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NEW OSTROWSKI TYPE INEQUALITIES FOR CO-ORDINATED (α, m)-CONVEX FUNCTIONS

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Abstract. In this paper, we establish some new Ostrowski type inequalities for functions of two variables whose derivatives in absolute value are co-ordinated (α, m)-convex.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'(x)| \le K$, then the inequality:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)\, du \right| \le K(b-a) \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right],$$

holds. This result is known in the literature as the *Ostrowski inequality* (see e.g. [28]). For recent results and generalizations concerning Ostrowski's inequality see [4, 5, 6, 7, 12, 15, 17, 18, 21, 26, 30, 33, 34, 35, 36] and the references therein.

Let $[0, b]$, where b is greater than 0, be an interval of the real line $\mathbb{R}$, and let $K(b)$ denote the class of all functions $f : [0, b] \to \mathbb{R}$ which are continuous and nonnegative on $[0, b]$ and such that $f(0) = 0$. A function f is said to be convex on $[0, b]$ if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y),$$

holds for all $x, y \in [0, b]$ and $t \in [0, 1]$. Let $K_C(b)$ denote the class of all functions $f \in K(b)$ convex on $[0, b]$, and let $K_F(b)$ be the class of all functions $f \in K(b)$ convex in mean on $[0, b]$, that is, the class of all functions $f \in K(b)$ for which $F \in K_C(b)$, where the mean function F of the function $f \in K(b)$ is defined by

$$F(x) = \begin{cases} \frac{1}{x}\int_0^x f(t)dt, & x \in (0, b] \\ 0, & x = 0 \end{cases}.$$

Let $K_S(b)$ denote the class of all functions $f \in K(b)$ which are starshaped with respect to the origin on $[0, b]$, that is, the class of all functions f with the property that

$$f(tx) \le tf(x),$$

holds for all $x \in [0, b]$ and $t \in [0, 1]$. In [8], Bruckner and Ostrow, among others, proved that

$$K_C(b) \subset K_F(b) \subset K_S(b).$$

In [39] G. Toader, (see also [9, Definition 1.1, Page 2]) defined m-convexity: another intermediate between the usual convexity and starshaped convexity.

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Definition 1. [9, Definition 1.1, Page 2] *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0, 1]$, if we have*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y),$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$. We say that f is m-concave if $-f$ is m-convex.

Denote by $K_m(b)$ the class of all m-convex functions on $[0, b]$ for which $f(0) \le 0$. Obviously, for $m = 1$, m-convexity is the standard convexity of functions on $[0, b]$, and for $m = 0$ the concept of starshaped functions. The following lemmas hold (see [39] see also [9, Lemma A & Lemma B, Page 2]):

Lemma 1. [9, Lemma A, Page 2] *If f is in the class $\mathrm{K}_m(b)$, then it is starshaped.*

Lemma 2. [9, Lemma B, Page 2] *If f is in the class $\mathrm{K}_m(b)$ and $0 < n < m \le 1$, then f is in the class $\mathrm{K}_n(b)$.*

From Lemma 1 and Lemma 2 it follows that

$$K_1(b) \subset K_m(b) \subset K_0(b),$$

whenever $m \in (0, 1)$. Note that in the class $K_1(b)$ are only convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \le 0$, that is, $K_1(b)$ is a proper subclass of the class of convex functions on $[0, b]$. It is interesting to point out that for any $m \in (0, 1)$ there are continuous and differentiable functions which are m-convex, but which are not convex in the standard sense (see [40]). The notion of m-convexity was further generalized by [27] in the following definition (see also [9, Definition 1.2, Page 3]).

Definition 2. [9, Definition 1.2, Page 3] *The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be (α, m)-convex, where $(\alpha, m) \in [0, 1]^2$, if we have*

$$f(tx + m(1-t)y) \le t^{\alpha} f(x) + m(1-t^{\alpha})f(y),$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Denote by $K_m^{\alpha}(b)$ the class of all (α, m)-convex functions on $[0, b]$ for which $f(0) \le 0$. It can be easily seen that for $(\alpha, m) \in \{(0, 0),\ (\alpha, 0),\ (1, 0),\ (1, m),\ (1, 1), (\alpha, 1)\}$ one obtains the following classes of functions: increasing, α-starshaped, starshaped, m-convex, convex and α-convex functions respectively. Note that in the class $K_1^1(b)$ are only convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \le 0$, that is $K_1^1(b)$ is a proper subclass of the class of all convex functions on $[0, b]$. For further results on inequalities related to m-convex and (α, m)-convex functions we refer the readers to [9, 10, 13, 21, 26, 29, 30].

Let us consider now a bidimensional interval $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$, a mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \le \lambda f(x, y) + (1-\lambda)f(z, w),$$

holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$. The mapping f is said to be concave on the co-ordinates on Δ if the above inequality holds in reversed direction, for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.

A modification for convex (concave) functions on Δ, which are also known as co-ordinated convex (concave) functions, was introduced by S.S. Dragomir [19] as follows:

A function $f : \Delta \to \mathbb{R}$ is said to be convex (concave) on the co-ordinates on Δ if the partial mappings $f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v)$ are convex (concave) where defined for all $x \in [a, b], y \in [c, d]$.

Clearly, every convex (concave) mapping $f : \Delta \to \mathbb{R}$ is convex (concave) on the co-ordinates. Furthermore, there exists co-ordinated convex (concave) function which is not convex (concave), (see for instance [19]).

Also, in [19], Dragomir proved the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$:

Theorem 1. *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
& \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\
& \leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
& \leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx\right. \\
& \left. + \frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
& \leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \qquad (1)
\end{aligned}$$

The above inequalities are sharp. The inequalities in (1) hold in reverse direction if the mapping f is concave.

In [31], Özdemir et al. gave the definition of m-convex and (α, m)-convex functions on the co-ordinates on rectangle from the plane $\mathbb{R}^2$ which generalize the notion of co-ordinated convex functions given in [19]:

Definition 3. [31, Definition 3, Page 3] *Let us consider a bidimensional interval $\Lambda =: [0, b] \times [0, d]$ in $[0, \infty)^2$. A mapping $f : \Lambda \to \mathbb{R}$ is said to be m-convex on Λ if the inequality*

$$f(\lambda x + m(1-\lambda)z, \lambda y + m(1-\lambda)w) \leq \lambda f(x, y) + m(1-\lambda)f(z, w),$$

holds for all $(x, y), (z, w) \in \Lambda$ with $\lambda \in [0, 1]$ and for some fixed $m \in [0, 1]$.
A function $f : \Lambda \to \mathbb{R}$ is said to be m-convex on the co-ordinates on Λ if the partial mappings $f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v)$ are m-convex where defined for all $x \in [a, b], y \in [c, d]$.

Definition 4. [31, Definition 4, Page 4] *Let us consider a bidimensional interval $\Lambda =: [0, b] \times [0, d]$ in $[0, \infty)^2$. A mapping $f : \Lambda \to \mathbb{R}$ is said to be (α, m)-convex on Λ if the inequality*

$$f(\lambda x + m(1-\lambda)z, \lambda y + m(1-\lambda)w) \leq \lambda^\alpha f(x, y) + m(1-\lambda^\alpha)f(z, w),$$

holds for all $(x, y), (z, w) \in \Lambda$ with $\lambda \in [0, 1]$ and for some fixed $(\alpha, m) \in [0, 1]^2$.
A function $f : \Lambda \to \mathbb{R}$ is said to be (α, m)-convex on the co-ordinates on Λ if the partial mappings $f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v)$ are (α, m)-convex where defined for all $x \in [a, b], y \in [c, d]$.

A formal definition of co-ordinated m-convex and co-ordinated (α, m)-convex functions may be stated in:

Definition 5. *Let us consider a bidimensional interval* $\Lambda =: [0,b] \times [0,d]$ *in* $[0,\infty)^2$. *A mapping* $f : \Lambda \to \mathbb{R}$ *is said to be co-ordinated* m*-convex on* Λ *if the inequality*

$$\begin{aligned} & f(tx + m(1-t)z, sy + m(1-s)w) \\ & \le stf(x,y) + mt(1-s)f(x,w) \\ & + ms(1-t)f(z,y) + m^2(1-t)(1-s)f(z,w), \end{aligned}$$

holds for all $(x,y),(z,w) \in \Lambda$ *with* $t,s \in [0,1]$ *and for some fixed* $m \in [0,1]$.
Similarly, A mapping $f : \Lambda \to \mathbb{R}$ *is said to be co-ordinated* (α,m)*-convex on* Λ *if the inequality*

$$\begin{aligned} & f(tx + m(1-t)z, sy + m(1-s)w) \\ & \le s^\alpha t^\alpha f(x,y) + mt^\alpha(1-s^\alpha)f(x,w) \\ & + ms^\alpha(1-t^\alpha)f(z,y) + m^2(1-t^\alpha)(1-s^\alpha)f(z,w), \end{aligned}$$

holds for all $(x,y),(z,w) \in \Lambda$ *with* $t,s \in [0,1]$ *and for some fixed* $(\alpha,m) \in [0,1]^2$.

It was also proved in [31] that every m-convex function $f : \Lambda \to \mathbb{R}$ is co-ordinated m-convex and every (α,m)-convex function $f : \Lambda \to \mathbb{R}$ is co-ordinated (α,m)-convex. Some Hermite-Hadamard type inequalities for co-ordinated m-convex and co-ordinated (α,m)-convex functions were also established in [31].

Note that for $(\alpha,m) \in \{(1,m),(1,1)\}$ one obtains the classes of co-ordinated m-convex and co-ordinated convex functions on Λ respectively.

For further results concerning co-ordinated convex functions, co-ordinated s-convex functions, co-ordinated m-convex functions and co-ordinated (α,m)-convex functions we refer the interested reader to [1, 2, 3, 16, 19, 20, 22, 23, 24, 25, 31] and the references therein.

The main purpose of this paper is to establish some new Ostrowski type inequalities for functions whose second order partial derivatives are co-ordinated (α,m)-convex on the rectangle from the plane$\mathbb{R}^2$. Our established results generalize those results proved in [25].

2. Main Results

To establish our main results we need the following identity:

Lemma 3. *Let* $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ *be a twice partial differentiable mapping on* Ω°. *If* $\frac{\partial^2 f}{\partial s \partial t} \in L([ma,mb] \times [mc,md])$ *with* $[ma,mb] \times [mc,md] \subseteq \Omega$, *then the following identity holds:*

$$\begin{aligned} & m^2 f(x,y) + \frac{1}{(b-a)(d-c)} \int_{ma}^{mb} \int_{mc}^{md} f(u,v)\,dvdu - A \\ & = \frac{(x-ma)^2(y-mc)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \frac{\partial^2}{\partial s \partial t} f(tx + m(1-t)a, sy + m(1-s)c)\,dsdt \\ & - \frac{(x-ma)^2(md-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \frac{\partial^2}{\partial s \partial t} f(tx + m(1-t)a, sy + m(1-s)d)\,dsdt \\ & - \frac{(mb-x)^2(y-mc)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \frac{\partial^2}{\partial s \partial t} f(tx + m(1-t)b, sy + m(1-s)c)\,dsdt \\ & + \frac{(mb-x)^2(md-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 st \frac{\partial^2}{\partial s \partial t} f(tx + m(1-t)b, sy + m(1-s)d)\,dsdt, \quad (2) \end{aligned}$$

for all $(x,y) \in [ma,mb] \times [mc,md]$*, where*

$$A = \frac{m}{d-c}\int_{mc}^{md} f(x,v)\,dv + \frac{m}{b-a}\int_{ma}^{mb} f(u,y)\,du.$$

Proof. Using integration by parts and the change of variables $u = tx + m(1-t)a$, $v = sy + m(1-s)c$, we have

$$\begin{aligned}
&\frac{(x-ma)^2(y-mc)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\frac{\partial^2}{\partial s\partial t}f(tx+m(1-t)a, sy+m(1-s)c)\,dsdt \\
&= \frac{(x-ma)^2(y-mc)^2}{(b-a)(d-c)}\int_0^1 t\left[s\frac{\partial^2}{\partial s\partial t}f(tx+m(1-t)a, sy+m(1-s)c)\,ds\right]dt \\
&= \frac{(x-ma)(y-mc)}{(b-a)(d-c)}f(x,y) - \frac{x-ma}{(b-a)(d-c)}\int_{mc}^{y} f(x,v)\,dv \\
&- \frac{y-mc}{(b-a)(d-c)}\int_{ma}^{x} f(u,y)\,du + \frac{1}{(b-a)(d-c)}\int_{ma}^{x}\int_{mc}^{y} f(u,v)\,dvdu \qquad (3)
\end{aligned}$$

Similarly, by integration by parts, we also have

$$\begin{aligned}
&\frac{(x-ma)^2(md-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\frac{\partial^2}{\partial s\partial t}f(tx+m(1-t)a, sy+m(1-s)d)\,dsdt \\
&= -\frac{(x-ma)(md-y)}{(b-a)(d-c)}f(x,y) - \frac{x-ma}{(b-a)(d-c)}\int_{md}^{y} f(x,v)\,dv \\
&+ \frac{md-y}{(b-a)(d-c)}\int_{ma}^{x} f(u,y)\,du + \frac{1}{(b-a)(d-c)}\int_{ma}^{x}\int_{md}^{y} f(u,v)\,dvdu, \qquad (4)
\end{aligned}$$

$$\begin{aligned}
&\frac{(mb-x)^2(y-mc)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\frac{\partial^2}{\partial s\partial t}f(tx+m(1-t)b, sy+m(1-s)c)\,dsdt \\
&= -\frac{(mb-x)(y-mc)}{(b-a)(d-c)}f(x,y) + \frac{mb-x}{(b-a)(d-c)}\int_{mc}^{y} f(x,v)\,dv \\
&- \frac{y-mc}{(b-a)(d-c)}\int_{mb}^{x} f(u,y)\,du + \frac{1}{(b-a)(d-c)}\int_{mb}^{x}\int_{mc}^{y} f(u,v)\,dvdu \qquad (5)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(mb-x)^2(md-y)^2}{(b-a)(d-c)}\int_0^1\int_0^1 st\frac{\partial^2}{\partial s\partial t}f(tx+m(1-t)b, sy+m(1-s)d)\,dsdt \\
&= \frac{(mb-x)(md-y)}{(b-a)(d-c)}f(x,y) + \frac{mb-x}{(b-a)(d-c)}\int_{md}^{y} f(x,v)\,dv \\
&+ \frac{md-y}{(b-a)(d-c)}\int_{mb}^{x} f(u,y)\,du + \frac{1}{(b-a)(d-c)}\int_{mb}^{x}\int_{md}^{y} f(u,v)\,dvdu. \qquad (6)
\end{aligned}$$

From (3)–(6), we get (2). This completes the proof. □

Theorem 2. *Let* Γ *be an open region in* $\mathbb{R}^2$ *such that* $[0,\infty)^2 \subset \Gamma$*. Let* $f:\Gamma\to\mathbb{R}$ *be a twice partial differentiable mapping on* Γ *such that* $\frac{\partial^2 f}{\partial s\partial t} \in L([ma,mb]\times[mc,md])$*, with* $[ma,mb]\times[mc,md] \subseteq \Gamma$*. If* $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ *is* (α,m)*-convex on the co-ordinates on* $[ma,mb]\times[mc,md]$ *for* $(\alpha,m)\in(0,1]^2$ *and* $\left|\frac{\partial^2}{\partial s\partial t}f(x,y)\right| \le M$*,* $(x,y)\in[ma,mb]\times[mc,md]$*, then* *the following inequality holds:*

$$\left|m^2 f(x,y) + \frac{1}{(b-a)(d-c)} \int_{ma}^{mb} \int_{mc}^{md} f(u,v)\, dvdu - A\right| \le M\left(\frac{\alpha+2m}{2(\alpha+2)}\right)^2 \left[\frac{(x-ma)^2+(mb-x)^2}{b-a}\right]\left[\frac{(y-mc)^2+(md-y)^2}{d-c}\right], \quad (7)$$

for all $(x,y) \in [ma,mb]\times[mc,md]$*, where* A *is defined in Lemma 3.*

Proof. By Lemma 3, we have that the following inequality holds:

$$\begin{aligned}
&\left|m^2 f(x,y) + \frac{1}{(b-a)(d-c)} \int_{ma}^{mb} \int_{mc}^{md} f(u,v)\, dvdu - A\right| \\
&\le \frac{(x-ma)^2(y-mc)^2}{(b-a)(d-c)} \int_0^1\int_0^1 st \left|\frac{\partial^2}{\partial s\partial t} f(tx+m(1-t)a, sy+m(1-s)c)\right| dsdt \\
&+ \frac{(x-ma)^2(md-y)^2}{(b-a)(d-c)} \int_0^1\int_0^1 st \left|\frac{\partial^2}{\partial s\partial t} f(tx+m(1-t)a, sy+m(1-s)d)\right| dsdt \\
&+ \frac{(mb-x)^2(y-mc)^2}{(b-a)(d-c)} \int_0^1\int_0^1 st \left|\frac{\partial^2}{\partial s\partial t} f(tx+m(1-t)b, sy+m(1-s)c)\right| dsdt \\
&+ \frac{(mb-x)^2(md-y)^2}{(b-a)(d-c)} \int_0^1\int_0^1 st \left|\frac{\partial^2}{\partial s\partial t} f(tx+m(1-t)b, sy+m(1-s)d)\right| dsdt, \quad (8)
\end{aligned}$$

for all $(x,y) \in [ma,mb]\times[mc,md]$.
Using the co-ordinated (α,m)-convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$, we get the following inequality holds:

$$\begin{aligned}
&\int_0^1\int_0^1 st \left|\frac{\partial^2}{\partial s\partial t} f(tx+m(1-t)a, sy+m(1-s)c)\right| dsdt \\
&\le m\left|\frac{\partial^2}{\partial s\partial t} f(x,c)\right| \int_0^1\int_0^1 t^{\alpha+1} s(1-s^\alpha)\, dsdt \\
&+ m\left|\frac{\partial^2}{\partial s\partial t} f(a,y)\right| \int_0^1\int_0^1 s^{\alpha+1} t(1-t^\alpha)\, dsdt \\
&+ \left|\frac{\partial^2}{\partial s\partial t} f(x,y)\right| \int_0^1\int_0^1 t^{\alpha+1} s^{\alpha+1} dsdt \\
&+ m^2\left|\frac{\partial^2}{\partial s\partial t} f(a,c)\right| \int_0^1\int_0^1 st(1-t^\alpha)(1-s^\alpha)\, dsdt. \quad (9)
\end{aligned}$$

Since

$$\int_0^1\int_0^1 t^{\alpha+1} s^{\alpha+1} dsdt = \frac{1}{(\alpha+2)^2},$$

$$\int_0^1\int_0^1 t^{\alpha+1} s(1-s^\alpha)\, dsdt = \int_0^1\int_0^1 s^{\alpha+1} t(1-t^\alpha)\, dsdt = \frac{\alpha}{2(\alpha+2)^2},$$

$$\int_0^1\int_0^1 st(1-t^\alpha)(1-s^\alpha)\, dsdt = \frac{\alpha^2}{4(\alpha+2)^2}$$

and

$$\left|\frac{\partial^2}{\partial s\partial t} f(x,y)\right| \le M, (x,y) \in [ma,mb]\times[mc,md].$$

Hence from (9), we obtain

$$\int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)a, sy + (1-s)c\right) \right| dsdt \leq \frac{M(2+\alpha m)^2}{4(\alpha+2)^2}. \tag{10}$$

Analogously, we also have

$$\int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)a, sy + (1-s)d\right) \right| dsdt \leq \frac{M(2+\alpha m)^2}{4(\alpha+2)^2}, \tag{11}$$

$$\int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)b, sy + (1-s)c\right) \right| dsdt \leq \frac{M(2+\alpha m)^2}{4(\alpha+2)^2} \tag{12}$$

and

$$\int_0^1 \int_0^1 st \left| \frac{\partial^2}{\partial s \partial t} f\left(tx + (1-t)b, sy + (1-s)d\right) \right| dsdt \leq \frac{M(2+\alpha m)^2}{4(\alpha+2)^2}. \tag{13}$$

Now by making use of the inequalities (10)-(13) and the fact that

$$\begin{aligned}
&(x-ma)^2(y-mc)^2 + (x-ma)^2(md-y)^2 \\
&+ (mb-x)^2(y-mc)^2 + (mb-x)^2(md-y)^2 \\
&= \left[(x-ma)^2 + (mb-x)^2\right]\left[(y-mc)^2 + (md-y)^2\right],
\end{aligned}$$

we get the inequality (7). This completes the proof. □

The following result is about the powers of the absolute value of the partial derivatives:

Theorem 3. *Let Γ be an open region in $\mathbb{R}^2$ such that $[0,\infty)^2 \subset \Gamma$. Let $f : \Gamma \to \mathbb{R}$ be a twice partial differentiable mapping on Γ such that $\frac{\partial^2 f}{\partial s \partial t} \in L([ma, mb] \times [mc, md])$, with $[ma, mb] \times [mc, md] \subseteq \Gamma$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ is (α, m)-convex on the co-ordinates on $[ma, mb] \times [mc, md]$ for $(\alpha, m) \in (0,1]^2$, p, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\left|\frac{\partial^2}{\partial s \partial t} f(x,y)\right| \leq M$, $(x,y) \in [ma, mb] \times [mc, md]$, then the following inequality holds:*

$$\begin{aligned}
&\left| m^2 f(x,y) + \frac{1}{(b-a)(d-c)} \int_{ma}^{mb} \int_{mc}^{md} f(u,v)\, dvdu - A \right| \\
&\leq \frac{M}{(1+p)^{\frac{2}{p}}} \left(\frac{\alpha m + 1}{\alpha + 1}\right)^{\frac{2}{q}} \left[\frac{(x-ma)^2 + (mb-x)^2}{b-a}\right] \left[\frac{(y-mc)^2 + (md-y)^2}{d-c}\right], \qquad (14)
\end{aligned}$$

for all $(x,y) \in [ma, mb] \times [mc, md]$, where A is defined in Lemma 3.

Proof. By Lemma 3 and using the Hölder inequality for double integrals, we have that inequality holds:

$$\begin{aligned}
&\left|m^2 f(x,y)+\frac{1}{(b-a)(d-c)}\int_{ma}^{mb}\int_{mc}^{md} f(u,v)\,dvdu - A\right| \le \left(\int_0^1\int_0^1 s^p t^p dsdt\right)^{\frac{1}{p}} \\
&\times\left[\frac{(x-ma)^2(y-mc)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+m(1-t)a, sy+m(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}}\right. \\
&+\frac{(x-ma)^2(md-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+m(1-t)a, sy+m(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}} \\
&+\frac{(mb-x)^2(y-mc)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+m(1-t)b, sy+m(1-s)c)\right|^q dsdt\right)^{\frac{1}{q}} \\
&\left.+\frac{(mb-x)^2(md-y)^2}{(b-a)(d-c)}\left(\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+m(1-t)b, sy+m(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}\right],
\end{aligned} \tag{15}$$

for all $(x,y)\in[ma,mb]\times[mc,md]$.
Since $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is (α,m)-convex on the co-ordinates on $[ma,mb]\times[mc,md]$ and $\left|\frac{\partial^2}{\partial s\partial t}f(x,y)\right| \le M$, $(x,y)\in[ma,mb]\times[mc,md]$, we have

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+m(1-t)a, sy+m(1-s)c)\right|^q dsdt \\
&\le m^2\left|\frac{\partial^2}{\partial s\partial t}f(a,c)\right|^q\int_0^1\int_0^1(1-t^\alpha)(1-s^\alpha)\,dsdt+\left|\frac{\partial^2}{\partial s\partial t}f(x,y)\right|^q\int_0^1\int_0^1 t^\alpha s^\alpha dsdt \\
&+m\left|\frac{\partial^2}{\partial s\partial t}f(a,y)\right|^q\int_0^1\int_0^1 s^\alpha(1-t^\alpha)\,dsdt++m\left|\frac{\partial^2}{\partial s\partial t}f(x,c)\right|^q\int_0^1\int_0^1 t^\alpha(1-s^\alpha)\,dsdt \\
&=\frac{M^q(\alpha m+1)^2}{(\alpha+1)^2}.
\end{aligned}$$

Similarly, we also have the following inequalities:

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a, sy+(1-s)d)\right|^q dsdt \le \frac{M^q(\alpha m+1)^2}{(\alpha+1)^2},$$

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b, sy+(1-s)c)\right|^q dsdt \le \frac{M^q(\alpha m+1)^2}{(\alpha+1)^2}$$

and

$$\int_0^1\int_0^1\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b, sy+(1-s)d)\right|^q dsdt \le \frac{M^q(\alpha m+1)^2}{(\alpha+1)^2}.$$

Using the fact

$$\int_0^1\int_0^1 s^p t^p dsdt = \frac{1}{(1+p)^{\frac{1}{p}}},$$

and the above inequalities in (15), we get (14). This completes the proof of the theorem. □

Remark 1. *Since for $p \in (1, \infty)$, we have*

$$\frac{1}{4} \leq \frac{1}{(1+p)^{\frac{2}{p}}} \leq 1,$$

if in Theorem 3, we put $m = 1$, we obtain

$$\begin{aligned} &\left| f(x,y) + \tfrac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v)\, dvdu - A \right| \\ &\leq M \left[\tfrac{(x-a)^2 + (b-x)^2}{b-a} \right] \left[\tfrac{(y-c)^2 + (d-y)^2}{d-c} \right] \end{aligned} \tag{16}$$

Now if we choose $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$ in (16), we get

$$\left| f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v)\, dvdu - A \right| \leq \frac{M(b-a)(d-c)}{4},$$

where

$$A = \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, v\right) dv + \frac{1}{b-a} \int_a^b f\left(u, \frac{c+d}{2}\right) du.$$

A different approach leads us to the following result:

Theorem 4. *Let Γ be an open region in $\mathbb{R}^2$ such that $[0,\infty)^2 \subset \Gamma$. Let $f : \Gamma \to \mathbb{R}$ be a twice partial differentiable mapping on Γ such that $\frac{\partial^2 f}{\partial s \partial t} \in L([ma, mb] \times [mc, md])$, with $[ma, mb] \times [mc, md] \subseteq \Gamma$. If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$ is (α, m)-convex on the co-ordinates on $[ma, mb] \times [mc, md]$ for $(\alpha, m) \in (0, 1]^2$, $q \geq 1$ and $\left|\frac{\partial^2}{\partial s \partial t} f(x,y)\right| \leq M$, $(x,y) \in [ma, mb] \times [mc, md]$, then the following inequality holds:*

$$\begin{aligned} &\left| m^2 f(x,y) + \frac{1}{(b-a)(d-c)} \int_{ma}^{mb} \int_{mc}^{md} f(u,v)\, dvdu - A \right| \\ &\leq \frac{M}{4} \left(\frac{2+\alpha m}{\alpha+2}\right)^{\frac{2}{q}} \left[\frac{(x-ma)^2 + (mb-x)^2}{b-a} \right] \left[\frac{(y-mc)^2 + (md-y)^2}{d-c} \right], \end{aligned} \tag{17}$$

for all $(x,y) \in [ma, mb] \times [mc, md]$, where A is defined in Lemma 3.

Proof. Suppose $q \geq 1$. From Lemma 3 and using the power mean inequality for double integrals, we have

$$\begin{aligned} &\left| m^2 f(x,y) + \frac{1}{(b-a)(d-c)} \int_{ma}^{mb} \int_{mc}^{md} f(u,v)\, dvdu - A \right| \leq \left(\int_0^1 \int_0^1 stdsdt \right)^{1-\frac{1}{q}} \\ &\times \left[\frac{(x-ma)^2 (y-mc)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f(tx + m(1-t)a, sy + m(1-s)c) \right|^q dsdt \right)^{\frac{1}{q}} \right. \\ &+ \frac{(x-ma)^2 (y-md)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f(tx + m(1-t)a, sy + m(1-s)d) \right|^q dsdt \right)^{\frac{1}{q}} \\ &+ \frac{(mb-x)^2 (y-mc)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f(tx + m(1-t)b, sy + m(1-s)c) \right|^q dsdt \right)^{\frac{1}{q}} \\ &\left. + \frac{(mb-x)^2 (md-y)^2}{(b-a)(d-c)} \left(\int_0^1 \int_0^1 ts \left| \frac{\partial^2}{\partial s \partial t} f(tx + m(1-t)b, sy + m(1-s)d) \right|^q dsdt \right)^{\frac{1}{q}} \right], \end{aligned} \tag{18}$$

for all $(x,y)\in[ma,mb]\times[mc,md]$.
By similar argument as in Theorem 3 that $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is (α,m)-convex on the co-ordinates on $[ma,mb]\times[mc,md]$ and $\left|\frac{\partial^2}{\partial s\partial t}f(x,y)\right|\leq M$, $(x,y)\in[ma,mb]\times[mc,md]$, we have

$$\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)c)\right|^q dsdt\leq\left(\frac{\alpha+2m}{2(\alpha+2)}\right)^2 M^q,$$

$$\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)a,sy+(1-s)d)\right|^q dsdt\leq\left(\frac{\alpha+2m}{2(\alpha+2)}\right)^2 M^q,$$

$$\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)c)\right|^q dsdt\leq\left(\frac{\alpha+2m}{2(\alpha+2)}\right)^2 M^q$$

and

$$\int_0^1\int_0^1 ts\left|\frac{\partial^2}{\partial s\partial t}f(tx+(1-t)b,sy+(1-s)d)\right|^q dsdt\leq\left(\frac{\alpha+2m}{2(\alpha+2)}\right)^2 M^q.$$

Now using the above inequalities and

$$\int_0^1\int_0^1 stdsdt=\frac{1}{4}$$

in (18), we get the desired inequality (17). This completes the proof. □

Remark 2. *Since for $p\in(1,\infty)$, we have*

$$\frac{1}{4}\leq\frac{1}{(1+p)^{\frac{2}{p}}}\leq 1,$$

if in Theorem 4, we put $m=1$, we obtain

$$\begin{aligned}&\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right|\\&\leq M\left[\frac{(x-a)^2+(b-x)^2}{b-a}\right]\left[\frac{(y-c)^2+(d-y)^2}{d-c}\right]\end{aligned}\tag{19}$$

Now if we choose $x=\frac{a+b}{2}$ and $y=\frac{c+d}{2}$ in (19), we get

$$\left|f(x,y)+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(u,v)\,dvdu-A\right|\leq\frac{M(b-a)(d-c)}{16},$$

where

$$A=\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},v\right)dv+\frac{1}{b-a}\int_a^b f\left(u,\frac{c+d}{2}\right)du.$$

This also reveals that the the inequality (17) gives tighter estimate than that of the inequality (14), i.e. approach via power mean inequality is a better approach than the one through Hölder's inequality.

Remark 3. *In Theorem 2-Theorem 4, if we choose $(\alpha,m)=(\alpha,1)$, we get those inequalities proved in [25].*

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