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NEW CLASS OF INTEGRAL OPERATORS PRESERVING SUBORDINATION AND SUPERORDINATION FOR ANALYTIC MEROMORPHIC FUNCTIONS

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ABSTRACT. A certain class of integral operators $\mathcal{J}_{\beta,\gamma}(f)$, $\gamma, \beta \in \mathbb{C}$, $\beta \in \mathbb{C}\backslash\{0\}$ and $\Re(\gamma - \beta) > 1/2$, of meromorphic functions in the punctured open unit disk is introduced. The main object of this paper is to investigate some subordination and superordination preserving properties of these integral operators with the sandwich type theorem.

1. INTRODUCTION

Let $\mathbb{H} = \mathbb{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ let

$$\mathbb{H}[a, n] = \left\{f \in \mathbb{H} : \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \right\}. \tag{1}$$

Let f and F be members of $\mathbb{H}$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case, we write $f \prec F$ or $f(z) \prec F(z)$. If the function F is univalent in $\mathbb{U}$, then we have $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

Definition 1 ([15]). *Let $\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the differential subordination*

$$\phi\left(p(z),\, zp'(z)\right) \prec h(z) \qquad (z \in \mathbb{U}), \tag{2}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant if $p \prec q$ for all p satisfying (2). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (2) is said to be the best dominant.

Definition 2 ([16]). *Let $\varphi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ and let h be analytic in $\mathbb{U}$. If p and $\varphi\left(p(z),\, zp'(z)\right)$ are univalent in $\mathbb{U}$ and satisfies the differential superordination*

$$h(z) \prec \varphi\left(p(z),\, zp'(z)\right) \qquad (z \in \mathbb{U}), \tag{3}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (3). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all dominants q of (3) is said to be the best subordinant.

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Definition 3 ([16]). *Let Q denote the set of all functions f that are analytic and injective on $\overline{\mathbb{U}}\backslash \mathrm{E}(f)$ where*

$$\mathrm{E}(f) = \{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} f(z) = \infty\}, \tag{4}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(f)$.

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{5}$$

which are analytic in the punctured open unit disk $\mathbb{U}^* = \mathbb{U}\backslash\{0\}$. Let $\Sigma^\star$ and Σ_k be the subclasses of Σ consisting of all functions which are, respectively, meromorphic starlike and meromorphic convex in $\mathbb{U}^*$ (see, for details, [1, 2]).

In the literature, several integral operators of meromorphic functions in the punctured open unit disk have been investigated and studied by many authors (cf., e.g., [3, 4, 5, 6, 7, 20, 21, 22]).

For a function $f \in \Sigma$, we now introduce a class of integral operators $\mathcal{J}_{\beta,\gamma}$ define by

$$\begin{array}{l} \mathcal{J}_{\beta,\gamma}(f)(z) = \left(\frac{2(\gamma-\beta)}{z^{2\gamma}} \int\limits_0^z t^{2\gamma-1} \left(-f'(t)\right)^\beta dt \right)^{\frac{1}{2\beta}} \\ \left(f \in \Sigma,\, \gamma,\beta \in \mathbb{C},\, \beta \in \mathbb{C}\backslash\{0\},\, \Re(\gamma-\beta) > \frac{1}{2}\right). \end{array} \tag{6}$$

The study of the subordination and superordination-preserving properties for certain integral operators has attracted and investigated by many authors (cf., [8, 9, 10, 11, 12, 13, 14]). In the present paper some subordination and superordination-preserving properties of the integral operators given by (6) are introduced. The sandwich-type theorem is also considered. The techniques adopted by Srivastava et al. [10] (see also [11] and [12]) are used.

The following lemmas will be required in our present investigation

Lemma 1 (see [17]). *Suppose that the function $H : \mathbb{C}^2 \longrightarrow \mathbb{C}$ satisfies the following condition*

$$\Re\{H(is,t)\} \leq 0, \tag{7}$$

for all real s and $t \leq -n\left(1+s^2\right)/2$, where n is a positive integer. If the function $p(z) = 1 + p_n z^n + \cdots$ is analytic in $\mathbb{U}$ and

$$\Re\{H(p(z), zp'(z))\} > 0 \qquad (z \in \mathbb{U}), \tag{8}$$

then $\Re\{p(z)\} > 0$ in $\mathbb{U}$.

Lemma 2 (see [18]). *Let $\beta,\gamma \in \mathbb{C}$ with $\beta \neq 0$ and let $h \in \mathbb{H}(\mathbb{U})$ with $h(0) = c$. If $\Re\{\beta h(z) + \gamma\} > 0$ $(z \in \mathbb{U})$, then the solution of the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \qquad (z \in \mathbb{U};\ q(0) = c) \tag{9}$$

is analytic in $\mathbb{U}$ and satisfies $\Re\{\beta q(z) + \gamma\} > 0$ $(z \in \mathbb{U})$.

Lemma 3 ([17]). *Let $p \in \mathcal{Q}$ with $p(0) = a$ and let $q(z) = a + a_n z^n + \cdots$ be analytic in $\mathbb{U}$ with $q(z) \neq a$ and $n \geq 1$. If q is not subordinate to p, then there exist points $z_0 = r_0 e^{i\theta} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash E(f)$, for which*

$$q(\mathbb{U}_{r_0}) \subset p(\mathbb{U}), \qquad q(z_0) = p(\zeta_0), \qquad z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \qquad (m \geq n). \tag{10}$$

A function $L(z,t)$ defined on $\mathbb{U}\times[0,\infty)$ is the subordination chain (or Löwner chain) if $L(.,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \in [0,\infty)$, $L(z,.)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$ and $L(z,s) \prec L(z,t)$ for $0 \leq s < t$.

Lemma 4 (see [16]). *Let $q \in \mathbb{H}[a,1]$, set $\varphi : \mathbb{C}^2 \longrightarrow \mathbb{C}$, and let $\varphi(q(z), zq'(z)) \equiv h(z)$ $(z \in \mathbb{U})$. If $L(z,t) = \varphi(q(z), tzq'(z))$ is a subordination chain and $p \in \mathbb{H}[a,1] \cap \mathcal{Q}$, then*

$$h(z) \prec \varphi(p(z), zp'(z)) \qquad (z \in \mathbb{U}), \tag{11}$$

implies that

$$q(z) \prec p(z). \tag{12}$$

Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{Q}$, then q is the best subordinant.

Lemma 5 (see [19]). *The function $L(z,t) = a_1(t)z + \cdots$ with $a_1(t) \neq 0$ and $\lim\limits_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if*

$$\Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} > 0 \quad (z \in \mathbb{U};\ 0 \le t < \infty). \tag{13}$$

2. Main result

Our first subordination theorem involving the integral operator $\mathcal{J}_{\beta,\gamma}$ defined by (6) is contained in Theorem 1 below.

Theorem 1. *Let $f, g \in \Sigma$. Suppose that*

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \qquad \left(z \in \mathbb{U},\ \phi(z) = z\left(-z^2 g'(z)\right)^{\beta}\right), \tag{14}$$

where

$$\delta = \frac{1 + |2(\gamma-\beta) - 1|^2 - \left|1 - (2(\gamma-\beta)-1)^2\right|}{4\Re(2(\gamma-\beta)-1)} \qquad \left(\Re(\gamma-\beta) > \frac{1}{2}\right). \tag{15}$$

Then the following subordination relation:

$$z\left(-z^2 f'(z)\right)^{\beta} \prec z\left(-z^2 g'(z)\right)^{\beta} \tag{16}$$

implies that

$$z\left(z\mathcal{J}_{\beta,\gamma}(f)(z)\right)^{2\beta} \prec z\left(z\mathcal{J}_{\beta,\gamma}(g)(z)\right)^{2\beta} \tag{17}$$

where $\mathcal{J}_{\beta,\gamma}(f)$ is the integral operator defined by (6).
Moreover, the function $z\left(z\mathcal{J}_{\beta,\gamma}(g)(z)\right)^{2\beta}$ is the best dominant.

Proof. Let

$$F(z) = z\left(z\mathcal{J}_{\beta,\gamma}(f)(z)\right)^{2\beta}, \quad G(z) = z\left(z\mathcal{J}_{\beta,\gamma}(g)(z)\right)^{2\beta}. \tag{18}$$

Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathbb{U}}$, and

$$G'(\xi) \neq 0 \quad (|\xi| = 1). \tag{19}$$

We first prove that if the function q is defined by

$$q(z) = 1 + \frac{zG''(z)}{G'(z)}, \tag{20}$$

then

$$\Re q(z) > 0, z \in \mathbb{U}. \tag{21}$$

From the definition of (6), we obtain

$$\beta\frac{z\left(\mathcal{J}_{\beta,\gamma}(g)(z)\right)'}{\mathcal{J}_{\beta,\gamma}(g)(z)} = -\gamma + \frac{(\gamma-\beta)\left(-g'(z)\right)^{\beta}}{\left(\mathcal{J}_{\beta,\gamma}(g)(z)\right)^{2\beta}}. \tag{22}$$

From (14), (18) and (22), we have

$$\beta \frac{z\left(\mathcal{J}_{\beta,\gamma}\left(g\right)\left(z\right)\right)'}{\mathcal{J}_{\beta,\gamma}(g)(z)} = -\gamma + (\gamma - \beta)\frac{\phi(z)}{G(z)}. \tag{23}$$

Since $G(z) = z\left(z\mathcal{J}_{\beta,\gamma}(g)(z)\right)^{2\beta}$, then we get

$$\beta \frac{z\left(\mathcal{J}_{\beta,\gamma}\left(g\right)\left(z\right)\right)'}{\mathcal{J}_{\beta,\gamma}(g)(z)} = -\left(\frac{1}{2} + \beta\right) + \frac{1}{2}\frac{zG'(z)}{G(z)}. \tag{24}$$

Now, using (23) and (24), one get

$$2(\gamma - \beta)\phi(z) = \left(2\left(\gamma - \beta\right) - 1\right)G(z) + zG'(z). \tag{25}$$

By differentiating (25) and through a little simplification, we obtain

$$2(\gamma - \beta)\phi'(z) = \left(2\left(\gamma - \beta\right) - 1\right)G'(z) + G'(z)q(z), \tag{26}$$

where the function $q(z)$ is defined by (20).

Logarithmic differentiation of (26) and a simplification yields

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{q(z) + \left(2\left(\gamma - \beta\right) - 1\right)} \equiv h(z). \tag{27}$$

From (14), we note that

$$\Re\left\{h(z) + \left(2\left(\gamma - \beta\right) - 1\right)\right\} > 0 \quad \left(z \in \mathbb{U}\right), \tag{28}$$

and by using Lemma 2, we conclude that the differential equation (27) has a solution $q \in \mathbb{H}(\mathbb{U})$ with $q(0) = h(0) = 1$.

Now, we will use Lemma 1 to prove that, under the assumption, the inequality (21) holds. Let us put

$$H(u,v) = u + \frac{v}{u + \left(2\left(\gamma - \beta\right) - 1\right)} + \delta, \tag{29}$$

where δ is given by (15). From (14), (27) and (29), we obtain

$$\Re\left\{H(q(z), zq'(z))\right\} > 0 \quad \left(z \in U\right). \tag{30}$$

We proceed to show that $\Re\left\{H(is,t)\right\} \leq 0$ for all real s and $t \leq -\left(1+s^2\right)/2$. From (29), we have

$$\begin{aligned}\Re\left\{H(is,t)\right\} = \Re\left\{is + \tfrac{t}{is+(2(\gamma-\beta)-1)} + \delta\right\} &= \tfrac{t\Re\{(2(\gamma-\beta)-1)\}}{|is+(2(\gamma-\beta)-1)|^2} + \delta \\ &\leq -\tfrac{E_\delta(s)}{2|is+(2(\gamma-\beta)-1)|^2},\end{aligned} \tag{31}$$

where

$$\begin{gathered}E_\delta\left(s\right) = \left(2\Re\left(\gamma - \beta\right) - 1 - 2\delta\right)s^2 \\ -4\delta \mathrm{Im}\left[\left(2\left(\gamma - \beta\right) - 1\right)\right]s - 2\delta\left|2\left(\gamma - \beta\right) - 1\right|^2 + 2\Re\left(\gamma - \beta\right) - 1.\end{gathered} \tag{32}$$

For δ given by (15), we note that the coefficient of s^2 in the quadratic expression for $E_\delta\left(s\right)$ defined by (32) is greater than or equal to zero. Moreover, the discriminant Δ of $E_\delta\left(s\right)$ in (32) is represented by

$$\begin{gathered}\tfrac{\Delta}{4} = -4\delta^2\left[2\Re\left(\gamma - \beta\right) - 1\right]^2 \\ +2\delta\left[1 + \left|2\left(\gamma - \beta\right) - 1\right|^2\right]\left[2\Re\left(\gamma - \beta\right) - 1\right] - \left[2\Re\left(\gamma - \beta\right) - 1\right]^2,\end{gathered} \tag{33}$$

which, for the assumed value of δ given by (15), yields

$$\Delta = 0,$$

and so the quadratic expression for s in $E_\delta(s)$ given by (32) is a perfect square. We thus see from (31) that

$$\Re\{H(is,t)\} \le 0; \quad \left(s \in \mathbb{R},\, t \le -\frac{1}{2}\left(1+s^2\right)\right).$$

Hence, by using Lemma 1, we conclude that

$$\Re\{q(z)\} > 0 \qquad (z \in \mathbb{U}),$$

that is, that the function G defined by (18) is convex in $\mathbb{U}$.

Next, we prove that (16) implies $F(z) \prec G(z)$. For this purpose, we consider the function $L(z,t)$ given by

$$L(z,t) = \frac{(2(\gamma-\beta)-1)}{2(\gamma-\beta)}G(z) + \left(\frac{1+t}{2(\gamma-\beta)}\right)zG'(z) \qquad (z \in \mathbb{U},\, 0 \le t < \infty). \tag{34}$$

Since G is convex in $\mathbb{U}$ and $\Re(\gamma-\beta) > \frac{1}{2}$ we obtain

$$a_1(t) = \left(\frac{\partial L(z,t)}{\partial z}\right)_{z=0} = G'(0)\left(\frac{(2(\gamma-\beta)-1)+1+t}{2(\gamma-\beta)}\right) = 1 + \frac{t}{2(\gamma-\beta)} \neq 0, \tag{35}$$

and

$$\lim_{t\to\infty} |a_1(t)| = \infty.$$

Also

$$\Re\left(\frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}}\right) = \Re\left\{2(\gamma-\beta) - 1 + (1+t)\left(1 + \frac{zG''(z)}{G'(z)}\right)\right\} > 0 \qquad (z \in \mathbb{U}). \tag{36}$$

By Lemma 5, we conclude that $L(z,t)$ is a subordination chain. We observe from the definition of a subordination chain that

$$\phi(z) = \frac{(2(\gamma-\beta)-1)}{2(\gamma-\beta)}G(z) + \frac{1}{2(\gamma-\beta)}zG'(z) = L(z,0)$$

and

$$L(z,0) \prec L(z,t) \qquad (z \in \mathbb{U}, \quad 0 \le t < \infty).$$

This implies that

$$L(\xi,t) \notin L(\mathbb{U},0) = \phi(\mathbb{U}), \qquad (\xi \in \partial\mathbb{U}, \quad 0 \le t < \infty).$$

Now, suppose that F is not subordinate to G, then by Lemma 3, there exists points $z_0 \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U}$ such that

$$F(z_0) = G(\xi_0), \qquad z_0 F(z_0) = (1+t)\,\xi_0 G'(\xi_0) \quad (0 \le t < \infty).$$

Hence, we have

$$\begin{aligned} L(\xi_0,t) &= \tfrac{(2(\gamma-\beta)-1)}{2(\gamma-\beta)}G(\xi_0) + \left(\tfrac{1+t}{2(\gamma-\beta)}\right)\xi_0 G'(\xi_0) \\ &= \tfrac{(2(\gamma-\beta)-1)}{2(\gamma-\beta)}F(\xi_0) + \tfrac{z_0 F(z_0)}{2(\gamma-\beta)} \\ &= z_0\left(-z_0^2 f'(z_0)\right)^{\beta} \in \phi(\mathbb{U}), \end{aligned}$$

by virtue of the subordination condition (16). This contradicts the above observation that $L(\xi,t) \notin \phi(\mathbb{U})$. Therefore we have $F(z) \prec G(z)$. Considering $F(z) = G(z)$, we see that the function G is the best dominant. Therefore, we complete the proof of Theorem 1. □

We next prove a dual problem of Theorem 1 in the sense that the subordinations are replaced by superordinations. Since the first part of the proof is similar to that of Theorem 1, then we will use the same notation as in the proof of Theorem 1.

Theorem 2. *Let $f, g \in \Sigma$. Suppose that*

$$\Re\left\{1+\frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \qquad \left(z \in \mathbb{U},\ \phi(z) = z\left(-z^2 g'(z)\right)^{\beta}\right), \tag{37}$$

where δ is given by (15), $z\left(-z^2 f'(z)\right)^{\beta}$ is univalent in $\mathbb{U}$, and $z\left(z\mathcal{J}_{\beta,\gamma}(f)(z)\right)^{2\beta} \in \mathcal{Q}$ where $\mathcal{J}_{\beta,\gamma}$ is the integral operator defined by (6). Then, the superordination

$$z\left(-z^2 g'(z)\right)^{\beta} \prec z\left(-z^2 f'(z)\right)^{\beta} \tag{38}$$

implies that

$$z\left(z\mathcal{J}_{\beta,\gamma}(g)(z)\right)^{2\beta} \prec z\left(z\mathcal{J}_{\beta,\gamma}(f)(z)\right)^{2\beta}. \tag{39}$$

Moreover, the function $z\left(z\mathcal{J}_{\beta,\gamma}(g)(z)\right)^{2\beta}$ is the best subordinant.

Proof. Let F and G defined, as before, by (18). From (25), we have

$$\begin{aligned} \phi(z) &= \tfrac{(2(\gamma-\beta)-1)}{2(\gamma-\beta)}G(z) + \tfrac{1}{2(\gamma-\beta)}zG'(z) \\ &= \varphi\left(G(z),\, zG'(z)\right), \end{aligned} \tag{40}$$

and therefore,

$$1+\frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{(2(\gamma-\beta)-1)+q(z)}, \tag{41}$$

where the function q is defined by (20). Then, by using the same method as in the proof of Theorem 1, we can prove that $\Re\{q(z)\} > 0$ for all $z \in \mathbb{U}$. That is, G defined by (18) is convex (univalent) in $\mathbb{U}$.

In order to prove that the superordination condition (38) implies that $F(z) \prec G(z)$, where the functions F and G, defined by (18), we consider the function $L(z,t)$ defined by

$$L(z,t) = \frac{(2(\gamma-\beta)-1)}{2(\gamma-\beta)}G(z) + \frac{t}{2(\gamma-\beta)}zG'(z) \qquad (z \in \mathbb{U}, \quad 0 \le t < \infty). \tag{42}$$

Following the same steps as in the proof of Theorem 1, we can prove easily that $L(z,t)$ is a subordination chain. Therefore, according to Lemma 4, we conclude that the superordination condition (38) must imply the superordination given by (39). Furthermore, since the differential equation (40) has the univalent solution G, it is the best subordinant of the given differential superordination. Therefore, we complete the proof of Theorem 2. □

If we combine Theorems 1 and 2, then we can obtain the following sandwich-type theorem.

Theorem 3. *Let $f, g_k \in \Sigma$, $(k = 1, 2)$. Suppose that*

$$\Re\left\{1+\frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\delta \qquad \left(z \in \mathbb{U},\ \phi_k(z) = z\left(-z^2 g_k'(z)\right)^{\beta}\right);\ (k = 1, 2), \tag{43}$$

where δ is given by (15), $z\left(-z^2 f'(z)\right)^{\beta}$ is univalent in $\mathbb{U}$, and $z\left(z\mathcal{J}_{\beta,\gamma}(f)(z)\right)^{2\beta} \in \mathcal{Q}$ where $\mathcal{J}_{\beta,\gamma}$ is the integral operator defined by (6). Then, the subordination

$$z\left(-z^2 g_1'(z)\right)^{\beta} \prec z\left(-z^2 f'(z)\right)^{\beta} \prec z\left(-z^2 g_2'(z)\right)^{\beta} \tag{44}$$

implies that

$$z\left(z\mathcal{J}_{\beta,\gamma}(g_1)(z)\right)^{2\beta} \prec z\left(z\mathcal{J}_{\beta,\gamma}(f)(z)\right)^{2\beta} \prec z\left(z\mathcal{J}_{\beta,\gamma}(g_2)(z)\right)^{2\beta}. \tag{45}$$

Moreover, the functions $z\left(z\mathcal{J}_{\beta,\gamma}(g_1)(z)\right)^{2\beta}$ and $z\left(z\mathcal{J}_{\beta,\gamma}(g_2)(z)\right)^{2\beta}$ are the best subordinant and the best dominant, respectively.

By setting $\gamma-\beta=1$ in Theorem 3, so that $\delta=\frac{1}{2}$, we deduce the following consequence of Theorem 3.

Corollary 1. *Let $f, g_k \in \Sigma$, $(k=1,2)$. Suppose that*

$$\Re\left\{1+\frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\frac{1}{2} \qquad \left(z\in\mathbb{U},\ \phi_k(z)=z\left(-z^2 g_k'(z)\right)^{\beta}\right);\ (k=1,2), \tag{46}$$

$z\left(-z^2 f'(z)\right)^{\beta}$ is univalent in $\mathbb{U}$, and $z\left(z\mathcal{J}_{\beta,\beta+1}(f)(z)\right)^{2\beta}\in\mathcal{Q}$ where $\mathcal{J}_{\beta,\beta+1}$ is the integral operator defined by (6) with $\gamma=\beta+1$. Then, the subordination

$$z\left(-z^2 g_1'(z)\right)^{\beta} \prec z\left(-z^2 f'(z)\right)^{\beta} \prec z\left(-z^2 g_2'(z)\right)^{\beta} \tag{47}$$

implies that

$$z\left(z\mathcal{J}_{\beta,\beta+1}(g_1)(z)\right)^{2\beta} \prec z\left(z\mathcal{J}_{\beta,\beta+1}(f)(z)\right)^{2\beta} \prec z\left(z\mathcal{J}_{\beta,\beta+1}(g_2)(z)\right)^{2\beta} \tag{48}$$

Moreover, the functions $z\left(z\mathcal{J}_{\beta,\beta+1}(g_1)(z)\right)^{2\beta}$ and $z\left(z\mathcal{J}_{\beta,\beta+1}(g_2)(z)\right)^{2\beta}$ are the best subordinant and the best dominant, respectively.

If we take $\gamma-\beta=1+\frac{1}{2}i$ in Theorem 3, then we are easily led to the following result.

Corollary 2. *Let $f, g_k \in \Sigma$, $(k=1,2)$. Suppose that*

$$\Re\left\{1+\frac{z\phi_k''(z)}{\phi_k'(z)}\right\} > -\frac{3-\sqrt{5}}{4} \qquad \left(z\in\mathbb{U},\ \phi_k(z)=z\left(-z^2 g_k'(z)\right)^{\beta}\right);\ (k=1,2), \tag{49}$$

$z\left(-z^2 f'(z)\right)^{\beta}$ is univalent in $\mathbb{U}$, and $z\left(z\mathcal{J}_{\beta,\beta+1+\frac{1}{2}i}(f)(z)\right)^{2\beta}\in\mathcal{Q}$ where $\mathcal{J}_{\beta,\beta+1+\frac{1}{2}i}$ is the integral operator defined by (6) with $\gamma=\beta+1+\frac{1}{2}i$. Then, the subordination

$$z\left(-z^2 g_1'(z)\right)^{\beta} \prec z\left(-z^2 f'(z)\right)^{\beta} \prec z\left(-z^2 g_2'(z)\right)^{\beta} \tag{50}$$

implies that

$$z\left(z\mathcal{J}_{\beta,\beta+1+\frac{1}{2}i}(g_1)(z)\right)^{2\beta} \prec z\left(z\mathcal{J}_{\beta,\beta+1+\frac{1}{2}i}(f)(z)\right)^{2\beta} \prec z\left(z\mathcal{J}_{\beta,\beta+1+\frac{1}{2}i}(g_2)(z)\right)^{2\beta}. \tag{51}$$

Moreover, the functions $z\left(z\mathcal{J}_{\beta,\beta+1+\frac{1}{2}i}(g_1)(z)\right)^{2\beta}$ and $z\left(z\mathcal{J}_{\beta,\beta+1+\frac{1}{2}i}(g_2)(z)\right)^{2\beta}$ are the best subordinant and the best dominant, respectively.

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