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POSITIVE PERIODIC SOLUTIONS FOR FIRST-ORDER NONLINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH PERIODIC DELAY

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Abstract. In this paper, we study the existence of positive periodic solutions of two classes for first-order nonlinear neutral functional differential equations with periodic delay. The main tool employed here is the Krasnoselskii's hybrid fixed point theorem dealing with a sum of two mappings, one is a contraction and the other is compact. Two examples are included to illustrate our results. The results obtained here generalize the work [12].

1. Introduction

Delay differential equations have attracted a rapidly growing attention in the field of nonlinear dynamics and have become a powerful tool for investigating the complexities of the real-world problems such as infectious diseases, biotic population, physics, population dynamics, industrial robotics, neuronal networks, and even economics and finance. Due to their importance in numerous applications, many authors are studying the existence, uniqueness, stability and positivity of solutions for delay differential equations, see the references in this article and the references therein.

Motivated by the papers [1]–[14] and references therein, we concentrate the study on the existence of positive periodic solutions for two new kinds of first-order nonlinear neutral functional differential equations with periodic delay

$$[g(t)(x(t)+G(t,x(t-\tau(t))))]' = -a(t)x(t)+f(t,x(t-\tau(t))), \tag{1}$$

$$\left[g(t)\left(x(t)+\int_{-\infty}^{0} Q(r)\,G(t,x(t+h(r)))\,dr\right)\right]' = -a(t)x(t)+b(t)\int_{-\infty}^{0} Q(r)\,f(t,x(t+h(r)))\,dr, \tag{2}$$

where τ, a, b is a continuous real-valued function. The functions $f, G : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous in their respective arguments, $h \in C(\mathbb{R}_-,\mathbb{R})$, $g \in C^1(\mathbb{R},\mathbb{R}^+\setminus\{0\})$, $Q \in C(\mathbb{R}_-,\mathbb{R}^+)$, and $\int_{-\infty}^{0} Q(r)\,dr = 1$. The aim of this paper is, by applying the Krasnoselskii's fixed point theorem and some techniques, to establish a set of sufficient conditions which guarantee the existence of positive periodic solutions of (1) and (2).

The organization of this paper is as follows. In Section 2, we present the Krasnoselskii's fixed point theorem and the inversion of (1) and (2). In Section 3, we present our main results on existence of positive periodic solutions of (1) and (2). Two examples are given to show the efficiency and applications of our results in Section 4. The results presented in this paper generalize the main results in [12].

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2. Preliminaries and inversion of the equations (1) and (2)

Krasnoselskii's fixed point theorem has been extensively used in differential and functional differential equations, for its proof we refer the reader to [15].

Theorem 1 (Krasnoselskii). *Let $\mathcal{M}$ be a closed convex nonempty subset of a Banach space $(\mathcal{B}, \|.\|)$. Suppose that A and B map $\mathcal{M}$ into $\mathcal{B}$ such that*

(i) $x, y \in \mathcal{M}$, implies $Ax + By \in \mathcal{M}$,

(ii) B is a contraction mapping,

(iii) A is continuous and compact.

Then there exists $z \in \mathcal{M}$ with $z = Az + Bz$.

Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{N}$ denotes the set of all positive integers. For $T > 0$ define

$$P_T = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t+T) = \phi(t)\}, \ t \in \mathbb{R}.$$

Then P_T is a Banach space when it is endowed with the supremum norm

$$\|x\| = \sup_{t \in [0,T]} |x(t)|, \text{ for each } x \in P_T.$$

Let

*beginequation** $\mathcal{M}(K, L) = \{x \in P_T : K \leq x(t) \leq L, \ \forall t \in [0, T]\}$, for any $L > K \geq 0$.

Since we are searching for the existence of periodic solutions for equations (1) and (2), it is natural to assume that

$$a(t+T) = a(t), \ b(t+T) = b(t), \ g(t+T) = g(t), \ \tau(t+T) = \tau(t), \tag{3}$$

with τ being scalar function, continuous and $\tau(t) \geq \tau^* > 0$. Also, we assume

$$\int_0^T \frac{g'(r) + a(r)}{g(r)} dr > 0. \tag{4}$$

Functions $f(t, x)$ and $G(t, x)$ are periodic in t of period T, that is

$$f(t+T, x) = f(t, x), \ G(t+T, x) = G(t, x). \tag{5}$$

It is easy to see that $\mathcal{M}(K, L)$ is a bounded closed and convex subset of the Banach space P_T.

The following lemmas are fundamental to our results.

Lemma 1. *Suppose* (3), (4),(5) *hold. If $x \in P_T$, then x is a solution of equation* (1) *if and only if*

$$x(t) = \int_t^{t+T} H(t,s) [f(s, x(s - \tau(s))) + a(s) G(s, x(s - \tau(s)))] ds - G(t, x(t - \tau(t))), \tag{6}$$

where

$$H(t,s) = \frac{\exp\left(\int_t^s \frac{g'(r)+a(r)}{g(r)} dr\right)}{g(s)\left[\exp\left(\int_0^T \frac{g'(r)+a(r)}{g(r)} dr\right) - 1\right]}, \ \forall (t,s) \in \mathbb{R}^2. \tag{7}$$

Proof. Let $x \in P_T$ be a solution of (1). First we write equation (1) as

$$[x(t) + G(t, x(t - \tau(t)))]' = -\frac{(g'(t) + a(t))}{g(t)} [x(t) + G(t, x(t - \tau(t)))]$$
$$+ \frac{1}{g(t)} [a(t) G(t, x(t - \tau(t))) + f(t, x(t - \tau(t)))].$$

Multiply both sides of the above equation by $\exp\left(\int_0^t \frac{g'(r)+a(r)}{g(r)}dr\right)$ and then integrate from t to $t+T$ to obtain

$$\int_t^{t+T}\left[x(s)+G(s,x(s-\tau(s)))\right]'\exp\left(\int_0^s \frac{g'(r)+a(r)}{g(r)}dr\right)ds$$
$$=-\int_t^{t+T}\frac{g'(s)+a(s)}{g(s)}\left[x(s)+G(s,x(s-\tau(s)))\right]\exp\left(\int_0^s \frac{g'(r)+a(r)}{g(r)}dr\right)ds$$
$$+\int_t^{t+T}\frac{1}{g(s)}\left[a(s)G(s,x(s-\tau(s)))+f(s,x(s-\tau(s)))\right]\exp\left(\int_0^s \frac{g'(r)+a(r)}{g(r)}dr\right)ds.$$

Integrating by part and using the fact that $x(t)=x(t+T)$, we arrive at

$$\left[x(t)+G(t,x(t-\tau(t)))\right]\left(\exp\left(\int_0^T \frac{g'(r)+a(r)}{g(r)}dr\right)-1\right)\exp\left(\int_0^t \frac{g'(r)+a(r)}{g(r)}dr\right)$$
$$=\int_t^{t+T}\frac{1}{g(s)}\left[a(s)G(s,x(s-\tau(s)))+f(s,x(s-\tau(s)))\right]\exp\left(\int_0^s \frac{g'(r)+a(r)}{g(r)}dr\right)ds.$$

Dividing both sides of the above equation by $\exp\left(\int_0^t \frac{g'(r)+a(r)}{g(r)}dr\right)$, we obtain

$$x(t)=\int_t^{t+T}H(t,s)\left[a(s)G(s,x(s-\tau(s)))+f(s,x(s-\tau(s)))\right]ds-G(t,x(t-\tau(t))).$$

The converse implication is easily obtained and the proof is complete. □

Lemma 2. *Suppose* (3), (4), (5) *hold. If $x\in P_T$, then x is a solution of equation* (2) *if and only if*

$$x(t)=-\int_{-\infty}^0 Q(r)G(t,x(t+h(r)))dr+\int_t^{t+T}H(t,s)$$
$$\times\left[a(s)\int_{-\infty}^0 Q(r)G(s,x(s+h(r)))dr+b(s)\int_{-\infty}^0 Q(r)f(s,x(s+h(r)))dr\right]ds, \tag{8}$$

where $H(t,s)$ is given by (7).

Proof. The proof is similar of Lemma 1. □

3. Existence of positive periodic solutions

In this section we obtain the existence of a positive periodic solution of (1) and (2). By applying Theorem 1, we need to define a Banach space $\mathcal{B}$, a closed convex subset $\mathcal{M}$ of $\mathcal{B}$ and construct two mappings, one is a contraction and the other is compact. So, we let $(\mathcal{B},\|.\|)=(P_T,\|.\|)$ and $\mathcal{M}(K,L)=\{x\in P_T : K\leq x(t)\leq L,\ \forall t\in[0,T]\}$, for any $0\leq K<L$.

We begin by the existence of positive solutions for (1). From (6) define a mapping S_1 by

$$(S_1\varphi)(t)=-G(t,\varphi(t-\tau(t)))+\int_t^{t+T}H(t,s)$$
$$\times\left[f(s,\varphi(s-\tau(s)))+a(s)G(s,\varphi(s-\tau(s)))\right]ds. \tag{9}$$

It is clear form (6) that $S_1:\mathcal{M}(K,L)\to P_T$ by the way it was constructed in Lemma 1. Therefore, we express equation (9) as

$$(S_1\varphi)(t)=(A_1\varphi)(t)+(B_1\varphi)(t),\ \forall t\in\mathbb{R},$$

where A_1, $B_1 : \mathcal{M}(K,L) \to P_T$ are given by

$$(A_1\varphi)(t) = \int_t^{t+T} H(t,s)\left[f(s,\varphi(s-\tau(s))) + a(s)\,G(s,\varphi(s-\tau(s)))\right] ds,\ \forall t \in \mathbb{R}, \quad (10)$$

and

$$(B_1\varphi)(t) = -G(t,\varphi(t-\tau(t))),\ \forall t \in \mathbb{R}. \quad (11)$$

It follows from (3) and (5) that for any $\varphi \in \mathcal{M}(K,L)$ and $t \in \mathbb{R}$

$$\begin{aligned}(A_1\varphi)(t+T) &= \int_{t+T}^{t+2T} H(t+T,s)\left[f(s,\varphi(s-\tau(s))) + a(s)\,G(s,\varphi(s-\tau(s)))\right] ds \\ &= \int_t^{t+T} H(t+T,s+T)\left[f(s+T,\varphi(s+T-\tau(s+T)))\right. \\ &\quad \left.+\ a(s+T)\,G(s+T,\varphi(s+T-\tau(s+T)))\right] ds \\ &= (A_1\varphi)(t) \end{aligned} \quad (12)$$

and

$$\begin{aligned}(B_1\varphi)(t+T) &= -G(t+T,\varphi(t+T-\tau(t+T))) \\ &= -G(t+T,\varphi(t+T-\tau(t+T))) \\ &= (B_1\varphi)(t). \end{aligned} \quad (13)$$

We assume that, for all $t \in [0,T]$, $x, y \in [K,L]^2$ there exist constants K, L, E, G_1, G_2, c_1 and c_2 satisfying

$$|G(t,x) - G(t,y)| \le E\,\|x-y\|, \quad (14)$$

$$0 \le c_1 < 1,\quad 0 \le c_2,\quad -c_1 x \le G(t,x) \le c_2 x, \quad (15)$$

$$0 < G_1 \le g'(t) + a(t) \le G_2, \quad (16)$$

$$(K + c_2 L)\,G_2 \le f(t,x) + a(t)\,G(t,x) \le (1-c_1)\,L G_1, \quad (17)$$

Lemma 3. *Suppose that the conditions* (3), (4), 5 *and* (14), (15), (16), (17) *hold, with* $0 < K < L$. *If* A_1 *is defined by* (10), *then* A_1 *is continuous and compact.*

Proof. First, we claim that A_1 is continuous in $\mathcal{M}(K,L)$, with $0 < K < L$. Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{M}(K,L)$ and $\varphi \in \mathcal{M}(K,L)$ with $\lim_{n\to\infty}\varphi_n = \varphi$. Then

$$\begin{aligned}\|A_1\varphi_n - A_1\varphi\| \le \sup_{t\in[0,T]} \int_t^{t+T} H(t,s)&\left[\left|f(s,\varphi_n(s-\tau(s))) - f(s,\varphi(s-\tau(s)))\right|\right. \\ &\left.+\ |a(s)|\left|G(s,\varphi_n(s-\tau(s))) - G(s,\varphi(s-\tau(s)))\right|\right] ds\end{aligned}$$

By the Dominated Convergence Theorem, $\lim_{n\to\infty}\left|(A_1\varphi_n)(t) - (A_1\varphi)(t)\right| = 0$. Then A_1 is continuous in $\mathcal{M}(K,L)$.

Second, we claim that A_1 is compact. It is sufficient to show that $A_1(\mathcal{M}(K,L))$ is uniformly bounded and equicontinuous in $[0,T]$. Notice that (10) and (15), (16), (17) ensure that

$$\begin{aligned}\|A_1\varphi\| &\le \sup_{t\in[0,T]}\left|\int_t^{t+T} H(t,s)\left[f(s,\varphi(s-\tau(s))) + a(s)\,G(s,\varphi(s-\tau(s)))\right]\right| ds \\ &\le (1-c_1)\,L G_1 \sup_{t\in[0,T]}\int_t^{t+T} H(t,s)\,ds \\ &\le (1-c_1)\,L,\ \forall \varphi \in [K,L],\end{aligned}$$

$$
\begin{aligned}
&(A_1\varphi)'(t) \\
&= -\frac{g'(t)+a(t)}{g(t)}(A_1\varphi)(t) - H(t,t)\left[f(t,\varphi(t-\tau(t))) + a(t)G(t,\varphi(t-\tau(t)))\right] \\
&+ H(t,t+T)\left[f(t+T,\varphi(t+T-\tau(t+T))) + a(t+T)G(t+T,\varphi(t+T-\tau(t+T)))\right]
\end{aligned}
$$

then

$$
\begin{aligned}
\left|(A_1\varphi)'(t)\right| &\leq \frac{g'(t)+a(t)}{g(t)}\left|(A_1\varphi)(t)\right| + \left[H(t,t+T) - H(t,t)\right] \\
&\quad \times \left|f(t,\varphi(t-\tau(t))) + a(t)G(t,\varphi(t-\tau(t)))\right| \\
&\leq \frac{G_2}{\theta}(1-c_1)L + \frac{\exp\left(\int_0^T \frac{g'(r)+a(r)}{g(r)}dr\right) - 1}{g(t)\left(\exp\left(\int_0^T \frac{g'(r)+a(r)}{g(r)}dr\right) - 1\right)}(1-c_1)G_1L \\
&\leq \frac{(1-c_1)(G_1+G_2)L}{\theta},\ \forall (t,\varphi) \in [0,T]\times[K,L],
\end{aligned}
$$

where $\theta = \min_{t\in[0,T]}\{g(t)\}$, which give that $A_1(\mathcal{M}(K,L))$ is uniformly bounded and equicontinuous sin $[0,T]$. Hence by Ascoli-Arzela's theorem A_1 is compact. □

Lemma 4. *Suppose that* (14) *holds. If* B_1 *is given by* (11) *with*

$$E < 1, \tag{18}$$

then B_1 *is a contraction.*

Proof. Let B_1 defined by (11), for all $\varphi, \psi \in \mathcal{M}(K,L)$ and $t \in \mathbb{R}$, we obtain by (14))

$$
\begin{aligned}
\left|(B_1\varphi)(t) - (B_1\psi)(t)\right| &= \left|G(t,\varphi(t-\tau(t))) - G(t,\psi(t-\tau(t)))\right| \\
&\leq E\left\|\varphi-\psi\right\|. \qquad (19)
\end{aligned}
$$

Then $\|B_1\varphi - B_1\psi\| \leq E\|\varphi-\psi\|$. Thus B_1 is a contraction by (18). □

Theorem 2. *Assume that* (3), (4), (5), (14), (15), (16), (17) *and* (18) *hold. Then* (1) *has at least one positive T-periodic solution in* $\mathcal{M}(K,L)$.

Proof. By Lemma 1, it is obvious that (1)) has a solution φ if and only if the equation $S_1\varphi = \varphi$ has a solution φ. Let A_1, B_1 defined by (10), (11) respectively. It follows from (12) and (13) that for any $\varphi \in \mathcal{M}(K,L)$ and $t \in \mathbb{R}$

$$A_1(\mathcal{M}(K,L)) \subseteq P_T, \quad B_1(\mathcal{M}(K,L)) \subseteq P_T. \tag{20}$$

By Lemma 3, the operator A_1 is compact and continuous. Also, from Lemma 4, the operator B_1 is a contraction. Moreover, by (10), (11) and (15), (16), (17), we infer that for all $\varphi, \psi \in \mathcal{M}(K,L)$ and $t \in \mathbb{R}$

$$
\begin{aligned}
&(A_1\varphi)(t) + (B_1\psi)(t) \\
&= \int_t^{t+T} H(t,s)\left[f(s,\varphi(s-\tau(s))) + a(s)\,G(s,\varphi(s-\tau(s)))\right] ds - G(t,\psi(t-\tau(t))) \\
&\le (1-c_1)\,L\int_t^{t+T} H(t,s)\left(g'(s)+a(s)\right) ds + c_1L \\
&= \frac{(1-c_1)\,L}{\exp\left(\int_0^T \frac{g'(r)+a(r)}{g(r)}dr\right)-1}\int_t^{t+T}\exp\left(\int_t^s \frac{g'(r)+a(r)}{g(r)}dr\right)\frac{(g'(s)+a(s))}{g(s)}ds + c_1L \\
&= \frac{(1-c_1)\,L}{\exp\left(\int_0^T \frac{g'(r)+a(r)}{g(r)}dr\right)-1}\left[\exp\left(\int_0^T \frac{g'(r)+a(r)}{g(r)}dr\right)-1\right] + c_1L \\
&= (1-c_1)\,L + c_1L = L,
\end{aligned}
$$

on the other hand,

$$
\begin{aligned}
&(A_1\varphi)(t) + (B_1\psi)(t) \\
&= \int_t^{t+T} H(t,s)\left[f(s,\varphi(s-\tau(s))) + a(s)\,G(s,\varphi(s-\tau(s)))\right] ds - G(t,\psi(t-\tau(t))) \\
&\ge (K+c_2)\,L\int_t^{t+T} H(t,s)\left(g'(s)+a(s)\right) ds - c_2L \\
&= (K+c_2)\,L - c_2L = K,
\end{aligned}
$$

which imply that

$$A_1x + B_1y \in \mathcal{M} \text{ for all } x,y \in \mathcal{M}(K,L) \text{ and } t \in \mathbb{R}. \tag{21}$$

Clearly, all the hypotheses of the Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}(K,L)$ such that $z = A_1z + B_1z$. By Lemma 1 this fixed point is a solution of (1). Hence (1) has a positive T-periodic solution. This completes the proof. ☐

Theorem 3. *Assume that there for all $x \in [K,L]$ exist constants K, L, G_1, G_2, E, c_1, c_2 and $t_0 \in [0,T]$ satisfying* (14), (15)*),* (16), (17)*:*

$$0 \le K < L, \tag{22}$$

and either

$$(K + c_2L)\,G_2 < f(t_0,x) + a(t_0)\,G(t_0,x), \tag{23}$$

or

$$g'(t_0) + a(t_0) < G_2. \tag{24}$$

Then (1) *has at least one positive T-periodic solution in $\mathcal{M}(K,L)$ with $K < x(t) \le L$ for each $t \in [0,T]$.*

Proof. As in the proof of Theorem 2, we conclude similarly that (1) has an T-periodic solution $x(t) \in \mathcal{M}(K,L)$. Now we assert that $x(t) > K$ for all $t \in [0,T]$. Otherwise, there exists $t^* \in [0,T]$ satisfying $x(t^*) = K$. In view of (6), (9) and (15), (22), we have

$$\begin{aligned}
K &= \int_{t^*}^{t^*+T} H(t^*, s)\left[f(s, x(s-\tau(s))) + a(s) G(s, x(s-\tau(s)))\right] ds \\
&\quad - G(t^*, x(t^* - \tau(t^*))) \\
&\geq \int_{t^*}^{t^*+T} H(t^*, s)\left[f(s, x(s-\tau(s))) + a(s) G(s, x(s-\tau(s)))\right] ds - c_2 L,
\end{aligned}$$

which implies that

$$\begin{aligned}
0 &\geq \int_{t^*}^{t^*+T} H(t^*, s)\left[f(s, x(s-\tau(s))) + a(s) G(s, x(s-\tau(s)))\right] ds - (K + c_2 L) \\
&= \int_{t^*}^{t^*+T} H(t^*, s) \\
&\quad \times \left[f(s, x(s-\tau(s))) + a(s) G(s, x(s-\tau(s))) - (K + c_2 L)(g'(s) + a(s))\right] ds. \qquad (25)
\end{aligned}$$

Assume that (23) holds. By means of (16), (17), (23), and the continuity of H, f, G, a, g , g', τ, and x, we get that

$$\begin{aligned}
&\int_{t^*}^{t^*+T} H(t^*, s) \\
&\times \left[f(s, x(s-\tau(s))) + a(s) G(s, x(s-\tau(s))) - (K + c_2 L)(g'(s) + a(s))\right] ds \\
&\geq \int_{t^*}^{t^*+T} H(t^*, s)\left[f(s, x(s-\tau(s))) + a(s) G(s, x(s-\tau(s))) - (K + c_2 L) G_2\right] ds > 0,
\end{aligned}$$

which contradicts (25).

Assume that (24) holds. In light of (16), (17), (24), and the continuity of H, f, G, a, g , g', τ, and x, we get that

$$\begin{aligned}
&\int_{t^*}^{t^*+T} H(t^*, s) \\
&\times \left[f(s, x(s-\tau(s))) + a(s) G(s, x(s-\tau(s))) - (K + c_2 L)(g'(s) + a(s))\right] ds \\
&> \int_{t^*}^{t^*+T} H(t^*, s)\left[f(s, x(s-\tau(s))) + a(s) G(s, x(s-\tau(s))) - (K + c_2 L) G_2\right] ds \geq 0,
\end{aligned}$$

which contradicts (25). This completes the proof. □

Now, we obtain the existence of a positive periodic solution of (2). From (8) define a mapping S_2 by

$$\begin{aligned}
&(S_2\varphi)(t) \\
&= -\int_{-\infty}^{0} Q(r) G(t, \varphi(t + h(r))) dr + \int_{t}^{t+T} H(t, s) \\
&\times \left[b(s) \int_{-\infty}^{0} Q(r) f(s, \varphi(s + h(r))) dr + a(s) \int_{-\infty}^{0} Q(r) G(s, \varphi(s + h(r))) dr\right] ds, \\
&\hfill (26)
\end{aligned}$$

It is clear form (3) and (5) that $S_2 : \mathcal{M}(K, L) \to P_T$ by the way it was constructed in Lemma (2)). Therefore, we express equation (26) as

$$(S_2\varphi)(t) = (A_2\varphi)(t) + (B_2\varphi)(t), \ \forall t \in \mathbb{R},$$

where $A_2, B_2 : \mathcal{M}(K, L) \to P_T$ are given by

$$\begin{aligned}
&(A_2\varphi)(t) \\
&= \int_t^{t+T} H(t,s) \\
&\times \left[b(s) \int_{-\infty}^0 Q(r) f(s, \varphi(s+h(r)))\, dr + a(s) \int_{-\infty}^0 Q(r) G(s, \varphi(s+h(r)))\, dr \right] ds,
\end{aligned} \tag{27}$$

and

$$(B_2\varphi)(t) = -\int_{-\infty}^0 Q(r) G(t, \varphi(t+h(r)))\, dr, \ \forall t \in \mathbb{R}. \tag{28}$$

Assume that for all $t \in [0, T]$, $x \in [K, L]$ there exist constants K, L, G_1, G_2, c_1 and c_2 satisfying

$$(K + c_2 L) G_2 \le b(t) f(t, x) + a(t) G(t, x) \le (1 - c_1) L G_1. \tag{29}$$

Lemma 5. *Suppose that the conditions* (3)*),* (4)*,* (5)*,* (14)*,* (15)*,* (16) *and* (29)*, with* $0 < K < L$*. If* A_2 *is defined by* (27)*, then* A_2 *is continuous and compact.*

Proof. First, we claim that A_2 is continuous in $\mathcal{M}(K, L)$. Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{M}(K, L)$ and $\varphi \in \mathcal{M}(K, L)$ with $\lim_{n\to\infty} \varphi_n = \varphi$. Then

$$\begin{aligned}
&\|A_2\varphi_n - A_2\varphi\| \\
&\le \sup_{t\in[0,T]} \int_t^{t+T} H(t,s) \left[|b(s)| \int_{-\infty}^0 Q(r) \left| f(s, \varphi_n(s+h(r))) - f(s, \varphi(s+h(r))) \right| dr \right. \\
&\left. + |a(s)| \int_{-\infty}^0 Q(r) \left| G(s, \varphi_n(s+h(r))) - G(s, \varphi(s+h(r))) \right| dr \right] ds.
\end{aligned}$$

By the Dominated Convergence Theorem, $\lim_{n\to\infty} |(A_2\varphi_n)(t) - (A_2\varphi)(t)| = 0$. Then A_1 is continuous in $\mathcal{M}(K, L)$.

Second, we claim that A_2 is compact. It is sufficient to show that $A_2(\mathcal{M}(K, L))$ is uniformly bounded and equicontinuous in $[0, T]$. Notice that (27), (15), (16) and (29) ensure that

$$\begin{aligned}
\|A_2\varphi\| &\le \sup_{t\in[0,T]} \left| \int_t^{t+T} H(t,s) \left[b(s) \int_{-\infty}^0 Q(r) f(s, \varphi(s+h(r)))\, dr \right. \right. \\
&\left. \left. + \ a(s) \int_{-\infty}^0 Q(r) G(s, \varphi(s+h(r)))\, dr \right] ds \right| \\
&\le (1 - c_1) L \int_{-\infty}^0 Q(r)\, dr G_1 \sup_{t\in[0,T]} \int_t^{t+T} H(t,s) ds \\
&\le (1 - c_1) L, \ \forall \varphi \in [K, L],
\end{aligned}$$

$$
\begin{aligned}
&(A_2\varphi)'(t)\\
&= -\frac{g'(t)+a(t)}{g(t)}(A_2\varphi)(t) + H(t,t+T)\\
&\times\left[b(t+T)\int_{-\infty}^{0} Q(r) f(t+T,\varphi(t+T+h(r)))\,dr\right.\\
&\left.+\,a(t+T)\int_{-\infty}^{0} Q(r) G(t+T,\varphi(t+T+h(r)))\,dr\right]\\
&-H(t,t)\left[b(t)\int_{-\infty}^{0} Q(r) f(t,\varphi(t+h(r)))\,dr + a(t)\int_{-\infty}^{0} Q(r) G(t,\varphi(t+h(r)))\,dr\right],
\end{aligned}
$$

then

$$
\begin{aligned}
\left|(A_2\varphi)'(t)\right| &\le \frac{g'(t)+a(t)}{g(t)}\left|(A_2\varphi)(t)\right| + [H(t,t+T)-H(t,t)]\\
&\times\left|b(t)\int_{-\infty}^{0} Q(r) f(t,\varphi(t+h(r)))\,dr + a(t)\int_{-\infty}^{0} Q(r) G(t,\varphi(t+h(r)))\,dr\right|\\
&\le \frac{G_2}{\theta}(1-c_1)L + \frac{\exp\left(\int_0^T \frac{g'(r)+a(r)}{g(r)}dr\right)-1}{g(t)\left[\exp\left(\int_0^T \frac{g'(r)+a(r)}{g(r)}dr\right)-1\right]}\int_{-\infty}^{0} Q(r)\,dr\,(1-c_1)G_1L\\
&\le \frac{(1-c_1)(G_1+G_2)L}{\theta},\ \forall (t,\varphi)\in[0,T]\times[K,L],
\end{aligned}
$$

were $\theta = \min_{t\in[0,T]}\{g(t)\}$, which give that $A_2(\mathcal{M}(K,L))$ is uniformly bounded and equicontinuous sin $[0,T]$. Hence by Ascoli-Arzela's theorem A_2 is compact. □

Lemma 6. *Suppose that* (14) *and* (18) *holds. If* B_2 *is given by* (28)*, then* B_2 *is a contraction.*

Proof. Let B_2 defined by (28), for all $\varphi,\psi\in\mathcal{M}(K,L)$ and $t\in\mathbb{R}$, we obtain by (14)

$$
\begin{aligned}
&\left|(B_2\varphi)(t)-(B_2\psi)(t)\right|\\
=&\left|\int_{-\infty}^{0} Q(r) G(t,\varphi(t-\tau(t)))\,dr - \int_{-\infty}^{0} Q(r) G(t,\psi(t-\tau(t)))\,dr\right|\\
\le& E\left\|\varphi-\psi\right\|.
\end{aligned}
$$

Then $\|B_2\varphi - B_2\psi\| \le E\|\varphi-\psi\|$. Thus B_2 is a contraction by (18). □

Theorem 4. *Suppose that the conditions* (3), (4), (5) *hold. Assume that there exist constants* K, L, G_1, G_2, E, c_1 *and* c_2 *satisfying* (14), (15), (16), (18) *and* (29)*. Then* (2) *has at least one positive* T*-periodic solution in* $\mathcal{M}(K,L)$*.*

Proof. By Lemma 2, it is obvious that (2) has a solution φ if and only if the equation $S_2\varphi=\varphi$ has a solution φ. Let A_2, $B_2:\mathcal{M}(K,L)\to P_T$ defined by (27), (28) respectively. It follows as in (12) and (13) that for any $\varphi\in\mathcal{M}(K,L)$ with $0<K<L$ and $t\in\mathbb{R}$

$$
A_2(\mathcal{M}(K,L))\subseteq P_T,\quad B_2(\mathcal{M}(K,L))\subseteq P_T.
$$

By Lemma 5, $A_2:\mathcal{M}(K,L)\to P_T$ defined by (27) is a compact and by Lemma 6, $B_2:\mathcal{M}(K,L)\to P_T$ defined by (28) is a contraction

The rest of the proof is similar to that of Theorem 2, and is omitted. This completes the proof. □

Theorem 5. *Suppose that the conditions* (3), (4), (5) *hold. Assume that there exist constants* K, L, G_1, G_2, E, c_1, c_2 *and* $t_0 \in [0,T]$ *satisfying* (14), (15), (16), (18), (29) *and either* (24) *or*

$$(K + c_2 L) G_2 < b(t_0) f(t_0, x) + a(t_0) G(t_0, x), \quad \forall x \in [K, L], \tag{30}$$

Then (2) *has at least one positive* T*-periodic solution in* $\mathcal{M}(K, L)$ *with* $K < x(t) \leq L$ *for each* $t \in [0,T]$.

Proof. The proof of this theorem is similar to that of Theorems 3 and 4 is omitted. □

Remark 1. *When* $G(t,x) = c(t)x$ *with* $c \in C(\mathbb{R}, \mathbb{R})$, *Theorems 2, 3, 4 and 5 reduce to Theorems 2.1, 2.2, 2.3 and 2.4 in* [12], *respectively.*

4. Examples

Now we construct two examples which illustrate the results obtained in Section 3.

Example 1. *Consider the first-order neutral functional differential equation with periodic delay*

$$\begin{aligned}
&\left[\left(1 + \frac{\cos t}{100}\right)\left(x(t) + \frac{x(t - 3\sin t - 2\cos t)}{1000} \cos\left(x(t - 3\sin t - 2\cos t)\right)\right)\right]' \\
&= -\left(1 + \frac{\sin t}{50}\right) x(t) + 20 + \cos^2 t + \sin^2\left(x(t - 3\sin t - 2\cos t)\cos t\right), \ \forall t \in \mathbb{R}.
\end{aligned} \tag{31}$$

Let $T = 2\pi$, $L = 100$, $K = 1$, $c_1 = c_2 = \frac{1}{1000}$, $G_1 = \frac{99}{100}$, $G_2 = \frac{101}{100}$, $E = \frac{101}{1000}$ *and*

$$g(t) = 1 + \frac{\cos t}{100}, \ a(t) = 1 + \frac{\sin t}{50}, \ \tau(t) = 3\sin t + 2\cos t,$$

$$G(t,x) = \frac{x}{1000}\cos(x), \quad \forall (t,x) \in \mathbb{R}^2,$$

$$f(t,x) = 20 + \cos^2 t + \sin^2(x\cos t), \quad \forall (t,x) \in \mathbb{R}^2.$$

It is easy to see that (14), (15), (16) *and* (18) *hold. Notice that*

$$\begin{aligned}
(K + c_2 L) G_2 &= 1.111 < 20 + \left(1 - \frac{1}{50}\right)\left(\frac{-1}{1000}\right) \cdot 100 \leq f(t,x) + a(t) G(t,x) \\
&\leq 22 + \left(1 + \frac{1}{50}\right)\frac{1}{1000} \cdot 100 < 98.901 = (1 - c_1) L G_1, \quad \forall (t,x) \in \mathbb{R}^2.
\end{aligned}$$

That is, (17) *is satisfied. Thus Theorem 2 yields that* (31) *has a positive* 2π*-periodic solution in* $\mathcal{M}(1, 100)$.

Example 2. *Consider the first-order neutral functional differential equation with periodic delay*

$$\begin{aligned}
&\left[\ln(1000 + 4\cos t)\left(x(t) + \int_{-\infty}^{0} e^r \frac{\cos(x(t+r))}{1000 - 4\cos t} x(t+r)\, dr\right)\right]' \\
&= -\frac{4}{1000 + 4\cos t} x(t) + \left(10^{-1} + \frac{\cos t + \sin t}{100}\right) \\
&\times \int_{-\infty}^{0} e^r \left[10^{-1} + \sin t + \frac{2x(t+r)\sin t + \cos^2 t}{5000 + 5\sin^2\left[t - \ln\left(1 + x^2(t+r)\right)\right]}\right] dr,
\end{aligned} \tag{32}$$

Let $T = 2\pi$, $L = 1000$, $K = 1$, $c_1 = \frac{1}{996}$, $c_2 = \frac{1}{1004}$, $G_1 = \frac{1}{996}$, $G_2 = \frac{7}{1004}$, $E = \frac{1001}{1004}$, $t_0 = \frac{\pi}{2}$ *and*

$$g(t) = \ln(1000 + 4\cos t), \; a(t) = \frac{4}{1000 + 4\cos t}, \; b(t) = 10^{-1} + \frac{\cos t + \sin t}{100}, \; \forall t \in \mathbb{R},$$

$$h(r) = -r, \; Q(r) = e^r, \; \forall r \in \mathbb{R}_-,$$

$$G(t, x) = \frac{\cos(x)}{1000 - 4\cos t} x, \quad \forall (t, x) \in \mathbb{R}^2,$$

$$f(t, x) = 10^{-1} + \sin t + \frac{2x\sin t + \cos^2 t}{5000 + 5\sin^2[t - \ln(1 + x^2)]}, \quad \forall (t, x) \in \mathbb{R}^2.$$

It is easy to see that (14), (15), (16), (18) *and* (22), (24) *hold. Notice that*

$$\begin{aligned}
(K + c_2 L) G_2 &= \frac{3507}{252004} \\
&< \frac{649988}{24999600} = \left(10^{-1} + \frac{-1-1}{10}\right)\left(10^{-1} + \frac{-2000+0}{5000}\right) \\
&\quad + \left(\frac{-4}{1000000 - 16}\right) \cdot 1000 \\
&\leq b(t) f(t, x) + a(t) G(t, x) \\
&\leq \left(10^{-1} + \frac{1+1}{10}\right)\left(10^{-1} + \frac{2000+1}{5000}\right) \\
&\quad + \left(\frac{4}{1000000 - 16}\right) \cdot 1000 \\
&= \frac{481429997}{3124950000} \\
&< \frac{995000}{992016} = (1 - c_1) L G_1, \quad \forall (t, x) \in \mathbb{R}^2,
\end{aligned}$$

that is, (29) *is satisfied. Clearly* (30) *follows from the above inequalities. Thus Theorem 5 yields that* (32) *has a positive* 2π*-periodic solution in* $\mathcal{M}(1, 1000)$.

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