, U_i(y,\xi)\, d\Gamma_M(y);\ i = 1,2,3, \end{aligned} \tag{1}$$

where Γ_D, Γ_N and Γ_M denote the surfaces on which the boundary conditions of Dirichlet's, Neumann's and mixed type are prescribed, respectively: temperature $T(y)$, heat flux $a\partial T(y)/\partial n_y$ and a heat exchange between exterior medium and surface of the body represented by $\alpha T(y) + a\partial T(y)/\partial n_y$ law; F is the heat source; a is thermal conductivity; α is the coefficient of convective heat conductivity; $\gamma = \alpha_t(2\mu + 3\lambda)$ is the thermoelastic constant; λ, μ are Lame's constants of elasticity; α_t is the coefficient of the linear thermal expansion. The introduced in Eq. (1) the main thermoelastic Green's functions (MTGFs) $U_i(x,\xi)$ have the physical sense as thermoelastic displacements at an arbitrary inner point $\xi \equiv (\xi_1, \xi_2, \xi_3)$ of observation, generated by a unit heat source applied at another arbitrary inner point $x \equiv (x_1, x_2, x_3)$ and described by δ–Dirac's function. To determine the MTGFs in the works [10, 19, 20, 21, 22, 23] was proposed the following integral formula [10, 19, 20, 21, 22, 23]:

$$U_i(x,\xi) = \gamma \int_V G_T(x,z)\ \Theta^{(i)}(z,\xi)\, dV(z);\ x, z, \xi \in V, \tag{2}$$

where G_T is the GF for a heat conduction BVP corresponding to an unit internal point heat source, and $\Theta^{(i)}$ are functions of influence of unit concentrated body forces on elastic volume dilatation.

The main advantage of the formulas (1) and (2) is that the searched thermoelastic displacements u_i are determined in the integral form directly via the prescribed inner heat source and other thermal data, given on the boundary. However the elegant Maysel's integral formula remain still very important because it was generalized to piezoelectric vibrations [24], to piezoelectric bodies [25], to anisotropic thermoelastic bodies [26] and to the dynamic eigenstrain problem [27].

Note, that, using the formulas (1) and (2), the author derived in elementary functions some new very useful thermoelastic GFs and Green's type integral formulas for quadrant [28, 29], half–space [30, 31], quarter-space [32, 33], wedge [34, 35] and half-wedge [36]. For all these BVPs the difficulties associated with the constructing of the additional influence functions for elastic volume dilatation and with the computing of the volume integral (2) have been successfully overcome. Furthermore, it was observed that for more complicated BVPs of thermoelasticity these difficulties are substantially. This is why the author of this article starts to find other methods to drive MTGFs. However, the preliminary his investigations have shown that the classical methods [11, 12, 13, 14, 15, 16, 17, 18] such as: method of body-force analogy [11, 16], Goodier's method [11], method of thermoelastic potentials [11, 15, 16] and many other methods [12, 13, 14, 15, 16, 17, 18] leads to necessity to solve additional elastic BVPs or to calculate the complicate volume integral (2). So, the mentioned above methods are not effective for constructing MTGFs. But a crucial moment of author's next investigations was, when he discovers that thermoelastic Lame equations permit to be presented in a form of three independent Poisson's type equations.

So, on this base he obtained the following integral representations [37, 38]:

$$\begin{gathered} U_i\left(x,\xi\right)+\tfrac{\lambda+\mu}{2\mu}\xi_i\Theta\left(x,\xi\right)= \\ -\tfrac{\gamma}{2(\lambda+2\mu)}x_iG_i\left(x,\xi\right)+\tfrac{\gamma\xi_i}{2\mu}G_T\left(x,\xi\right)-\int_\Gamma\left[V_i\left(x,y\right)\tfrac{\partial}{\partial n_\Gamma}-\tfrac{\partial V_i(x,y)}{\partial n_\Gamma}\right]G_i\left(y,\xi\right)d\Gamma\left(y\right) \\ V_i\left(x,y\right)=U_i\left(x,y\right)+\tfrac{y_i}{2\mu}\left[\left(\lambda+\mu\right)\Theta\left(x,y\right)-\gamma G_T\ \left(x,y\right)\right] \end{gathered} \quad (3)$$

– for MTGFs, and

$$\Theta\left(x,\xi\right)=\frac{\gamma}{\lambda+2\mu}G_\Theta\left(x,\xi\right)+\int_\Gamma\left[\frac{\partial\Theta\left(x,y\right)}{\partial n_\Gamma}-\Theta\left(x,y\right)\frac{\partial}{\partial n_\Gamma}\right]G_\Theta\left(y,\xi\right)d\Gamma\left(y\right) \quad (4)$$

– for thermoelastic volume dilatation $\Theta\left(x,\xi\right)=U_{j,j}\left(x,\xi\right)$, $j=1,2,3$.

Furthermore, the next author's preliminary investigations have shown that on the base of these integral representations it become possible to develop a new very efficient unified technique of constructing new MTGFs. In turn using this technique it becomes possible to create a large very useful in applications database of these functions. This is explained by the fact that, unlike existing classical methods, where each problem may be solved in isolation (in this case, for each task should be selected his appropriate method), the expected new technique will solve immediately the whole class of thermoelastic BVPs.

So, the main objective of this paper is developing a new very efficient unified technique of constructing new MTGFs. Also, on the base of this technique we prove a theorem about derivation the constructive formulas for MTGFs to a general BVP for an octant $V\left(0\leq x_1,x_2,x_3<\infty\right)$ (or quadrant $V\left(0\leq x_1,\ x_2<\infty\right)$) which is bounded by the quarter-planes $\Gamma_{10}\left(y_1=0,\ 0\leq y_2,\ y_3<\infty\right)$, $\Gamma_{20}(0\leq y_1,\ y_3<\infty,\ y_2=0)$ and $\Gamma_{30}\left(0\leq y_1,\ y_2<\infty,\ y_3=0\right)$ (or half-planes $\Gamma_{10}\left(y_1=0,\ 0\leq y_2<\infty\right)$, $\Gamma_{20}(0\leq y_1<\infty,\ y_2=0)$) and with different types of homogeneous mechanical and thermal boundary conditions. The main advantage of obtained constructive formulas is that by changes of GFPE and by calculating some simplest integrals it is possible easily to write MTGFs in elementary functions for about 12 BVPs of thermoelasticity.

2. Constructive Formulas for MTGFs in Terms of GFPE

Let us consider some canonical semi-infinite domains, whose surfaces represent planes or their parts (straight lines or their parts) of Cartesian system of coordinates. Let also these domains have not parallel planes or their parts (parallel straight lines or their parts).

For considered domains, if on the mentioned-above boundaries are given homogeneous locally-mixed boundary conditions (zero normal stresses and tangential displacements or zero normal stresses and tangential displacements are given in any combinations) are true some theorems about the constructing MTGFs $U_i\left(x,\xi\right)$ and derivation of Green-type integral formula, by using integral representations (3) and (4). So, in the work [37] is derived MTGFs and Green-type integral formula for a special BVP of thermoelasticity for an octant. Also in [38] are derived constructive formulas for MTGFs $U_i\left(x,\xi\right)$ for a class of BVPs of thermoelasticity for an octant, when boundary conditions for thermal GF $G_T\left(x,\xi\right)$ are linked with mechanical boundary conditions in such a way that derived MTGFs $U_i\left(x,\xi\right)$ are expressed in terms of GFPE only. In the present study boundary conditions for thermal GF $G_T\left(x,\xi\right)$ are linked with mechanical boundary conditions in more complicated form, so that derived MTGFs $U_i\left(x,\xi\right)$ are not expressed in terms of GFPE only, but appear some new additional harmonic functions, so that an obtained result differs substantially from those, presented in [37] and [38]. For these cases is true the following

Theorem 1. *Let the field of MTGFs for displacements* $U_i\left(x,\xi\right)$ *and temperature* $G_T\left(x,\xi\right)$ *at inner points* $\xi\equiv\left(\xi_1,\xi_2,\xi_3\right)$ *of the generalized BVP for thermoelastic octant* $V(0\leq$

$x_1, x_2, x_3 < \infty$) be determined by non-homogeneous Lame thermoelastic equations $\mu\nabla^2_\xi U_i(x,\xi) + (\lambda+\mu)\Theta_{,\xi_i}(x,\xi) - \gamma G_{T,\xi_i}(x,\xi) = 0$, Poisson equation $\nabla^2_x G_T(x,\xi) = -\delta(x-\xi)$ and in the points $y \equiv (0, y_2, y_3)$, $y \equiv (y_1, 0, y_3)$ and $y \equiv (y_1, y_2, 0)$ of boundary quadrants $\Gamma_{10}(y_1 = 0, 0 \le y_2, y_3 < \infty)$, $\Gamma_{20}(0 \le y_1 < \infty, y_2 = 0, 0 \le y_3 < \infty)$ and $\Gamma_{30}(0 \le y_1, y_2 < \infty, y_3 = 0)$ the following homogeneous locally-mixed mechanical and thermal conditions are given:

$$U_1(x;0,\xi_2,\xi_3) = U_2(x;0,\xi_2,\xi_3) = U_3(x;0,\xi_2,\xi_3) = 0; \partial G_T(x;0,\xi_2,\xi_3)/\partial n_{\xi_1} = 0 \tag{5a}$$

– rigid fixation of the points of boundary quadrant $\Gamma_{10}(y_1 = 0, 0 \le y_2, y_3 < \infty)$;

$$\begin{gathered}\sigma_{21}(x;\xi_1,0,\xi_3) = U_2(x;\xi_1,0,\xi_3) = \sigma_{23}(x;\xi_1,0,\xi_3) = 0;\\ \partial G_T(x;\xi_1,0,\xi_3)/\partial n_{\xi_2} = 0\end{gathered} \tag{5b}$$

or

$$U_1(x;\xi_1,0,\xi_3) = \sigma_{22}(x;\xi_1,0,\xi_3) = U_3(x;\xi_1,0,\xi_3) = 0; G_T(x;\xi_1,0,\xi_3) = 0 \tag{5c}$$

– on the boundary quadrant $\Gamma_{20}(0 \le y_1 < \infty, y_2 = 0, 0 \le y_3 < \infty)$, and

$$\begin{gathered}\sigma_{31}(x;\xi_1,\xi_2,0) = \sigma_{32}(x;\xi_1,\xi_2,0) = U_3(x;\xi_1,\xi_2,0) = 0;\\ \partial G_T(x;\xi_1,\xi_2,0)/\partial n_{\xi_3} = 0\end{gathered} \tag{5d}$$

or

$$U_1(x;\xi_1,\xi_2,0) = U_2(x;\xi_1,\xi_2,0) = \sigma_{33}(x;\xi_1,\xi_2,0) = 0; G_T(x;\xi_1,\xi_2,0) = 0 \tag{5e}$$

– on the boundary quadrant $\Gamma_{30}(0 \le y_1, y_2 < \infty, y_3 = 0)$, where σ_{33} and σ_{21} σ_{31} σ_{23} are the normal and the tangential stresses which are determined by the well-known Duhamel-Neumann law

$$\sigma_{ij} = \mu(U_{i,j} + U_{j,i}) + \delta_{ij}(\lambda U_{k,k} - \gamma G_T); i,j = 1,2,3. \tag{6}$$

Then the constructive formulae for MTGFs $U_i(x,\xi)$ and thermoelastic volume dilatation $\Theta(x,\xi)$ for this class of BVPs of thermoelasticity are the following:

$$\begin{gathered}U_i(x,\xi) = \tfrac{\gamma}{2(\lambda+2\mu)}\Big\{\Big[\tfrac{\xi_i}{\mu}\left((\lambda+2\mu)G_T(x,\xi) - (\lambda+\mu)G_1(x,\xi)\right) - x_i G_i(x,\xi)\Big]\\ -2\xi_k\tfrac{\lambda+2\mu}{\mu}W_T(x,\xi) - 2B^{-1}\Big(x_1\tfrac{\partial}{\partial x_1} - 2\tfrac{\lambda+2\mu}{\lambda+\mu}\Big)\xi_1\tfrac{\partial}{\partial\xi_i}\int W_T(x,\xi)\,d\xi_1\Big\}\\ \Theta(x,\xi) = \tfrac{\gamma}{\lambda+2\mu}\Big[G_1(x,\xi) + \tfrac{2B^{-1}}{\lambda+\mu}\Big(\mu x_1\tfrac{\partial}{\partial x_1} + \lambda + 2\mu\Big)W_T(x,\xi)\Big]\\ i = 1,2,3;\ B = \tfrac{\lambda+3\mu}{\lambda+\mu},\end{gathered} \tag{7}$$

were $G_T(x,\xi)$, $G_i(x,\xi)$ are GFPE and $W_T(x,\xi)$ is that regular part of the Green's functions $G_T(x,\xi)$ which contains inferior index 1 (that part of the $G_T(x,\xi)$ which are reflected via boundary Γ_{10}). For functions $G_T(x,\xi)$, $G_i(x,\xi)$ on the marginal planes or their parts (straight lines or their parts) are given homogeneous conditions that are similar to boundary conditions for temperature and thermoelastic displacements $U_i(x,\xi)$ respectively. So, as example, under boundary conditions for $G_i(x,\xi)$ it mean that, if $U_i = 0$, then $G_i = 0$ and if $U_{i,n} = 0$, then $G_{i,n} = 0$.

Proof. First, we use the general representations (3) and (4) that in the case of the octant $V \equiv (0 \le x_1, x_2, x_3 \le \infty)$ can be rewritten in the following form:

$$\Theta(x,\xi) = \frac{\gamma}{\lambda+2\mu}G_\Theta(x,\xi) + \sum_{j=1}^{3}\int_{\Gamma_{j0}}\left[\frac{\partial\Theta(x,y)}{\partial n_{y_j}} - \Theta(x,y)\frac{\partial}{\partial n_{y_j}}\right]G_\Theta(y,\xi)\,d\Gamma_{j0}(y) \tag{8}$$

– for thermoelastic volume dilatation $\Theta(x,\xi)$, and

$$U_i(x,\xi) = -\frac{\lambda+\mu}{2\mu}\xi_i\Theta(x,\xi) - \frac{\gamma}{2(\lambda+2\mu)}x_iG_i(x,\xi) + \frac{\gamma\xi_i}{2\mu}G_T(x,\xi) - \sum_{j=1}^{3}\int_{\Gamma_{j0}}\left[V_i(x,y)\frac{\partial}{\partial n_{y_j}} - \frac{\partial V_i(x,y)}{\partial n_{y_j}}\right]G_i(y,\xi)\,d\Gamma_{j0}(y) \tag{9}$$

– for MTGFs $U_i(x,\xi)$.

Second, we use the following hypotheses presented in works [37] and [38]:

a) Let the surfaces of some domains represent planes or their parts (straight lines or their parts) of Cartesian system of coordinates. If on the marginal planes or their parts (straight lines or their parts) are given zero normal displacements, zero tangential stresses and zero normal derivative of Green's function $G_{T,n} = 0$ for temperature (see Eqs. (5b) and (5d), then the normal derivative of volume dilatation is equal to zero, $\Theta_{,n} = 0$;

b) Respectively, if on the marginal planes or their parts (straight lines or their parts) are given zero normal stresses, zero tangential displacements and zero Green's function $G_T = 0$ for temperature (see Eqs. (5c) and (5e), then volume dilatation $\Theta = 0$.

Also here we prove and use in future the following hypothesis:

c) If on the marginal planes or their parts (straight lines or their parts) are given zero normal and tangential displacements and zero Green's function for temperature (see Eqs. (5a), then volume dilatation on these kind of surfaces (or counters), as example boundary quadrant $\Gamma_{10}\,(y_1 = 0, 0 \le y_2, y_3 < \infty)$, is determined as follows:

$$\Theta\,|_{\xi_1=0} = \frac{2\gamma}{(\lambda+2\mu)}B^{-1}\left[\frac{\mu}{\lambda+\mu}x_1\frac{\partial}{\partial x_1} + \frac{\lambda+2\mu}{\lambda+\mu}\right]W_T\,|_{\xi_1=0} \tag{10a}$$

where W_T is that regular part of the Green's function $G_T(x,\xi)$ which contains inferior index 1 (that part of the $G_T(x,\xi)$ which are reflected via boundary Γ_{10}).

To prove Eq. (10a) let us use the relations:

$$\Theta|_{\xi_1=0} = (U_{1,\xi_1} + U_{2,\xi_2} + U_{3,\xi_3})\,|_{\xi_1=0} = = U_{1,\xi_1}\,|_{\xi_1=0} + U_{2,\xi_2}\,|_{\xi_1=0} + U_{3,\xi_3}\,|_{\xi_1=0} = U_{1,\xi_1}\,|_{\xi_1=0} \tag{10b}$$

that follows from boundary conditions (5a): $U_2 = U_3 = 0 \Rightarrow U_{2,\xi_2}|_{\xi_1=0} = U_{3,\xi_3}|_{\xi_1=0} = 0$.

Next, let in the representations (3) and (4) the functions G_i, G_Θ and G_T are the GFPE those homogeneous boundary conditions are the similar to the boundary conditions for U_i, Θ and G_T, respectively. So, it mean that, if on a marginal quadrant are known U_i and Θ, T, then $G_i = 0$ and $G_\Theta = G_T = 0$; and if on a marginal quadrant are known $U_{i,n}$ and $\Theta_{,n}$, $T_{,n}$, then $G_{i,n} = 0$ and $G_{\Theta,n} = G_{T,n} = 0$. In these cases, the hypothesis c) boundary conditions (5a) lead to following equivalent boundary conditions:

$$U_1 = U_2 = U_3 = 0;\ \partial G_T/\partial n_{10} = 0 \Rightarrow G_1 = G_2 = G_3 = G_\Theta = \partial G_T/\partial n_{10} = 0 \tag{11a}$$

– on the marginal quadrant $\Gamma_{10}\,(y_1 = 0, 0 \le y_2, y_3 < \infty)$.

Also, in the mentioned-above cases, using the hypotheses a) and b) in the works [37, 38] is proved that the boundary conditions (5b), (5c) and (5d), (5e) lead to following equivalent locally-mixed boundary conditions:

$$\sigma_{21} = U_2 = \sigma_{23} = 0; G_{T,2} = 0 \Rightarrow U_{1,2} = 0; U_2 = 0; U_{2,1} = U_{2,3} = U_{3,2} = 0 \Rightarrow \Theta_{,2} = 0; G_{1,2} = 0; G_2 = 0; G_{3,2} = 0; G_{\Theta,2} = 0; G_{T,2} = 0 \tag{11b}$$

or

$$U_1 = \sigma_{22} = U_3 = 0; G_T = 0; \Rightarrow U_1 = U_{1,1} = U_{1,3} = U_3 = = U_{3,1} = U_{3,3} = U_{2,2} = 0 \Rightarrow \Theta = 0; G_1 = G_{2,2} = G_3 = G_\Theta = G_T = 0 \tag{11c}$$

– on the marginal quadrant $\Gamma_{20}(0 \le y_1 < \infty, y_2 = 0, 0 \le y_3 < \infty)$, and

$$\begin{gathered}\sigma_{31} = \sigma_{32} = U_3 = 0; \partial G_T/\partial n_{\xi_3} = 0; \Rightarrow U_{1,3} = U_{2,3} = U_3 = U_{3,1} = U_{3,2} \\ \Rightarrow \Theta_{,3} = 0; G_{1,3} = 0; G_3 = 0; G_{2,3} = 0; G_{\Theta_{,3}} = 0; G_{T_{,3}} = 0;\end{gathered} \tag{11d}$$

or

$$\begin{gathered}U_1 = U_2 = \sigma_{33} = 0; G_T = 0; \Rightarrow U_1 = U_{1,1} = U_{1,2} = U_2 = U_{2,2} = U_{2,1} = U_{3,3} \\ \Rightarrow \Theta = G_1 = G_2 = G_{3,3} = G_\Theta = G_T = 0\end{gathered} \tag{11e}$$

– on the marginal quadrant $\Gamma_{30}(0 \le y_1, y_2 < \infty, y_3 = 0)$.

Now substituting the boundary conditions (11c), (11d) and (11e) into Eq. (3) we obtain the following simplified integral representations for MTGFs:

$$\begin{gathered}U_i(x,\xi) = -\tfrac{\lambda+\mu}{2\mu}\xi_i\Theta(x,\xi) - \tfrac{\gamma}{2(\lambda+2\mu)}x_iG_i(x,\xi) + \tfrac{\gamma\xi_i}{2\mu}G_T(x,\xi) - \\ -\int_{\Gamma_{10}}\left[V_i(x,y)\tfrac{\partial}{\partial n_{y_1}} - \tfrac{\partial V_i(x,y)}{\partial n_{y_1}}\right]G_i(y,\xi)\,d\Gamma_{10}(y)\end{gathered} \tag{12a}$$

Taking into account expression (3) for $V_1(x,y)$ and boundary conditions (11a) we can rewrite Eq. (12a) at $i = 1$ as follows below:

$$\begin{gathered}U_1(x,\xi) = -\tfrac{\lambda+\mu}{2\mu}\xi_1\Theta(x,\xi) - \tfrac{\gamma}{2(\lambda+2\mu)}x_1G_1(x,\xi) + \tfrac{\gamma\xi_1}{2\mu}G_T(x,\xi) \\ -\int_{\Gamma_{10}}\left[U_1(x,y) + \tfrac{y_1}{2\mu}\left[(\lambda+\mu)\Theta(x,y) - \gamma G_T(x,y)\right]\right]\tfrac{\partial}{\partial n_{y_1}}G_1(y,\xi)\,d\Gamma_{10}(y)\end{gathered} \tag{12b}$$

or in the final form:

$$U_1(x,\xi) = -\frac{\lambda+\mu}{2\mu}\xi_1\Theta(x,\xi) - \frac{\gamma}{2(\lambda+2\mu)}x_1G_1(x,\xi) + \frac{\gamma\xi_1}{2\mu}G_T(x,\xi) \tag{12c}$$

because on marginal quadrant Γ_{10} $U_1(x,y) = 0$ and $y_1 = 0$.

According to (10b) from (12c) we obtain:

$$U_{1,\xi_1}|_{\xi_1=0} = \left[-\frac{\lambda+\mu}{2\mu}\Theta - \frac{\gamma}{2(\lambda+2\mu)}x_1G_{1,\xi_1} + \frac{\gamma}{2\mu}G_T\right]|_{\xi_1=0} \tag{12d}$$

or

$$\Theta|_{\xi_1=0} = \frac{\gamma}{(\lambda+2\mu)}B^{-1}\left[\frac{\lambda+2\mu}{\lambda+\mu}G_T - \frac{\mu}{\lambda+\mu}x_1G_{1,\xi_1}\right]|_{\xi_1=0} \tag{12e}$$

Next, due equality $G_T|_{\xi_1=0} = 2W_T$ and due the fact that $G_1(x,\xi)$ has the same boundary conditions on the marginal quadrants Γ_{20} and Γ_{30} as the function $G_T(x,\xi)$ except on marginal quadrant Γ_{10} where the function $G_1|_{\xi_1=0} = 0$ and $G_{T,\xi_1}|_{\xi_1=0} = 0$, follows that $G_{1,\xi_1}|_{\xi_1=0} = -G_{T,x_1}|_{\xi_1=0} = -2W_{T,x_1}|_{\xi_1=0}$. So, applying the last result in (12e) we obtain volume dilatation on Γ_{10} given in Eq. (10a).

Substituting the boundary values of the volume dilatation Θ and respective GFPE G_Θ from equations (10a) and (11b), (11c), (11d) and (11e) into representation (4) we can see that integrals on marginal quadrants Γ_{20} and Γ_{30} are zero, so that we obtain:

$$\begin{gathered}\Theta(x,\xi) = \tfrac{\gamma}{\lambda+2\mu}\left[G_\Theta(x,\xi) - \tfrac{2B^{-1}}{\lambda+\mu}\left(\mu x_1\tfrac{\partial}{\partial x_1} + \lambda + 2\mu\right)\right. \\ \left.\int_{\Gamma_{10}}W_T(x;0,y_2,y_3)\tfrac{\partial G_\Theta(0,y_2,y_3,\xi)}{\partial n_{y_1}}d\Gamma_{10}(y)\right]\end{gathered} \tag{13a}$$

As boundary conditions (11a), (11b), (11c), (11d) and (11e) for $G_\Theta(x,\xi)$ and $G_1(x,\xi)$ on all marginal quadrants are the same, then $G_\Theta(x,\xi) = G_1(x,\xi)$ and from (13a) follows:

$$\Theta(x,\xi) = \frac{\gamma}{\lambda+2\mu}\left[G_1(x,\xi) + \frac{2B^{-1}}{\lambda+\mu}\left(\mu x_1\frac{\partial}{\partial x_1} + \lambda + 2\mu\right)W_T(x,\xi)\right], \tag{13b}$$

where $W_T(x,\xi)$ is that regular part of the Green's functions $G_T(x,\xi)$ which contains inferior index 1 (that part of the $G_T(x,\xi)$ which are reflected via boundary Γ_{10}). So, the formula (13b) coincides with the constructive formula for volume dilatation in the Eq.

(7). Substituting (13b) into (12c) we obtain the final constructive formula for MTGFs $U_1(x,\xi)$:

$$\begin{gathered} U_1(x,\xi)=\frac{\gamma}{2(\lambda+2\mu)}\left[\frac{\xi_1}{\mu}\left((\lambda+2\mu)\,G_T(x,\xi)-(\lambda+\mu)\,G_1(x,\xi)\right)-\right. \\ \left.-x_1G_1(x,\xi)-2B^{-1}\xi_1\left(x_1\frac{\partial}{\partial x_1}+\frac{\lambda+2\mu}{\mu}\right)W_T(x,\xi)\right] \end{gathered} \tag{14}$$

that coincides with the result in the Eq. (7) of the theorem obtained at $i=1$.

Next, applying (13b) into (12a) at $i=k=2,3$ we obtain:

$$\begin{gathered} U_k(x,\xi)=\frac{\gamma}{2(\lambda+2\mu)}\left[\frac{\xi_k}{\mu}\left((\lambda+2\mu)\,G_T(x,\xi)-(\lambda+\mu)\,G_1(x,\xi)\right)-\right. \\ \left.-x_kG_k(x,\xi)-2B^{-1}\xi_k\left(x_1\frac{\partial}{\partial x_1}+\frac{\lambda+2\mu}{\mu}\right)W_T(x,\xi)\right]+I_k(x,\xi), \end{gathered} \tag{15a}$$

where taking into account expressions (10a) and (11a) into (12a) the integral I_k can be written as follows:

$$\begin{gathered} I_k(x,\xi)= \\ =-\gamma(\lambda+2\mu)^{-1}B^{-1}\left(x_1\frac{\partial}{\partial x_1}-2\frac{\lambda+2\mu}{\lambda+\mu}\right)\int_{\Gamma_{10}}y_kW_T(x,y)\,\frac{\partial}{\partial n_{y_1}}G_k(y,\xi)\,d\Gamma_{10}(y) \\ =\gamma(\lambda+2\mu)^{-1}B^{-1}\left(x_1\frac{\partial}{\partial x_1}-2\frac{\lambda+2\mu}{\lambda+\mu}\right)\left[\xi_kW_T(x,\xi)-\xi_1\frac{\partial}{\partial\xi_k}\int W_T(x,\xi)\,d\xi_1\right] \end{gathered} \tag{15b}$$

Finally, introducing integral (15b) into Eq. (15a) we obtain constructive formula

$$\begin{gathered} U_k(x,\xi)=\frac{\gamma}{2(\lambda+2\mu)}\left\{\frac{\xi_k}{\mu}\left[(\lambda+2\mu)\,G_T(x,\xi)-(\lambda+\mu)\,G_1(x,\xi)\right]-x_kG_k(x,\xi)\right. \\ -2\xi_k\frac{\lambda+2\mu}{\mu}W_T(x,\xi)-2B^{-1}\left(x_1\frac{\partial}{\partial x_1}-2\frac{\lambda+2\mu}{\lambda+\mu}\right)\xi_1\frac{\partial}{\partial\xi_k}\int W_T(x,\xi)\,d\xi_1 \end{gathered} \tag{15c}$$

that coincide with the result in Eq. (7) at $i=k=2,3$. □

Note, that one of the most difficult moments of our investigations was the calculation of the integral (15b). But this moment was avoided successfully, when we established the following properties of integral I_k:

- the integral I_k is a harmonic function with respect to coordinates of both points: $x\equiv(x_1,x_2,x_3)$ and $\xi\equiv(\xi_1,\xi_2,\xi_3)$;
- the values of integral I_k on marginal quadrants are determined by the boundary conditions of his integrands: the integrand $G_k(y,\xi)$ (with respect to coordinates of the point $\xi\equiv(\xi_1,\xi_2,\xi_3)$) and of the integrand $G_T(x,y)$ (with respect to coordinates of the point $x\equiv(x_1,x_2,x_3)$. These boundary conditions just are given in the Eqs. (11a), (11b), (11c), (11d) and (11e). So, these two properties of the left part of integral I_k help us to write his right part as is shown in Eq. (15b).

Note that constructive formulas for MTGFs (7) at $i=1,2$ are applicable also for 2D BVPs of thermoelasticity. Thus the proposed technique permits to derive many MTGFs in thermoelasticity. Indeed, on the base of constructive formula (7) we can easy (by changing the respective well-known analytical expressions for GFPEs $G_T(x,\xi)$, $G_i(x,\xi)$ [10] and calculating some simplest integrals) to write MTGFs $U_i(x,\xi)$ and volume dilatation $\Theta(x,\xi)$ in elementary functions for many BVPs of thermoelasticity: eight for 3D BVPs (one for space, one for half-space, two for quarter-space and four for octant) and four for 2D BVPs (one for plane, one for half-plane and two for quadrant). However, in this study we give only one example for constructing new MTGFs $U_i(x,\xi)$ and volume dilatation $\Theta(x,\xi)$ in elementary functions for a BVP of thermoelasticity for an octant.

3. New Explicit MTGFs $U_i(x,\xi)$ and Green-type Integral Formula for a Thermoelastic Octant

Let the field of displacements $U_i(x,\xi)$ and temperature $G_T(x,\xi)$ at inner points $\xi \equiv (\xi_1,\xi_2,\xi_3)$ of the thermoelastic octant $V\,(0 \le x_1, x_2, x_3 < \infty)$ be determined by non-homogeneous Lame equations $\mu\nabla_\xi^2 U_i(x,\xi) + (\lambda+\mu)\,\Theta_{,\xi_i}(x,\xi) - \gamma G_{T,\xi_i}(x,\xi) = 0$ and Poisson equation $\nabla_x^2 G_T(x,\xi) = -\delta(x-\xi)$, but in the points $y \equiv (0, y_2, y_3)$, $y \equiv (y_1, 0, y_3)$ and $y \equiv (y_1 y_2, 0)$ of marginal quadrants $\Gamma_{10}\,(y_1 = 0, 0 \le y_2, y_3 < \infty)$, $\Gamma_{20}(0 \le y_1 < \infty, y_2 = 0, 0 \le y_3 < \infty)$ and $\Gamma_{30}(0 \le y_1, y_2 < \infty, y_3 = 0)$ the following homogeneous mechanical and thermal conditions are given:

$$U_1 = U_2 = U_3 = 0; \partial G_T/\partial n_{10} = 0 \tag{16a}$$

– rigid fixation of the points of boundary quadrant $\Gamma_{10}\,(y_1 = 0, 0 \le y_2, y_3 < \infty)$

$$\sigma_{21} = U_2 = \sigma_{23} = 0; \partial G_T/\partial n_{\xi_2} = 0 \tag{16b}$$

– locally mixed boundary conditions on the marginal quadrant $\Gamma_{20}(0 \le y_1 < \infty, y_2 = 0, 0 \le y_3 < \infty)$ and

$$\sigma_{31} = \sigma_{32} = U_3 = 0; \partial G_T/\partial n_{30} = 0 \tag{16c}$$

– locally mixed boundary conditions on the boundary quadrant $\Gamma_{30}(0 \le y_1, y_2 < \infty, y_3 = 0)$.

Then, according to the boundary conditions (11a), (11b), (11d) and (16a), (16b), (16c) the respective boundary conditions for Green's functions $G_i(x,\xi)$ for Poison's equation are the following:

$$G_1 = G_2 = G_3 = \partial G_T/\partial n_{10} = G_\Theta = 0 \tag{17a}$$

on the boundary quadrant Γ_{10},

$$G_{1,2} = G_2 = G_{3,2} = G_{T,2} = G_{\Theta,2} = 0 \tag{17b}$$

on the boundary quadrant Γ_{20} and

$$G_{1,3} = G_{2,3} = G_3 = G_{T,3} = G_{\Theta,3} = 0 \tag{17c}$$

on the boundary quadrant Γ_{30}.

So, the expressions of GFPEs with boundary conditions (17a), (17b) and (17c) for octant V are the following [10]:

$$\begin{aligned} G_\Theta(x,\xi) = G_1(x,\xi) = \\ = (4\pi)^{-1}\left(R^{-1} - R_1^{-1} + R_2^{-1} - R_{12}^{-1} + R_3^{-1} - R_{13}^{-1} + R_{23}^{-1} - R_{123}^{-1}\right), \end{aligned} \tag{18a}$$

$$G_2(x,\xi) = (4\pi)^{-1}\left(R^{-1} - R_1^{-1} - R_2^{-1} + R_{12}^{-1} + R_3^{-1} - R_{13}^{-1} - R_{23}^{-1} + R_{123}^{-1}\right) \tag{18b}$$

$$G_3(x,\xi) = (4\pi)^{-1}\left(R^{-1} - R_1^{-1} + R_2^{-1} - R_{12}^{-1} - R_3^{-1} + R_{13}^{-1} - R_{23}^{-1} + R_{123}^{-1}\right) \tag{18c}$$

$$G_T(x,\xi) = (4\pi)^{-1}\left(R^{-1} + R_1^{-1} + R_2^{-1} + R_{12}^{-1} + R_3^{-1} + R_{13}^{-1} + R_{23}^{-1} + R_{123}^{-1}\right) \tag{18d}$$

where

$$\begin{aligned} R &= \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2} \\ R_1 &= \sqrt{(x_1+\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2} \\ R_2 &= \sqrt{(x_1-\xi_1)^2 + (x_2+\xi_2)^2 + (x_3-\xi_3)^2} \\ R_{12} &= \sqrt{(x_1+\xi_1)^2 + (x_2+\xi_2)^2 + (x_3-\xi_3)^2} \\ R_3 &= \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3+\xi_3)^2} \end{aligned} \tag{18e}$$

$$\begin{aligned}
R_{13} &= \sqrt{(x_1+\xi_1)^2+(x_2-\xi_2)^2+(x_3+\xi_3)^2}\\
R_{23} &= \sqrt{(x_1-\xi_1)^2+(x_2+\xi_2)^2+(x_3+\xi_3)^2}\\
R_{123} &= \sqrt{(x_1+\xi_1)^2+(x_2+\xi_2)^2+(x_3+\xi_3)^2}
\end{aligned}$$

On the base of GFPE $G_T(x,\xi)$ (18d), (18e) and of the proved theorem, we can rewrite its regular part $W_T(x,\xi)$ that contains inferior index 1 (that part of function $G_T(x,\xi)$ that is reflected via boundary Γ_{10}). So, the function $W_T(x,\xi)$ has the following expression:

$$W_T(x,\xi) = (2\pi)^{-1}\left(R_1^{-1}+R_{12}^{-1}+R_{13}^{-1}+R_{123}^{-1}\right). \tag{19}$$

Substituting expressions (18a), (18b), (18c), (18d), (18e) and (19) in the constructive formula (7) we obtain the final explicit expressions for MTGFs $U_i(x,\xi)\,; i=1,2,3$ and volume dilatation $\Theta(x,\xi)$ as follows:

$$\begin{gathered}
U_1(x,\xi) = \tfrac{\gamma}{8\pi(\lambda+2\mu)}\times\\
\Big[\tfrac{\xi_1}{\mu}(\lambda+2\mu)\left(R^{-1}+R_1^{-1}+R_2^{-1}+R_{12}^{-1}+R_3^{-1}+R_{13}^{-1}+R_{23}^{-1}+R_{123}^{-1}\right)-\\
\left(x_1+\tfrac{\lambda+\mu}{\mu}\xi_1\right)\left(R^{-1}-R_1^{-1}+R_2^{-1}-R_{12}^{-1}+R_3^{-1}-R_{13}^{-1}+R_{23}^{-1}-R_{123}^{-1}\right)-\\
2B^{-1}\xi_1\left(x_1\tfrac{\partial}{\partial x_1}+\tfrac{\lambda+2\mu}{\mu}\right)\left(R_1^{-1}+R_{12}^{-1}+R_{13}^{-1}+R_{123}^{-1}\right)\Big].
\end{gathered}\tag{20a}$$

$$\begin{gathered}
U_2(x,\xi) = \tfrac{\gamma}{8\pi(\lambda+2\mu)}\times\\
\Big\{\tfrac{\xi_2}{\mu}\left[(\lambda+2\mu)\left(R^{-1}+R_1^{-1}+R_2^{-1}+R_{12}^{-1}+R_3^{-1}+R_{13}^{-1}+R_{23}^{-1}+R_{123}^{-1}\right)-\right.\\
\left.(\lambda+\mu)\left(R^{-1}-R_1^{-1}+R_2^{-1}-R_{12}^{-1}+R_3^{-1}-R_{13}^{-1}+R_{23}^{-1}-R_{123}^{-1}\right)\right]-\\
x_2\left(R^{-1}-R_1^{-1}-R_2^{-1}+R_{12}^{-1}+R_3^{-1}-R_{13}^{-1}-R_{23}^{-1}+R_{123}^{-1}\right)-\\
2\xi_2\tfrac{\lambda+2\mu}{\mu}\left(R_1^{-1}+R_{12}^{-1}+R_{13}^{-1}+R_{123}^{-1}\right)-2B^{-1}\left(x_1\tfrac{\partial}{\partial x_1}-2\tfrac{\lambda+2\mu}{\lambda+\mu}\right)\times\\
\xi_1\tfrac{\partial}{\partial\xi_2}\ln\left(|x_1+\xi_1+R_1|\cdot|x_1+\xi_1+R_{12}|\cdot|x_1+\xi_1+R_{13}|\cdot|x_1+\xi_1+R_{1123}|\right)\Big\}
\end{gathered}\tag{20b}$$

$$\begin{gathered}
U_3(x,\xi) = \tfrac{\gamma}{8\pi(\lambda+2\mu)}\times\\
\Big\{\tfrac{\xi_3}{\mu}\left[(\lambda+2\mu)\left(R^{-1}+R_1^{-1}+R_2^{-1}+R_{12}^{-1}+R_3^{-1}+R_{13}^{-1}+R_{23}^{-1}+R_{123}^{-1}\right)-\right.\\
\left.(\lambda+\mu)\left(R^{-1}-R_1^{-1}+R_2^{-1}-R_{12}^{-1}+R_3^{-1}-R_{13}^{-1}+R_{23}^{-1}-R_{123}^{-1}\right)\right]-\\
x_3\left(R^{-1}-R_1^{-1}+R_2^{-1}-R_{12}^{-1}-R_3^{-1}+R_{13}^{-1}-R_{23}^{-1}+R_{123}^{-1}\right)-\\
2\xi_3\tfrac{\lambda+2\mu}{\mu}\left(R_1^{-1}+R_{12}^{-1}+R_{13}^{-1}+R_{123}^{-1}\right)-2B^{-1}\left(x_1\tfrac{\partial}{\partial x_1}-2\tfrac{\lambda+2\mu}{\lambda+\mu}\right)\times\\
\xi_1\tfrac{\partial}{\partial\xi_3}\ln\left(|x_1+\xi_1+R_1|\cdot|x_1+\xi_1+R_{12}|\cdot|x_1+\xi_1+R_{13}|\cdot|x_1+\xi_1+R_{1123}|\right)\Big\}
\end{gathered}\tag{20c}$$

– for MTGFs, and

$$\begin{gathered}
\Theta(x,\xi) = \tfrac{\gamma}{4\pi(\lambda+2\mu)}\Big[\left(R^{-1}-R_1^{-1}+R_2^{-1}-R_{12}^{-1}+R_3^{-1}-R_{13}^{-1}+R_{23}^{-1}-R_{123}^{-1}\right)+\\
\tfrac{2B^{-1}}{\lambda+\mu}\left(\mu x_1\tfrac{\partial}{\partial x_1}+\lambda+2\mu\right)\left(R_1^{-1}+R_{12}^{-1}+R_{13}^{-1}+R_{123}^{-1}\right)\Big]
\end{gathered}\tag{21}$$

– and for thermoelastic volume dilatation.

Also, the final explicit expressions for MTGFs $U_i(x,\xi)$ may be presented in the following compact form:

$$\begin{gathered}
U_i(x,\xi) = \tfrac{\gamma}{8\pi(\lambda+2\mu)}\Big[\tfrac{\partial}{\partial\xi_i}\left(R+R_1+R_2+R_{12}+R_3+R_{13}+R_{23}+R_{123}\right)\\
+2\left(\tfrac{\partial}{\partial\xi_i}-L_\Theta^{(i)}\tfrac{\partial}{\partial\xi_1}\right)\left(\xi_1\ln|x_1+\xi_1+R_1|\,|x_1+\xi_1+R_{12}|\,|x_1+\xi_1+R_{13}|\right.\\
\left.|x_1+\xi_1+R_{123}|-R_1-R_{12}-R_{13}-R_{123}\right)\Big]\\
L_\Theta^{(i)} = \left(\delta_{1i}-B^{-1}\xi_1\left(\partial/\partial\xi_i\right)\right).
\end{gathered}\tag{22}$$

Indeed, taking the derivatives in (22) it is observed that they coincide with expressions (20a), (20b), (20c).

As example of validation of the obtained MTGFs (20a), (20b), (20c) or (22) in Appendix is presented their graphs, constructed using Maple 15 software.

Finally, calculating on the basis of the functions (22) the other influence functions ($U_i(0, y_2, y_3, \xi)$ on marginal quadrant Γ_{10}, $U_i(y, \xi) = U_i(y_1, 0, y_3; \xi)$ on marginal quadrant Γ_{20} and $U_i(y, \xi) = U_i(y_1, y_2, 0; \xi)$ on marginal quadrant Γ_{30}) and substituting them in the formula (1) we obtain the following integral solution of above mentioned BVP for the thermoelastic octant in the form:

$$\begin{gathered} u_i(\xi) = a^{-1} \int_0^\infty \int_0^\infty \int_0^\infty F(z)\, U_i(z, \xi)\, dz_1 dz_2 dz_3 - \\ \int_0^{+\infty} \int_0^{+\infty} \frac{\partial T(0, y_2, y_3)}{\partial y_1} U_i(0, y_2, y_3; \xi)\, dy_2 dy_3 - \\ \int_0^{+\infty} \int_0^{+\infty} \frac{\partial T(y_1, 0, y_3)}{\partial y_2} U_i(y_1, 0, y_3; \xi)\, dy_1 dy_3 \\ - \int_0^\infty \int_0^\infty \frac{\partial T(y_1, y_2, 0)}{\partial y_3} U_i(y_1, y_2, 0; \xi)\, dy_1 dy_2 ; z \equiv (z_1, z_2, z_3) ; \xi \equiv (\xi_1, \xi_2, \xi_3), \end{gathered} \tag{23a}$$

where $U_i(z, \xi)$ are determined by Eq. (22); the other kernels are determined by the following expressions:

$$\begin{gathered} U_i(0, y_2, y_3; \xi) = \frac{\gamma}{8\pi(\lambda+2\mu)} \left[2\frac{\partial}{\partial \xi_i}(R + R_2 + R_3 + R_{23}) + 2\left(\frac{\partial}{\partial \xi_i} - L_\Theta^{(i)} \frac{\partial}{\partial \xi_1}\right) \right. \\ \left. \times (\xi_1 \ln|\xi_1 + R|\,|\xi_1 + R_2|\,|\xi_1 + R_3|\,|\xi_1 + R_{23}| - R_1 - R_{12} - R_{13} - R_{123}) \right] \end{gathered} \tag{23b}$$

$$\begin{gathered} U_i(y_1, 0, y_3; \xi) = \frac{\gamma}{8\pi(\lambda+2\mu)} \left[2\frac{\partial}{\partial \xi_i}(R + R_1 + R_3 + R_{13}) + \right. \\ \left. 4\left(\frac{\partial}{\partial \xi_i} - L_\Theta^{(i)} \frac{\partial}{\partial \xi_1}\right)(\xi_1 \ln|x_1 + \xi_1 + R_1|\,|x_1 + \xi_1 + R_{13}| - R_1 - R_{13}) \right] \end{gathered} \tag{23c}$$

$$\begin{gathered} U_i(x, \xi) = \frac{\gamma}{8\pi(\lambda+2\mu)} \left[2\frac{\partial}{\partial \xi_i}(R + R_1 + R_2 + R_{12}) + \right. \\ \left. 4\left(\frac{\partial}{\partial \xi_i} - L_\Theta^{(i)} \frac{\partial}{\partial \xi_1}\right)(\xi_1 \ln|x_1 + \xi_1 + R_1|\,|x_1 + \xi_1 + R_{12}| - R_1 - R_{12}) \right] \end{gathered} \tag{23d}$$

Note, that from formulas (21) and (22) for thermoelastic octant we can obtain respective new volume dilatation and MTGFs for half-space $V\,(0 \le x_1 < +\infty, -\infty < x_2, x_3 < +\infty)$ and also for quarter-space $V\,(0 \le x_1, x_2 < +\infty, -\infty < x_3 < +\infty)$. To obtain these results is enough to omit in the formulas (21) and (22) for thermoelastic octant the terms that contain inferior indexes $1, 3$ and index 3 respectively.

4. Checking the Derived MTGFs for Thermoelastic Octant

Here we check the obtained MTGFs $U_i(x, \xi)$ for octant by using their proprieties as functions of double influence which take in consideration both physical phenomena of solid body: heat conduction and elasticity [10, 19, 20, 21, 22, 23]. Also in Appendix is presented MTGFs $U_i(x, \xi)$ and GFPE $G_T(x, \xi)$ for octant which are evaluated numerically and graphically using Maple 15 software.

4.1. **Checking the derived MTGFs for thermoelastic octant with respect to point** $x \equiv (x_1, x_2, x_3)$**.** Here we check the MTGFs for thermoelastic octant derived in the section 3. So, according to works [10, 19, 20, 21, 22, 23], MTGFs $U_i(x, \xi)$ determined by (22) must satisfy over the coordinates of the point of observation $x \equiv (x_1, x_2, x_3)$ the equation

$$\nabla_x^2 U_i(x, \xi) = -\gamma \Theta^{(i)}(x, \xi) \tag{24}$$

and the homogeneous boundary conditions similar to the GFPE G_T for octant shown in Eqs. (17a), (17b), (17c):

$$\partial U_i(0, y_2, y_3; \xi) / \partial n_{10} = 0, y \equiv (0, y_2, y_3) \in \Gamma_{10} \tag{25a}$$

$$\partial U_i(y_1, 0, y_3; \xi) / \partial n_{20} = 0, y \equiv (y_1, 0, y_3; \xi) \in \Gamma_{20} \tag{25b}$$

$$\partial U_i\left(y_1,y_2,0;\xi\right)/\partial n_{30}=0,y\equiv\left(0,y_2,y_3\right)\in\Gamma_{30} \tag{25c}$$

So, calculating Laplace operator from $U_i\left(x,\xi\right)$, determined by equation (22), we obtain:

$$\begin{gathered}\nabla_x^2U_i\left(x,\xi\right)=\gamma\left[8\pi\left(\lambda+2\mu\right)\right]^{-1}\nabla_x^2U_i\left(x,\xi\right)=\gamma\left[8\pi\left(\lambda+2\mu\right)\right]^{-1}\times\\ \Big[2\tfrac{\partial}{\partial\xi_i}\left(R^{-1}+R_1^{-1}+R_2^{-1}+R_{12}^{-1}+R_3^{-1}+R_{13}^{-1}+R_{23}^{-1}+R_{123}^{-1}\right)+\\ +4\left(\tfrac{\partial}{\partial\xi_i}-L_\Theta^{(i)}\tfrac{\partial}{\partial\xi_1}\right)\left(-R_1^{-1}-R_{12}^{-1}-R_{13}^{-1}-R_{123}^{-1}\right)\Big]=\\ \gamma\left[4\pi\left(\lambda+2\mu\right)\right]^{-1}\Big[\tfrac{\partial}{\partial\xi_i}\left(R^{-1}-R_1^{-1}+R_2^{-1}-R_{12}^{-1}+R_3^{-1}-R_{13}^{-1}+R_{23}^{-1}-R_{123}^{-1}\right)\\ +2L_\Theta^{(i)}\tfrac{\partial}{\partial x_1}\left(R_1^{-1}+R_{12}^{-1}+R_{13}^{-1}+R_{123}^{-1}\right)\Big]\end{gathered} \tag{26a}$$

But according to handbook [10] (see answer to problem 16.L.9) the obtained in Eq. (26a) last expression may be written as follows:

$$\begin{gathered}\gamma\left[4\pi\left(\lambda+2\mu\right)\right]^{-1}\Big[\tfrac{\partial}{\partial\xi_i}\left(R^{-1}-R_1^{-1}+R_2^{-1}-R_{12}^{-1}+R_3^{-1}-R_{13}^{-1}+R_{23}^{-1}-R_{123}^{-1}\right)\\ +2L_\Theta^{(i)}\tfrac{\partial}{\partial\xi_1}\left(R_1^{-1}+R_{12}^{-1}+R_{13}^{-1}+R_{123}^{-1}\right)\Big]=-\gamma\Theta^{(i)}\end{gathered} \tag{26b}$$

So, substituting the result (26a) in (26b) in equation (24) we see that it is satisfied.

Next taking the derivative in direction to normal n_{10} on the marginal quadrant Γ_{10} of MTGFs $U_i\left(x,\xi\right)$, determined by (22), at $x\to y\equiv\left(0,y_2,y_3\right)$ we can see that boundary condition in (25a) is satisfied. Indeed,

$$\begin{gathered}\tfrac{\partial U_i(x,\xi)}{\partial n_{10}}\Big|_{x\to y\equiv(0,y_2,y_3)}=-\tfrac{\partial U_i(x,\xi)}{\partial x_1}\Big|_{x\to y\equiv(0,y_2,y_3)}=\\ -\tfrac{\gamma}{8\pi(\lambda+2\mu)}\Big[\tfrac{\partial^2}{\partial\xi_i\partial\xi_1}\left(-R+R_1-R_2+R_{12}-R_3+R_{13}-R_{23}+R_{123}\right)+\\ 2\left(\tfrac{\partial}{\partial\xi_i}-L_\Theta^{(i)}\tfrac{\partial}{\partial\xi_1}\right)\Big(\xi_1\tfrac{\partial}{\partial\xi_1}\ln\left|x_1+\xi_1+R_1\right|\left|x_1+\xi_1+R_{12}\right|\left|x_1+\xi_1+R_{13}\right|\\ \left|x_1+\xi_1+R_{123}\right|-\tfrac{\partial}{\partial\xi_1}\left(R_1+R_{12}+R_{13}+R_{123}\right)\Big)\Big]\Big|_{x\to y\equiv(0,y_2,y_3)}=0\end{gathered} \tag{27a}$$

Taking the derivative in direction to normal n_{20} on the marginal quadrant Γ_{20} of MTGFs $U_i\left(x,\xi\right)$, determined by (22), at $x\to y\equiv\left(y_1,0,y_3\right)$ we can see that boundary condition in (25b) also is satisfied:

$$\begin{gathered}\tfrac{\partial U_i(x,\xi)}{\partial n_{20}}\Big|_{x\to y\equiv(y_1,0,y_3)}=-\tfrac{\partial U_i(x,\xi)}{\partial x_2}\Big|_{x\to y\equiv(y_1,0,y_3)}=-\tfrac{\gamma}{8\pi(\lambda+2\mu)}\times\\ \Big[\tfrac{\partial^2}{\partial\xi_i\partial\xi_2}\left(-R-R_1+R_2+R_{12}-R_3-R_{13}+R_{23}+R_{123}\right)+2\left(\tfrac{\partial}{\partial\xi_i}-L_\Theta^{(i)}\tfrac{\partial}{\partial\xi_1}\right)\\ \tfrac{\partial}{\partial\xi_2}\left(\xi_1\ln\tfrac{|x_1+\xi_1+R_{12}|\cdot|x_1+\xi_1+R_{123}|}{|x_1+\xi_1+R_1|\cdot|x_1+\xi_1+R_{13}|}+R_1-R_{12}+R_{13}-R_{123}\right)\Big]_{x\to y\equiv(y_1,0,y_3)}=0\end{gathered} \tag{27b}$$

Finally, taking the derivative in direction to normal n_{30} on the marginal quadrant Γ_{30} of MTGFs $U_i\left(x,\xi\right)$, determined by (22), at $x\to y\equiv\left(y_1,y_2,0\right)$ we can see that boundary condition in (25c) also is satisfied:

$$\begin{gathered}\tfrac{\partial U_i(x,\xi)}{\partial n_{30}}\Big|_{x\to y\equiv(y_1,y_2,0)}=-\tfrac{\partial U_i(x,\xi)}{\partial x_3}\Big|_{x\to y\equiv(y_1,y_2,0)}=-\tfrac{\gamma}{8\pi(\lambda+2\mu)}\times\\ \Big[\tfrac{\partial^2}{\partial\xi_i\partial\xi_3}\left(-R-R_1-R_2-R_{12}+R_3+R_{13}+R_{23}+R_{123}\right)+2\left(\tfrac{\partial}{\partial\xi_i}-L_\Theta^{(i)}\tfrac{\partial}{\partial\xi_1}\right)\\ \tfrac{\partial}{\partial\xi_3}\left(\xi_1\ln\tfrac{|x_1+\xi_1+R_{13}||x_1+\xi_1+R_{123}|}{|x_1+\xi_1+R_1||x_1+\xi_1+R_{12}|}+R_1+R_{12}-R_{13}-R_{123}\right)\Big]_{x\to y\equiv(y_1,y_2,0)}=0.\end{gathered} \tag{27c}$$

4.2. **Checking the derived MTGFs for thermoelastic octant with respect to point $\xi\equiv\left(\xi_1,\xi_2,\xi_3\right)$.** According to works [10, 19, 20, 21, 22, 23], MTGFs $U_i\left(x,\xi\right)$ determined by (22) must satisfy over the coordinates of the point of application $\xi\equiv\left(\xi_1,\xi_2,\xi_3\right)$ of the unit point heat source to Lame thermoelastic equations

$$\mu\nabla_\xi^2U_i\left(x,\xi\right)+\left(\lambda+\mu\right)\Theta_{,\,\xi_i}\left(x,\xi\right)-\gamma G_{T,\,\xi_i}\left(x,\xi\right)=0 \tag{28a}$$

and boundary conditions in Eqs (16a), (16b), (16c). Indeed, taking Laplace operator from MTGFs $U_i(x,\xi)$ (22) we obtain:

$$\begin{gathered}\mu\nabla_\xi^2 U_i(x,\xi)=\mu\tfrac{\gamma}{4\pi(\lambda+2\mu)}\times\\ \Big[\tfrac{\partial}{\partial\xi_i}\left(R^{-1}+R_1^{-1}+R_2^{-1}+R_{12}^{-1}+R_3^{-1}+R_{13}^{-1}+R_{23}^{-1}+R_{123}^{-1}\right)+\\ 2B^{-1}\tfrac{\partial}{\partial\xi_i}\left(\ln|x_1+\xi_1+R_1|\,|x_1+\xi_1+R_{12}|\,|x_1+\xi_1+R_{13}|\,|x_1+\xi_1+R_{123}|-\right.\\ \left.x_1\left(R_1^{-1}+R_{12}^{-1}+R_{13}^{-1}+R_{123}^{-1}\right)\right)\Big]\end{gathered}\tag{28b}$$

Then substituting (28b), (18d), (18e), and (21) into equation (28a) we can see that Lame thermoelastic equations are satisfied.

Also we checked and confirm that the MTGFs $U_i(x,\xi)$ determined by Eq. (22) with respect to points $\xi\equiv(\xi_1,\xi_2,\xi_3)$ satisfy to boundary conditions (16a), (16b), (16c) or to following identical conditions:

$$U_1(x;0,\xi_2,\xi_3)=U_2(x;0,\xi_2,\xi_3)=U_3(x;0,\xi_2,\xi_3)=0\tag{29a}$$

$$\begin{gathered}\sigma_{21}(x;\xi_1,0,\xi_3)=U_2(x;\xi_1,0,\xi_3)=\sigma_{23}(x;\xi_1,0,\xi_3)=0\Rightarrow\\ U_{1,2}(x;\xi_1,0,\xi_3)=U_2(x;\xi_1,0,\xi_3)=U_{3,2}(x;\xi_1,0,\xi_3)=0;\xi\equiv(\xi_1,0,\xi_3)\in\Gamma_{20}\end{gathered}\tag{29b}$$

$$\begin{gathered}\sigma_{31}(x;\xi_1,\xi_2,0)=\sigma_{32}(x;\xi_1,\xi_2,0)=U_3(x;\xi_1,\xi_2,0)=0\Rightarrow\\ U_{1,3}(x;\xi_1,\xi_2,0)=U_{2,3}(x;\xi_1,\xi_2,0)=U_3(x;\xi_1,\xi_2,0)=0;\xi\equiv(\xi_1,\xi_2,0)\in\Gamma_{30}.\end{gathered}\tag{29c}$$

Finally we have proved that derived MTGFs for octant in Eq. (22) satisfy to respective BVPs of thermoelasticity described by Eqs. (28a) and (16a), (16b).

5. Conclusions

A new technique is proposed for derivation MTGFs $U_i(x,\xi)$ directly from respective Lame's equations (28a). This technique is based on new general integral representations for MTGFs $U_i(x,\xi)$ which are presented in Eqs. (3) and (4). A theorem about constructive formulas (7) for MTGFs $U_i(x,\xi)$ in terms of GFPE and other regular harmonic functions is proved. According to the obtained constructive formulas the MTGFs for a group of two-and three dimensional BVPs for a plane, a half-plane, a quadrant, a space, a quarter-space and an octant may be obtained by changing the respective well-known GFPEs and by calculating simple integrals. New MTGFs for octant, quarter-space and half-space are derived. All results are obtained in terms of elementary functions. The checking of correctitude of obtained results is given for a new BVP for thermoelastic octant. The derived MTGFs $U_i(x,\xi)$ and GFPE $G_T(x,\xi)$ for octant were evaluated numerically and graphically by using Maple 15 software. The main advantages of the proposed method in comparison with the $G\Theta$ convolution method for MTGFs constructing are: a. It is not necessary to derive the functions of influence of a unit concentrated force onto elastic volume dilatation - $\Theta^{(i)}$ and, b. It is not necessary to calculate a complicate volume integral of the product of the function $\Theta^{(i)}$ and Green's function G_T in the heat conduction theory. Also the proposed technique may be extended to different canonical domains of Cartesian system of coordinates.

References

[1] Duffy, D.G., *Green's Functions with Applications*, Chapman and Hall/CRC Press, Boca Raton, Florida, 2001.
[2] Greenberg, M.D., *Application of Green's Functions in Science and Engineering*, Prentice-Hall, Upper Saddle River, NJ, 1971.
[3] Stakgold, I., Holst, M., *Green's Functions and Boundary Value Problems*, Third Edition, John Wiley & Sons, New York, NY, 2011.
[4] Roach, G.F., *Green's Functions*, Cambridge University Press, New York, N.Y., 1982.

[5] Melnikov, Yu.A., *Green's Functions in Applied Mechanics*, Computational Mechanics Publications, Southampton, UK, 1995.

[6] Melnikov, Yu.A., *Influence Functions and Matrices*, Marcel Dekker, New York, N.Y., 1999.

[7] Berger, J.R., Tewary, V.K., *Green's functions and boundary element analysis for modeling of mechanical behavior of advanced materials*, in: Proc. of a workshop, NIST, Boulder, Colorado, Diane Publish., 1996.

[8] Qin, Q.H., *Green's function and boundary elements in multifield materials*, Elsevier, Oxford, UK, 2007.

[9] Şeremet, V., Bonnet, G., *Encyclopedia of Domain Green's Functions (Thermomagneto-electrostatics of solids in rectangular and polar coordinates)*, Edit. Center, Agrar. State Univ. of Moldova, Chișinău, 2008.

[10] Şeremet, V.D., *Handbook on Green's Functions and Matrices*, WIT Press, Southampton, 2003.

[11] Boley, B.A., Weiner, J.F., *Theory of Thermal Stresses*, Wiley, New York, 1960.

[12] Kovalenko, A.D., *Fundamentals of Thermoelasticity* (Russian), Naukova Dumka, Kiev, 1970.

[13] Mayzel, V.M., *The Temperature Problem of the Theory of Elasticity*, Publisher AN SSSR, Kiev, 1951.

[14] Melan E., Parkus H., *Thermoelastic stresses caused by the stationary heat fields*, Fizmatgiz, Moscow, 1958.

[15] Nowacki, W., *The Theory of Elasticity*, Mir, Moscow, 1975.

[16] Nowacki, W., *Thermoelasticity*, Pergamon Press and Polish Sc. Publishers, Oxford, Warszawa, 1962.

[17] Nowinski, J.L., *Theory of Thermoelasticity with Applications*, Sijthoff and Noordhoff International Publishers, Alphen Aan Den Rijn, 1978.

[18] Hetnarski, R.B., Eslami, M.R., *Thermal Stresses - Advanced Theory and Applications*, XXXII+, 559 pages, Springer, Dordrecht, 2009.

[19] Şeremet, V.D., *The modification of Maysel's formula in the stationary thermoelasticity*, Bulletin of Academy of Science of Republic of Moldova, Mathematics, 25 **(3)**, pp. 19 – 22, 1997.

[20] Şeremet, V., *New Results in 3-D Thermoelasticity*, in Proc. 14th U.S. National Congress of Theoretical and Applied Mechanics, p.29, 2002.

[21] Şeremet, V., *Some New Influence Functions and Integral Solutions in Theory of Thermal Stresses*, in Proc. IV-th International Congress on Thermal Stresses, p. 423, 2001.

[22] Şeremet, V., *The Integral Equations and Green's Matrices of the Influence Elements Method in the Mechanics of Solids* (Romanian), Dr. Habilitat Thesis, Technical University of Moldova, Chișinău, 1995.

[23] Şeremet, V., *Generalization of Green's formulae in thermoelasticity*, Collection: Multiscale Green's Functions for Nanostructures, National Science Digital Library of USA, NSF, 4p, 2003.

[24] Irschik, H., Ziegler F., *Maysel's formula generalized for piezoelectric vibrations - Application to thin shells of revolution*, AIAA Journal, 34 **(11)**, 2402–2405, 1996.

[25] Irschik, H., Pichler, U., *Maysel's Formula for Small Vibrations Superimposed Upon Large Static Deformations of Piezoelectric Bodies*, in IUTAM Symposium on Dynamics of Advanced Materials and Smart Structures, 106, 125–136, 2003.

[26] Irschik, H., Pichler, U., Gerstmayr, J., *Maysel's formula of thermoelasticity extended to anisotropic materials at finite strain*, International Journal of Solids and Structures 38 **(52)**, pp. 9479–9492, 2001.

[27] Irschik, H., Fotiu, P., Ziegler F., *Extension of Maysel's Formula to the Dynamic Eigenstrain Problem*, Journal of the Mechanical Behavior of Materials 5 **(1)**, pp. 59–66, 2011.

[28] Șeremet, V., Bonnet, G., *New closed-form thermoelastostatic Green function and Poisson-type integral formula for a quarter-plane*, Mathematical and Computer Modeling 53 **(1-2)**, pp. 347–358, 2011.

[29] Șeremet, V., *New closed-form Green function and integral formula for a thermoelastic quadrant*, Applied Mathematical Modeling 36 **(2)**, pp. 799–812, 2012.

[30] Șeremet, V., *A new technique to derive the Green's type integral formula in thermoelasticity*, Engineering Mathematics 69 **(4)**, pp. 313–326, 2011.

[31] Șeremet, V., *New Poisson's integral formulas for thermoelastic half-space and other canonical domains*, Engineering Analysis with Boundary Elements 34 **(2)**, pp. 158–162, 2010.

[32] Șeremet, V., *New explicit Green's functions and Poisson's integral formula for a thermoelastic quarter-space*, Journal of Thermal Stresses 33 **(4)**, pp. 356–386, 2010.

[33] Șeremet, V., *Thermoelastostatic equilibrium of a spatial quadrant: Green's function and solution in integrals*, Archive of Applied Mechanics 82 **(12)**, pp. 1771–1793, 2012.

[34] Șeremet, V., *Deriving exact Green's functions and integral formulas for a thermoelastic wedge*, Engineering Analysis with Boundary Elements 35 **(3)**, pp. 327–332, 2011.

[35] Șeremet, V., Crețu, I., *Influence functions, integral formulas, and explicit solutions for thermoelastic spherical wedges*, Acta Mechanica 224 **(4)**, pp. 893–918, 2013.

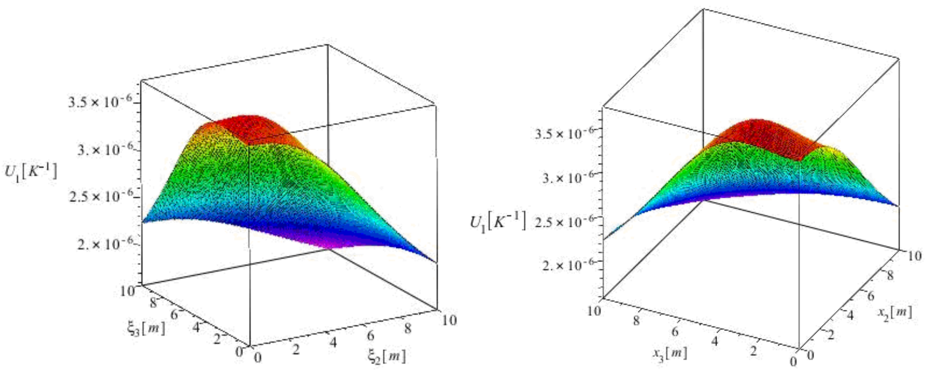
[36] Șeremet, V., *Exact elementary Green functions and Poisson-type integral formulas for a thermoelastic half-wedge with applications*, Journal of Thermal Stresses 33 **(12)**, pp. 1156–118, 2010.

[37] Șeremet, V., *A new approach to constructing Green's functions and integral solutions in thermoelasticity*, Acta Mechanica, DOI 10.1007/s00707-013-0974-3, 2013.

[38] Șeremet, V., *Recent Integral Representations for Thermoelastic Green's Functions and Many Examples of Their Exact Analytical Expressions*, Journal of Thermal Stresses, Accepted, 2014.

Appendix A. Graphs of MTGFs U_i and GFPE G_T within a Thermoelastic Octant

Here we present Figures 1, 2, 3, 4, 5 and 6, showing the MTGFs $U_i(x,\xi)$, determined by Eqs. (20a), (20b) and (20c) or (22) for 3D BVP of thermoelasticity (28a), (29a), (29b) and (29c) for the octant $V\,(0 \le x_1, x_2, x_3 < \infty)$. Also we present Figure 7 showing the GFPE $G_T(x,\xi)$ for a 3D BVP of heat conduction theory within octant V that consist from Poisson equation $\nabla_x^2 G_T(x,\xi) = -\delta(x-\xi)$ and boundary conditions (17a), (17b) and (17c) obtained using Maple 15 software. The MTGFs $U_i(x,\xi)$ are generated by the unitary inner point heat source $F = \delta(x-\xi)$. All twelve graphs for the MTGFs $U_i(x,\xi)$ were constructed at the following values of the constants: Poisson's ratio $\nu = 0.3$; elasticity modulus $E = 2,1 \cdot 10^5 Mpa$ and coefficient of linear thermal expansion $\alpha_t = 1.2 \cdot 10^{-5}\,(\mathrm{K})^{-1}$. Graphs of changing MTGFs $U_1(x,\xi)$ in dependence of $0 \le \xi_1 \le 10m$; $0 \le \xi_3 \le 10m$ constructed at $\xi_1 = 2m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ - Figure 1a and in dependence of $0 \le x_1 \le 10m$; $0 \le x_3 \le 10m$ constructed at $x_1 = 2m$; $\xi_1 = 2m$; $\xi_2 = 3m$; $\xi_3 = 4m$ - Figure 1b are showed in the Figure 1.



(A) (B)

Figure 1. Graphs of changing MTGFs U_1 in dependence of ξ_2 ,ξ_3 constructed at $\xi_1 = 2m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ – Figure 1a; and in dependence of x_2, x_3 constructed at $x_1 = 2m$; $\xi_1 = 2m$; $\xi_2 = 3m$; $\xi_3 = 4m$ – Figure 1b.

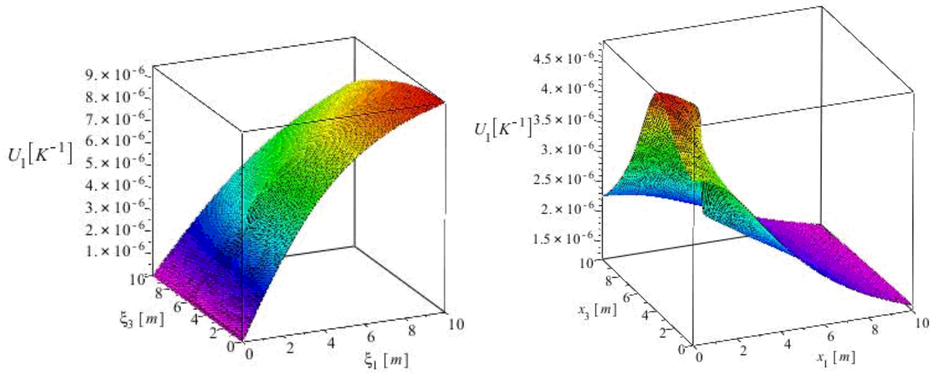
In the Figure 1 it can be observed:

(1) At $\xi_2 = 0$ and $\xi_3 = 0$ the lines tangential to surface U_1 in direction ξ_2 and ξ_3 are parallel to axes $0\xi_2$ and $0\xi_3$. This means that boundary condition $U_{1,\xi_2}(x;\xi_1,\xi_2 = 0,\xi_3) = 0$ (29b) on the marginal quadrant Γ_{20} and boundary condition $U_{1,\xi_3}(x;\xi_1,\xi_2,\xi_3 = 0) = 0$ (29c) on the marginal quadrant Γ_{30} are satisfied (Figure 1a);

(2) At $x_2 = 0$ and $x_3 = 0$ the lines tangential to surface U_1 in direction x_2 and x_3 are parallel to axes $0x_2$ and $0x_3$. This means that boundary condition $U_{1,x_2}(x,x_1,x_2 = 0,x_3;\xi) = 0$ (27b) on the marginal quadrant Γ_{20} and boundary condition $U_{1,x_3}(x_1,x_2,x_3 = 0;\xi) = 0$ (27c) on the marginal quadrant Γ_{30} are satisfied (Figure 1a); This is explaining by fact that boundary conditions for U_1 with respect to $x \equiv (x_1,x_2,x_3)$ are similar to those for function G_T showed in Eqs. (17a), (17b) and (17c);

(3) In the both Figures (Figure 1a and Figure 1b) the MTGF U_1 is positive. These graphs are identical. This is explains by Eq. (20a) that at $x_1 = \xi_1 = 2mU_1(x,\xi) = U_1(\xi, x)$.

Graphs of changing MTGFs $U_1\,(x,\xi)$ in dependence of $0 \le \xi_1 \le 10m$; $0 \le \xi_3 \le 10m$, constructed at $x_1 = 2m$; $\xi_2 = 3m$;$x_2 = 3m$; $x_3 = 4m$ – Figure 2a and in dependence of $0 \le x_1 \le 10m$; $0 \le x_3 \le 10m$ constructed at $x_2 = 3m$; $\xi_1 = 2m$; $\xi_2 = 3m$; $\xi_3 = 4m$ - Figure 2b is showed in the Figure 2.



(A) (B)

FIGURE 2. Graphs of changing MTGFs U_1 in dependence of ξ_1, ξ_3, constructed at $\xi_2 = 3m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ – Figure 2a; and in dependence of x_1, x_3, constructed at $x_2 = 3m$; $\xi_1 = 2m$; $\xi_2 = 3m$; $\xi_3 = 4m$ – Figure 2b.

In Figure 2 it can be observed:

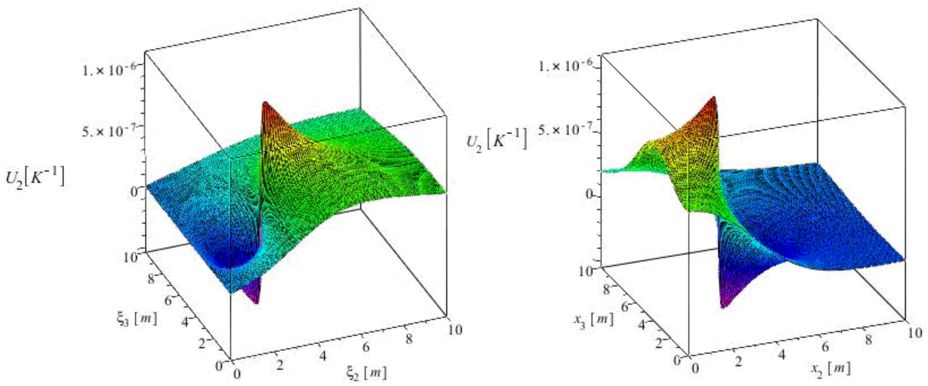
(1) At $\xi_1 = 0$ the MTGF U_1 is zero. This means that boundary conditions and $U_1(x; \xi_1 = 0, \xi_2, \xi_3) = 0$ (29a) on the marginal quadrant Γ_{10} is satisfied. At $\xi_3 = 0$ the lines tangential to surface U_1 in direction ξ_3 are parallel to axis $0\xi_3$. This means that $U_{1,\xi 3}\,(x; \xi_1, \xi_2, \xi_3 = 0) = 0$ on the marginal quadrant Γ_{30}, so that boundary condition (29c) is satisfied (Figure 2a);

(2) At $x_1 = 0$ and $x_3 = 0$ the lines tangential to surface U_1 in directions x_1 and x_3 are parallel to axes $0x_1$ and $0x_3$. This means that $U_{1,x_1}\,(x_1 = 0, x_2, x_3; \xi) = 0$ and $U_{1,x_3}\,(x_1, x_2, x_3 = 0; \xi) = 0$ on the marginal quadrants Γ_{10} and Γ_{30}, so that boundary conditions (27a) and (27c) are satisfied (Figure 2b). This is explaining by fact that boundary conditions for U_1 with respect to $x \equiv (x_1, x_2, x_3)$ are similar to those for function G_T, shown in equations (17a), (17b) and (17c);

Graphs of changing MTGFs $U_2\,(x,\xi)$ in dependence of $0 \le \xi_2 \le 10m$; $0 \le \xi_3 \le 10m$, constructed at $\xi_1 = 2m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ – Figure 3a and in dependence of $0 \le x_2 \le 10m$; $0 \le x_3 \le 10m$, constructed at $x_1 = 2m$; $\xi_1 = 2m$; $\xi_2 = 3m$; $\xi_3 = 4m$ – Figure 3b are showed in the Figure 3.

In the Figure 3 it can be observed:

(1) At $\xi_2 = 0$ the MTGF U_2 is zero. This means that boundary conditions $U_2(x; \xi_1, \xi_2 = 0, \xi_3) = 0$ (29b) on the marginal quadrants Γ_{20} are satisfied (Figure 3a); At $\xi_3 = 0$ the lines tangential to surface U_2 in direction ξ_3 are parallel to axis $0\xi_3$. This means that boundary conditions $U_{2,\xi_3}(x; \xi_1, \xi_2, \xi_3 = 0) = 0$ (29c) on the marginal quadrants Γ_{30} are satisfied (Figure 3a);



(A) (B)

FIGURE 3. Graphs of changing MTGFs U_2 in dependence of ξ_2, ξ_3 constructed at $\xi_1 = 2m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ – Figure 3a; and in dependence of x_2, x_3 constructed at $x_1 = 2m$; $\xi_1 = 2m$; $\xi_2 = 3m$; $\xi_3 = 4m$ – Figure 3b.

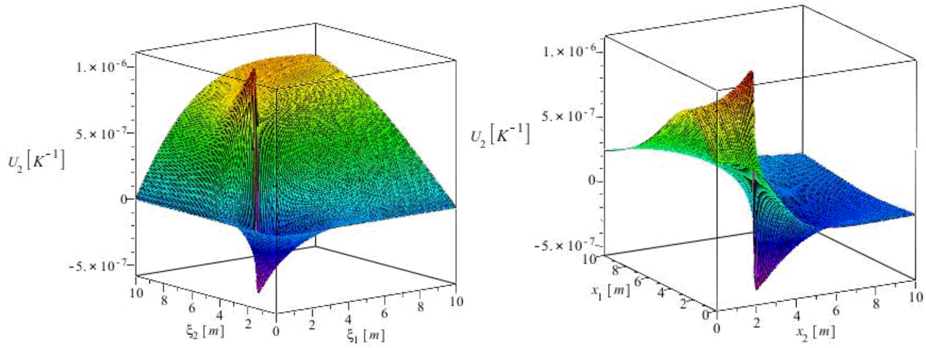
(2) At $x_3 = 0$ the lines tangential to surface U_2 in direction x_3 are parallel to axis $0x_3$. This means that boundary conditions $U_{2,x_3}(x_1, x_2, x_3 = 0; \xi) = 0$ (27c) on the marginal quadrants Γ_{30} are satisfied (Figure 3b); Also, at $x_2 = 0$ the lines tangential to surface U_2 in direction x_2 are parallel to axis $0x_2$. This means that boundary condition $U_{2,x_2}(x_1, x_2 = 0, x_3; \xi) = 0$ (27b) on the marginal quadrant Γ_{20} are satisfied (Figure 3b). This is explaining by fact that boundary conditions for U_2 with respect to $x \equiv (x_1, x_2, x_3)$ are similar to those for function G_T, shown in equations (17a) and (17c);
(3) In the Figure 3 both graphs (in the Figures 3a and 3b) have jumps: at $\xi_1 = 2m$, $\xi_3 = 4m$ (Figure 3a) and at $x_1 = 2m$, $x_3 = 4m$ (Figure 3b);

Graphs of changing MTGFs $U_2(x, \xi)$ in dependence of $0 \le \xi_1 \le 10m$; $0 \le \xi_2 \le 10m$, constructed at $\xi_3 = 1m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ – Figure 4a, and in dependence of $0 \le x_1 \le 10m$; $0 \le x_2 \le 10m$, constructed at $x_3 = 1m; \xi_1 = 2m$; $\xi_2 = 3m$; $\xi_3 = 4m$ – Figure 4b are showed in the Figure 4.

In Figure 4 it can be observed:

(1) At $\xi_1 = 0$ and at $\xi_2 = 0$ the MTGF U_2 is zero. This means that boundary conditions $U_2(x; \xi_1 = 0, \xi_2, \xi_3) = 0$ (29a) and $U_2(x; \xi_1, \xi_2 = 0, \xi_3) = 0$ (29b) on the marginal quadrants Γ_{10} and Γ_{20} are satisfied (Figure 4a);
(2) At $x_1 = 0$ and $x_2 = 0$ (Figures 4a and 4b) the lines tangential to surfaces U_2 in direction x_1 and x_2 are parallel to axes $0x_1$ and $0x_2$. This means that boundary condition $U_{2,x_1}(x_1 = 0, x_2, x_3; \xi) = 0$ (27a) and $U_{2,x_2}(x_1, x_2 = 0, x_3; \xi) = 0$ (27b) on the marginal quadrant Γ_{20} are satisfied (Figure 4b). Note that these boundary conditions follows from similar boundary conditions for function G_T, shown in equations (17a) and (17c);
(3) In the Figure 4 both graphs (in the Figures 4a and 4b) have jumps at $\xi_1 = 2m$, $\xi_3 = 4m$ (Figure 4a) and at $x_1 = 2m$, $x_3 = 4m$ (Figure 4b);

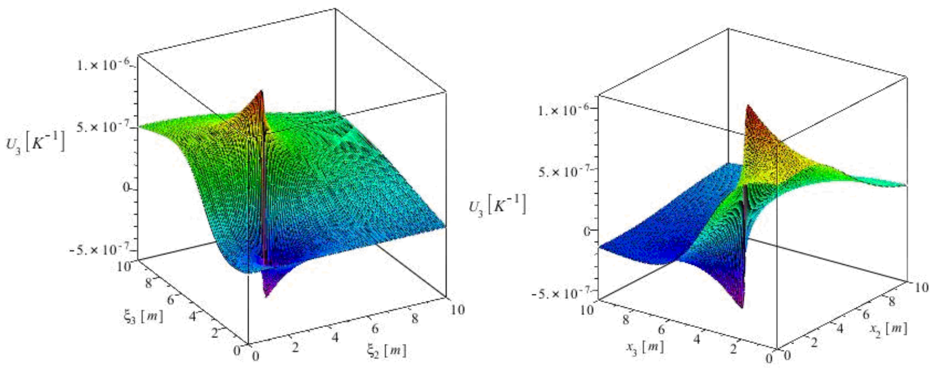
Graphs of changing MTGFs $U_3(x, \xi)$ in dependence of $0 \le \xi_2 \le 10m$; $0 \le \xi_3 \le 10m$, constructed at $\xi_1 = 2m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ – Figure 5a and in dependence of



(A) (B)

FIGURE 4. Graphs of changing MTGFs U_2 in dependence of ξ_1, ξ_2 constructed at $\xi_3 = 4m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ – Figure 4a; and in dependence of x_1, x_2, constructed at $x_3 = 4m$; $\xi_1 = 2m$; $\xi_2 = 3m$; $\xi_3 = 4m$ – Figure 4b.

$0 \leq x_2 \leq 10m$; $0 \leq x_3 \leq 10m$, constructed at $x_1 = 2m$; $\xi_1 = 2m$; $\xi_2 = 3m$; $\xi_3 = 4m$ – Figure 5b are showed in the Figure 5.



(A) (B)

FIGURE 5. Graphs of changing MTGFs U_3 in dependence of ξ_2, ξ_3 constructed at $\xi_1 = 2m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ – Figure 5a; and in dependence of x_2, x_3, constructed at $x_1 = 2m$; $\xi_1 = 2m$; $\xi_2 = 3m$; $\xi_3 = 4m$ – Figure 5b.

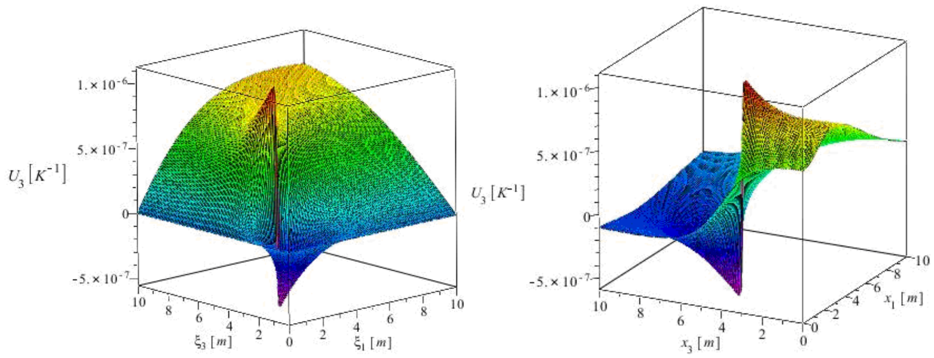
In the Figure 5 it can be observed:

(1) At $x_2 = 0$ and $x_3 = 0$ the lines tangential to surfaces U_3 in direction x_2 and in direction x_3 are parallel to axis $0x_2$ and $0x_3$ respectively . This means that boundary condition $U_{3,x_2}(x_1, x_2 = 0, x_3; \xi) = 0$ (27b) on the marginal quadrant Γ_{20} and boundary condition $U_{3,x_3}(x_1, x_2, x_3 = 0; \xi) = 0$ (27c), on the marginal quadrant Γ_{30} are satisfied (Figure 5b). This is explaining by fact that boundary conditions for U_3 with respect to $x \equiv (x_1, x_2, x_3)$ are similar to those for function G_T, shown in equations (17a), (17b) and (17c);

(2) At $\xi_3 = 0$ the MTGF U_3 is zero. This means that boundary condition $U_3(x; \xi_1, \xi_2, \xi_3 = 0) = 0$ (29c) on the marginal quadrant Γ_{30} is satisfied (Figure 5a). At $\xi_2 = 0$ the lines tangential to surfaces U_3 in direction ξ_2 are parallel to axis $0\xi_2$. This means that the boundary condition $U_{3,\xi_2}(x; \xi_1, \xi_2 = 0, \xi_3) = 0$ (29b) on the marginal quadrant Γ_{20} is satisfied (Figure 5a);

(3) In the Figure 5 both graphs (in the Figures 5a and 5b) have jumps at $\xi_1 = 2m$, $\xi_3 = 4m$ (Figure 5a) and at $x_1 = 2m$, $x_3 = 4m$ (Figure 5b);

Graphs of changing MTGF $U_3\,(x, \xi)$ in dependence of $0 \le \xi_1 \le 10m$; $0 \le \xi_3 \le 10m$, constructed at $\xi_2 = 3m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ – Figure 6a and in dependence of $0 \le x_1 \le 10m$; $0 \le x_3 \le 10m$, constructed at $x_2 = 3m$; $\xi_1 = 2m$; $\xi_2 = 3m$; $\xi_3 = 4m$ – Figure 6b are showed in the Figure 6.



(A) (B)

FIGURE 6. Graphs of changing MTGFs U_3 in dependence of ξ_1, ξ_3 constructed at $\xi_2 = 3m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ – Figure 6a; and in dependence of x_1, x_3, constructed at $x_2 = 3m$; $\xi_1 = 2m$; $\xi_2 = 3m$; $\xi_3 = 4m$ – Figure 6b.

In Figure 6 it can be observed:

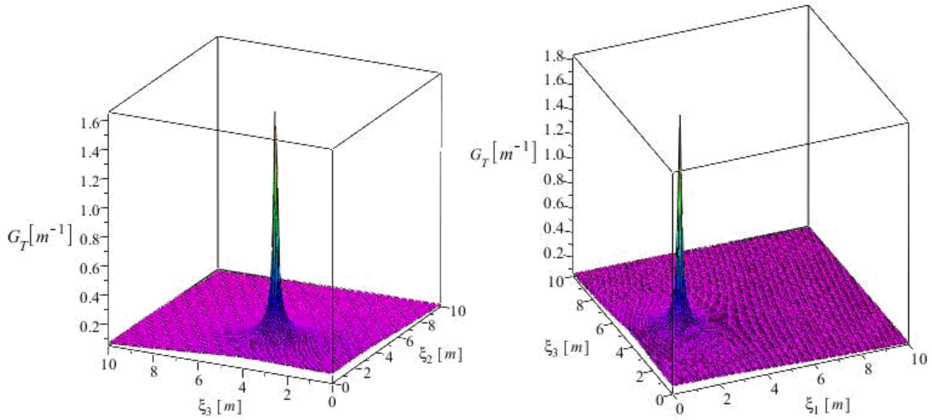
(1) At $\xi_1 = 0$ and $\xi_3 = 0$ the the MTGF U_3 is zero. This means that boundary conditions $U_3\,(x; \xi_1 = 0, \xi_2, \xi_3) = 0$ (29a) and $U_3\,(\xi_1, \xi_2, \xi_3 = 0; x) = 0$ (29c) on the marginal quadrants Γ_{10} and Γ_{30} are satisfied (Figure 6a);

(2) At $x_1 = 0$ and $x_3 = 0$ the lines tangential to surfaces U_3 in directions x_1 and x_3 are parallel to axes $0x_1$ and $0x_3$. This means that boundary conditions $U_{3,x_1}\,(x_1 = 0, x_2, x_3; \xi) = 0$ (27a) and $U_{3,x_3}\,(x_1, x_2, x_3 = 0; \xi) = 0$ (27c) on the marginal quadrants Γ_{10} and Γ_{30} are satisfied (Figure 6b); This is explaining by fact that boundary conditions for U_3 with respect to $x \equiv (x_1, x_2, x_3)$ are similar to those for function G_T, showed in Eqs (17a), (17b) and (17c).

(3) In the Figure 6 both graphs (in the Figures 6a and 6b) have jumps at $\xi_1 = 2m$, $\xi_3 = 4m$ (Figure 6a) and at $x_1 = 2m$, $x_3 = 4m$ (Figure 6b);

Graphs of changing GFPE G_T in dependence of $0 \le \xi_2 \le 10m$; $0 \le \xi_3 \le 10m$, constructed at $\xi_1 = 2m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ – Figure 7a and in dependence of $0 \le \xi_1 \le 10m$; $0 \le \xi_3 \le 10m$, constructed at $\xi_2 = 3m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ – Figure 7b are showed in the Figure 7.

In Figure 7 it can be observed:



(A) (B)

FIGURE 7. Graphs of changing GFPE G_T in dependence of ξ_2, ξ_3, constructed at $\xi_1 = 2m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ – Figure 7a; and in dependence of ξ_1, ξ_3, constructed at $\xi_2 = 3m$; $x_1 = 2m$; $x_2 = 3m$; $x_3 = 4m$ – Figure 7b.

(1) At $\xi_2 = 0$ (Figure 7a) and at $\xi_3 = 0$ (Figure 7a and Figure 7b) the lines tangential to surface G_T in directions ξ_2 and ξ_3 are parallel to axes $0\xi_2$ and $0\xi_3$. This means that boundary condition $G_{T,\xi_2}(x;\xi_1,\xi_2=0,\xi_3)=0$ (17b) on the marginal quadrant Γ_{20} (Figure 7a) and boundary condition $G_{T,\xi_3}(x;\xi_1,\xi_2,\xi_3=0)=0$ (17c) on the marginal quadrant Γ_{30} are satisfied (Figure 7a and Figure 7b);
(2) At $\xi_1 = 0$ the lines tangential to surface G_T in directions ξ_1 are parallel to axis $0\xi_1$. This means that boundary condition $G_{T,\xi_1}(x;\xi_1=0,\xi_2,\xi_3)=0$ (17a) on the marginal quadrant Γ_{10} is satisfied (Figure 7b);
(3) In the Figures 7a and 7b the GFPE $G_T \geq 0$. When one of the coordinates of the point $\xi \equiv (\xi_1,\xi_2,\xi_3)$ increases the GFPE G_T decreases and very soon vanishes. At singular point, when $\xi_1 = x_1 = 2m$, $\xi_2 = x_2 = 3m$, $\xi_3 = x_3 = 4m$ the GFPE G_T tends to $+\infty$ (Figure 7a and Figure 7b).

LABORATORY OF GREEN'S FUNCTIONS
AGRARIAN STATE UNIVERSITY OF MOLDOVA
MIRCEŞTI STREET 44, CHIŞINĂU 2049, MOLDOVA

LABORATORY OF MATHEMATICAL MODELING
INSTITUTE OF MATHEMATICS AND INFORMATICS
ACADEMY OF SCIENCE OF MOLDOVA
E-mail address: v.seremet@uasm.md