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HERMITE-HADAMARD INEQUALITIES FOR MODIFIED h-CONVEX FUNCTIONS

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Abstract. In this paper, we consider the class of modified h-convex functions, which was introduced by Toader [14]. We derive Hermite-Hadamard type inequalities for the modified h-convex functions. Some special cases are also discussed. We try to show that this class enjoys some nice properties which the convex functions have.

1. Introduction

Convexity plays a pivotal role in different fields of applied and pure sciences. In recent years, several extensions and generalizations of the concept of convexity have been considered using some novel and innovative techniques, see [2, 4, 5, 7, 8, 12, 13, 14, 15, 16]. Varošanec [15] introduced a significant class of nonconvex function, which is called h-convex function. She has noticed that this class generalizes several other classes of nonconvex functions along with classical convexity. For some recent studies on h-convex functions, see [9, 10, 11]. Toader [14] introduced a new class of nonconvex functions, which is called as (h, λ, μ)-convex functions. Toader studied the basic properties for this class of nonconvex functions.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then the following double inequality is known as Hermite-Hadamard inequality in the literature

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}.$$

named after C. Hermite and J. Hadamard. This result can be considered as a necessary and sufficient condition for a function to be convex. Interested readers are referred to [1] for useful details on Hermite-Hadamard inequalities and its variant forms.

Fejér [3], had given a generalization of the Hermite-Hadamard inequality as, if $f : [a, b] \to \mathbb{R}$ is a convex function and $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric about $\frac{a+b}{2}$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\mathrm{d}x \leq \int_a^b f(x)w(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)\mathrm{d}x.$$

For some recent investigations on Hermite-Hadamard type inequalities and on its variant forms, see [1, 2, 3, 7, 8, 9, 10, 11].

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In this paper, we discuss some basic properties of modified h-convex functions, which is basically a special case of (h, λ, μ)-convex functions. We also derive some Hermite-Hadamard type inequalities for modified h-convex functions. This is the main motivation of this paper.

2. Main results

In this section, we recall some basic results and concepts, which are useful in proving our results.

Definition 1 ([15]). *Let $f, h : J \to \mathbb{R}$ be a non-negative functions. We say that f is an h-convex function, or that f belongs to the class $SX(h, I)$, if*

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y), \quad \forall x, y \in I,\, t \in (0,1). \tag{1}$$

If the inequality (1) *is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$.*
This class along with classical convex functions also generalizes several other classes of nonconvex functions such as s-Breckner convex functions, P-functions and Q-functions [15].

Definition 2 ([14]). *Let $f, h : J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called modified h-convex function if*

$$f(tx + (1-t)y) \leq h(t)f(x) + (1-h(t))f(y), \quad \forall x, y \in J,\, t \in [0,1]. \tag{2}$$

Special Cases.

(i) For $h(t) = t$, the definition of modified h-convexity reduces to the definition of classical convexity.
(ii) For $h(t) = t^{\alpha}$, where $\alpha \in [0,1]$, then the definition of modified h-convexity reduces to the definition of $(\alpha, 1)$-convex function.

Definition 3 ([13]). *The function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be $(\alpha, 1)$-convex function if for all $x, y \in I$ we have*

$$f(tx + (1-t)y) \leq t^{\alpha} f(x) + (1-t^{\alpha})f(y). \tag{3}$$

Definition 4 ([16]). *A function $f : D \subset \mathbb{R} \to \mathbb{R}$ is said to be Wright-convex, if*

$$f((1-t)x + ty) + f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in D, t \in [0,1].$$

Definition 5 ([4]). *A function $f : D \subset \mathbb{R} \to \mathbb{R}_+$ is said to be Wright type multiplicatively convex, if*

$$f(x^{1-t}y^{t})f(x^{t}y^{1-t}) \leq f(x)f(y), \quad \forall x, y \in D, t \in [0,1].$$

Definition 6 ([8]). *A function $h : J \to \mathbb{R}$ is said to be supermultiplicative function if*

$$h(xy) \geq h(x)h(y), \quad \forall x, y \in J.$$

Definition 7 ([8]). *Two functions f and g are said to be similarly ordered (f is g-monotone) on $I \subseteq \mathbb{R}$, if $\langle f(x) - f(y), g(x) - g(y) \rangle \geq 0$, $\forall x, y \in I$.*

Definition 8 ([6]). *Let $f \in L_1[a, b]$, the Riemann-Liouville integrals $J^{\alpha}_{a^+} f$ and $J^{\alpha}_{b^-} f$ of order $\alpha > 0$ with $\alpha \geq 0$ are defined by*

$$J^{\alpha}_{a^+}(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad x > a$$

and

$$J_{b^-}^{\alpha}(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\mathrm{d}t. \quad x < b$$

Where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} x^{\alpha-1}\mathrm{d}x,$$

is the Gamma function, and also

$$J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x).$$

3. Main results

In this section, we will prove our main results.

Proposition 1. *If f and g are modified h-convex functions, then $\alpha f + \beta g$, for all $\alpha, \beta \in \mathbb{R}$, is also a modified h-convex functions.*

Proposition 2. *Let h_1, h_2 be non-negative functions defined on I with the property $h_2(t) \leq h_1(t)$. If f is modified h_2-convex function then f is modified h_1-convex function.*

Proof. Let f be modified h_2-convex function, then for all $a, b \in I$ and $t \in [0,1]$, we have

$$\begin{aligned} f(ta + (1-t)b) &\leq h_2(t)f(a) + (1 - h_2(t))f(b), \\ &\leq h_1(t)f(a) + (1 - h_1(t))f(b). \end{aligned}$$

This implies that f is modified h_1-convex function. □

Proposition 3. *Let g be linear function and f be modified h-convex function, then fog will be modified h-convex function.*

Proof. Let g be linear and f be modified h-convex function, then for all $a, b \in I$ and $t \in [0,1]$, we have

$$\begin{aligned} (fog)(ta + (1-t)b) &= f(tg(a) + (1-t)g(b)), \\ &\leq h(t)(fog)(a) + (1 - h(t))(fog)(b). \end{aligned}$$

This shows that fog is modified h-convex function. □

Proposition 4. *Let $f_i : I \subset \mathbb{R} \to \mathbb{R}$ be modified h-convex function, suppose $\mu_1, \ldots, \mu_n$ be positive scalers. Consider a function $g : \mathbb{R} \to \mathbb{R}$ such that*

$$g(x) = \sum_{i=1}^{n} \mu_i f_i(x),$$

then g is also modified h-convex function.

Proof. Since $f_i : I \subset \mathbb{R} \to \mathbb{R}$ be modified h-convex functions. Then for all $x, y \in I$ and $t \in [0,1]$, we have

$$\begin{aligned} g(tx + (1-t)y) &= \sum_{i=1}^{n} \mu_i f_i(tx + (1-t)y), \\ &\leq \sum_{i=1}^{n} \mu_i (h(t) f_i(x) + (1 - h(t)) f_i(y)), \\ &= h(t) \sum_{i=1}^{n} \mu_i f_i(x) + (1 - h(t)) \sum_{i=1}^{n} \mu_i f_i(y), \\ &= h(t) g(x) + (1 - h(t)) g(y). \end{aligned}$$

This completes the proof. □

Theorem 1. *Let f and g are two modified h-convex functions. Then the product of f and g will be a modified h-convex function if f and g are similarly ordered.*

Proof. Since f and g are modified h-convex function. Then

$$\begin{aligned} & f((1-t)a + tb) g((1-t)a + tb) \\ &\leq [(1 - h(t)) f(a) + h(t) f(b)][(1 - h(t)) g(a) + h(t) g(b)] \\ &= [1 - h(t)]^2 f(a) g(a) + h(t)(1 - h(t)) f(a) g(b) + h(t)(1 - h(t)) f(b) g(a) + [h(t)]^2 f(b) g(b) \\ &= (1 - h(t)) f(a) g(a) + h(t) f(b) g(b) - (1 - h(t)) f(a) g(a) - h(t) f(b) + [1 - h(t)]^2 f(a) g(a) \\ &\quad + h(t)(1 - h(t)) f(a) g(b) + h(t)(1 - h(t)) f(b) g(a) + [h(t)]^2 f(b) g(b) \\ &= (1 - h(t)) f(a) g(a) + h(t) f(b) g(b) - h(t)(1 - h(t)) \\ &\quad \times [f(a) g(a) + f(b) g(b) - f(b) g(a) - f(a) g(b)] \\ &\leq (1 - h(t)) f(a) g(a) + h(t) f(b) g(b). \end{aligned}$$

This completes the proof. □

Theorem 2. *Let h supermultiplicative function and suppose $f : I \to \mathbb{R}$ be modified h-convex function, then for $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$ such that $x_3 - x_1, x_3 - x_2, x_2 - x_1 \in I$, then*

$$h(x_3 - x_2) f(x_1) - h(x_3 - x_1) f(x_2) + [h(x_3 - x_1) - h(x_3 - x_2)] f(x_3) \geq 0, \tag{4}$$

if and only if f is a modified h-convex function.

Proof. Let f be modified h-convex function and $x_1, x_2, x_3 \in I$ be numbers. Then we have

$$\frac{x_3 - x_2}{x_3 - x_1} \in (0,1) \subseteq I, \quad \frac{x_2 - x_1}{x_3 - x_1} \in (0,1) \subseteq I,$$

and

$$\frac{x_3 - x_2}{x_3 - x_1} + \frac{x_2 - x_1}{x_3 - x_1} = 1$$

Also since h is supermultiplicative, then

$$h(x_3 - x_2) = h\left(\frac{x_3 - x_2}{x_3 - x_1}(x_3 - x_1)\right) \geq h\left(\frac{x_3 - x_2}{x_3 - x_1}\right) h(x_3 - x_1) \tag{5}$$

Let $h(x_3 - x_1) > 0$ and also

$$f(tx + (1-t)y) \leq h(t) f(x) + (1 - h(t)) f(y) \quad \forall\, x, y \in I,\ t \in (0,1). \tag{6}$$

Putting $t = \frac{x_3-x_2}{x_3-x_1}$, $x = x_1$ and $y = x_3$ in (6), we have

$$\begin{aligned} f(x_2) &\le h\left(\frac{x_3-x_2}{x_3-x_1}\right) f(x_1) + \left(1 - h\left(\frac{x_3-x_2}{x_3-x_1}\right)\right) f(x_3), \\ &\le \frac{h(x_3-x_2)}{h(x_3-x_1)} f(x_1) + \left(1 - \frac{h(x_3-x_2)}{h(x_3-x_1)}\right) f(x_3), \end{aligned}$$

that is,

$$h(x_3-x_1)f(x_2) \le h(x_3-x_2)f(x_1) + [h(x_3-x_1) - h(x_3-x_2)]f(x_3),$$

which implies that

$$h(x_3-x_2)f(x_1) - h(x_3-x_1)f(x_2) + [h(x_3-x_1) - h(x_3-x_2)]f(x_3) \ge 0,$$

which is the required (4).
Conversely let (4) holds, putting $x_1 = x$, $x_2 = tx + (1-t)y$ and $x_3 = y$ in (4), we have

$$\begin{aligned} h(y-x)f(tx+(1-t)y) &\le h(t(y-x))f(x) + [h(y-x) - h(t(y-x))]f(y), \\ &\le h(t)h(y-x)f(x) + h(y-x)[1-h(t)]f(y), \end{aligned}$$

from which, we have

$$f(tx+(1-t)y) \le h(t)f(x) + (1-h(t))f(y),$$

which shows that f is a modified h-convex function. □

We would like to emphasize that the modified h-convex function defined by (4) are similar to the g-convex functions introduced by Lackovic [5] in 1979.

We now present some integral inequalities of Hermite-Hadamard type for modified h-convex functions.

Theorem 3. *Let $f : I \to \mathbb{R}$ be modified h-convex function on the interval $[a, b]$ with $a < b$, then we have*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le f(a) + \{f(b) - f(a)\}\int_0^1 h(t)\mathrm{d}t. \tag{7}$$

Proof. Let $u = ta + (1-t)b$ and $v = (1-t)a + tb$, then we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) = f\left(\frac{u+v}{2}\right) &= f\left(\frac{ta+(1-t)b+(1-t)a+tb}{2}\right) \\ &\le \left\{1 - h(\frac{1}{2})\right\} f(ta+(1-t)b) + h(\frac{1}{2})f((1-t)a+tb). \end{aligned}$$

Integrating both sides of above inequality with respect to t on $[0, 1]$, we have

$$f\left(\frac{a+b}{2}\right) \le \left\{1 - h(\frac{1}{2})\right\}\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x + h(\frac{1}{2})\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \tag{8}$$

Now, we know that

$$\begin{aligned}\int_a^b f(x)\mathrm{d}x &= (b-a)\int_0^1 f(ta+(1-t)b)\mathrm{d}t,\\ &\leq (b-a)\int_0^1 \{(1-h(t))f(a)+h(t)f(b)\}\mathrm{d}t,\\ &= (b-a)\left[f(a)+\{f(b)-f(a)\}\int_0^1 h(t)\mathrm{d}t\right].\end{aligned}$$

Thus

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq f(a)+\{f(b)-f(a)\}\int_0^1 h(t)\mathrm{d}t. \tag{9}$$

Combining (8) and (9), we have the required result. □

Remark 1. *If we take $h(t) = t$ in (7), then we have classical Hermite-Hadamard inequality.*

Remark 2. *If we take $h(t) = t^\alpha$ in (7), then we have Hermite-Hadamard inequality for α-convex function.*

Corollary 1. *Let $f : I \to \mathbb{R}$ be α-convex function where $\alpha \in [0, 1]$ on the interval $[a, b]$ with $a < b$, then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{\alpha f(a)+f(b)}{\alpha+1}. \tag{10}$$

Here we note that on taking $\alpha = 1$, we have classical Hermite-Hadamard inequality.

Also one can easily obtain the following result by elementary analysis.

Corollary 2. *Let $\sum_{i=1}^{n} f_i : [a, b] \to \mathbb{R}$ be the sum of modified h-convex functions. Then*

$$\sum_{i=1}^{n} f_i\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\sum_{i=1}^{n}\int_a^b f_i(x)\mathrm{d}x \leq \sum_{i=1}^{n} f_i(a) + \left(\sum_{i=1}^{n} f_i(b) - \sum_{i=1}^{n} f_i(b)\right)\int_0^1 h(t)\mathrm{d}t.$$

Theorem 4. *Let f and g are two modified h-convex functions such that f and g are similarly ordered functions. If $h(\frac{1}{2})(1 + h(\frac{1}{2})) \neq \frac{1}{2}$, then*

$$\begin{aligned}&\frac{1}{1-2h(\frac{1}{2})+2h(\frac{1}{2})^2}\left[f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - h(\frac{1}{2})\left(1-h(\frac{1}{2})\right)Q(a,b,t)\right]\\ &\leq \frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \leq f(a)g(a)+[f(b)g(b)-f(a)g(a)]\int_o^1 h(t)\mathrm{d}t,\end{aligned}$$

where

$$Q(a,b,t) = M(a,b)\int_0^1 [1-h(t)+(h(t))^2]\mathrm{d}t,$$

and

$$M(a,b) = f(a)g(a) + f(b)g(b).$$

Proof. Let f and g be modified h-convex functions. Then

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right)\\
&= f\left(\frac{((1-t)a+tb)+(ta+(1-t)b)}{2}\right) g\left(\frac{((1-t)a+tb)+(ta+(1-t)b)}{2}\right)\\
&\le \left[\left\{1-h(\frac{1}{2})\right\} f((1-t)a+tb) + h(\frac{1}{2})f(ta+(1-t)b)\right]\\
&\qquad \left[\left\{1-h(\frac{1}{2})\right\} g((1-t)a+tb) + h(\frac{1}{2})g(ta+(1-t)b)\right]\\
&\le \left(1-h(\frac{1}{2})\right)^2 f((1-t)a+tb)g((1-t)a+tb) + \left(h(\frac{1}{2})\right)^2 f(ta+(1-t)b)g(ta+(1-t)b)\\
&\quad + h(\frac{1}{2})\left(1-h(\frac{1}{2})\right)[\{(1-h(t))f(a)+h(t)f(b)\}\{h(t)g(a)+(1-h(t))g(b)\}\\
&\qquad\qquad + \{h(t)f(a)+(1-h(t))f(b)\}\{(1-h(t))g(a)+h(t)g(b)\}]\\
&= \left(1-h(\frac{1}{2})\right)^2 f((1-t)a+tb)g((1-t)a+tb) + \left(h(\frac{1}{2})\right)^2 f(ta+(1-t)b)g(ta+(1-t)b)\\
&\quad + h(\frac{1}{2})\left(1-h(\frac{1}{2})\right)[h(t)(1-h(t))\{2f(a)g(a)+2f(b)g(b)\}\\
&\quad + \{(1-h(t))^2+(h(t))^2\}\{f(a)g(b)+f(b)g(a)\}]\\
&\le \left(1-h(\frac{1}{2})\right)^2 f((1-t)a+tb)g((1-t)a+tb) + \left(h(\frac{1}{2})\right)^2 f(ta+(1-t)b)g(ta+(1-t)b)\\
&\quad + h(\frac{1}{2})\left(1-h(\frac{1}{2})\right)[h(t)(1-h(t))\{2f(a)g(a)+2f(b)g(b)\}\\
&\quad + \{(1-h(t))^2+(h(t))^2\}\{f(a)g(a)+f(b)g(b)\}]\\
&= \left(1-h(\frac{1}{2})\right)^2 f((1-t)a+tb)g((1-t)a+tb) + \left(h(\frac{1}{2})\right)^2 f(ta+(1-t)b)g(ta+(1-t)b)\\
&\quad + h(\frac{1}{2})\left(1-h(\frac{1}{2})\right)[1-h(t)+(h(t))^2]M(a,b).
\end{aligned}$$

Integrating above inequality with respect to $t \in [0,1]$, we have

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - h(\frac{1}{2})\left(1-h(\frac{1}{2})\right) Q(a,b,t)\\
&\le \left[1-2h(\frac{1}{2})+2h(\frac{1}{2})^2\right]\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x
\end{aligned}$$

After suitable rearrangement we have the required left hand side.

Now we will prove the right hand side. Since f and g are modified h-convex functions. Then we have

$$f((1-t)a+tb)g((1-t)a+tb) \le \{(1-h(t))f(a)+h(t)f(b)\}\{(1-h(t))g(a)+h(t)g(b)\}$$

Since f and g are similarly ordered, then

$$f((1-t)a+tb)g((1-t)a+tb) \leq (1-h(t))f(a)g(a)+h(t)f(b)g(b).$$

Integrating above inequality with respect to $t \in [0,1]$, we have

$$\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \leq f(a)g(a)+\{f(b)g(b)-f(a)g(a)\}\int_0^1 h(t)\mathrm{d}t.$$

This completes the proof. □

Using the technique of Sarikaya [11], one can prove the following result. We include the proof for the sake of completeness and to convey the main idea.

Lemma 1. *Let f be modified h-convex function. Then*

$$f(a+b-x) \leq f(a)+f(b)-f(x), \ quad \forall x \in [a,b],$$

where $x = ta+(1-t)b$, $t \in [0,1]$.

Proof. Let f be modified h-convex function then for $x = ta+(1-t)b$, we have

$$\begin{aligned} f(a+b-x) &= f((1-t)a+tb) \\ &\leq (1-h(t))f(a)+h(t)f(b) \\ &= f(a)+f(b)-[h(t)f(a)+(1-h(t))f(b)] \\ &\leq f(a)+f(b)-f(ta+(1-t)b) \\ &= f(a)+f(b)-f(x). \end{aligned}$$

This completes the proof. □

Lemma 1 shows that the modified h-convex functions have the same nice property of convex functions.

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be a modified h-convex function, $w : [a,b] \to \mathbb{R}$, $w \geq 0$ be a integrable, symmetric with respect to $\frac{a+b}{2}$, then*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\mathrm{d}x \leq \int_a^b f(x)w(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}\int_a^b w(x)\mathrm{d}x. \tag{11}$$

Proof. Let f be a modified-h-convex function. Then, using Lemma 1, we have

$$\begin{aligned} f\left(\tfrac{a+b}{2}\right)\int_a^b w(x)\mathrm{d}x &= \int_a^b f\left(\tfrac{a+b-x+x}{2}\right)w(x)\mathrm{d}x \\ &\leq \int_a^b \left\{\left(1-h\left(\tfrac{1}{2}\right)\right)f(a+b-x)+h\left(\tfrac{1}{2}\right)f(x)\right\}w(x)\mathrm{d}x \\ &= \left(1-h\left(\tfrac{1}{2}\right)\right)\int_a^b f(a+b-x)w(a+b-x)\mathrm{d}x + h\left(\tfrac{1}{2}\right)\int_a^b f(x)w(x)\mathrm{d}x \\ &= \int_a^b f(x)w(x)\mathrm{d}x = \tfrac{1}{2}\int_a^b f(a+b-x)w(x)\mathrm{d}x+\tfrac{1}{2}\int_a^b f(x)w(x)\mathrm{d}x \\ &\leq \tfrac{1}{2}\int_a^b (f(a)+f(b)-f(x))w(x)\mathrm{d}x+\tfrac{1}{2}\int_a^b f(x)w(x)\mathrm{d}x = \tfrac{f(a)+f(b)}{2}\int_a^b w(x)\mathrm{d}x. \end{aligned}$$

This completes the proof. □

Now we derive the Hermite-Hadamard inequalities for modified h-convex functions via fractional integrals.

Theorem 6. *Let f be a modified h-convex function and $f \in L_1[a,b]$, with $a<b$. Then*

$$\frac{1}{\alpha} f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[\left(1-h(\frac{1}{2})\right) J^{\alpha}_{a^+} f(b) + h(\frac{1}{2}) J^{\alpha}_{b^-} f(a)\right], \tag{12}$$

and

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a^+} f(b) + J^{\alpha}_{b^-} f(a)\right] \le \frac{f(a)+f(b)}{\alpha}. \tag{13}$$

Proof. Since f is modified h-convex function. Then

$$f\left(\frac{x+y}{2}\right) \le \left\{1-h(\frac{1}{2})\right\} f(x) + h(\frac{1}{2}) f(y).$$

Let $x = ta + (1-t)b$ and $y = (1-t)a + tb$, then

$$f\left(\frac{a+b}{2}\right) \le \left\{1-h(\frac{1}{2})\right\} f(ta+(1-t)b) + h(\frac{1}{2}) f((1-t)a+tb).$$

Multiplying both sides of above inequality by $t^{\alpha-1}$, and then integrating with respect to t on $[0,1]$, we have

$$\begin{aligned}
\frac{1}{\alpha} f\left(\frac{a+b}{2}\right) &= f\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}\mathrm{d}t \\
&\le \left\{1-h(\frac{1}{2})\right\}\int_0^1 t^{\alpha-1} f(ta+(1-t)b)\mathrm{d}t + h(\frac{1}{2})\int_0^1 t^{\alpha-1} f((1-t)a+tb)\mathrm{d}t \\
&= \left\{1-h(\frac{1}{2})\right\}\int_a^b \left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u)\frac{du}{b-a} + h(\frac{1}{2})\int_a^b \left(\frac{v-u}{b-a}\right)^{\alpha-1} f(v)\frac{dv}{b-a} \\
&= \frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}}\left[\left(1-h(\frac{1}{2})\right) J^{\alpha}_{a^+} f(b) + h(\frac{1}{2}) J^{\alpha}_{b^-} f(a)\right],
\end{aligned}$$

which is (12).
Since f is a modified h-convex function, then

$$\begin{aligned}
f(ta+(1-t)b) + f((1-t)a+tb) &\le h(t)f(a) + (1-h(t))f(b) + (1-h(t))f(a) + h(t)f(b) \\
&= f(a) + f(b).
\end{aligned}$$

Multiplying both sides of above inequality by $t^{\alpha-1}$ and the integrating with respect to $t \in [0,1]$, we have

$$\int_0^1 t^{\alpha-1} f(ta+(1-t)b)\mathrm{d}t + \int_0^1 t^{\alpha-1} f((1-t)a+tb)\mathrm{d}t \le [f(a)+f(b)]\int_0^1 t^{\alpha-1}\mathrm{d}t.$$

From which we have

$$\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a^+} f(b) + J^{\alpha}_{b^-} f(a)\right] \le \frac{f(a)+f(b)}{\alpha}.$$

This completes the proof. □

Remark 3. *It is worth to mention here that our main results of this paper continue to hold for the the class of modified (m,h)-convex functions, which is defined as below*

Definition 9. *Let $f, h : J \subset \mathbb{R} \to \mathbb{R}$ be a non-negative function. A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called modified (m, h)-convex function if*

$$f(tx + (1-t)y) \leq h(t)f(x) + m(1-h(t))f(y/m), \quad \forall x, y \in J,\ t \in [0,1].$$

Remark 4. *It is known that, a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is Wright-convex function if and only if there exists a convex function $C : I \subseteq \mathbb{R} \to \mathbb{R}$ and an additive function $A : \mathbb{R} \to \mathbb{R}$, such that $f(x) = C(x) + A(x)$, $x \in I$. Using the technique of Ng.* [4], *one can show that the Wright type of modified h-convex functions can be represented as:*

$$f(x) = F(x) + A(x), x \in I,$$

where $F(x)$ is the modified h-convex functions and $A(x)$ is the additive function. This is an interesting problem.

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