TJMM
6 (2014), No. 2, 129-137

SPACE AND TIME QUANTIZATION. PARTICLES WITH SPIN 1/2

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Abstract. The dynamical state of a particle with the spin 1/2 is described by a (4-dimensional) bispinor wave function in the momentum representation. The time and the coordinates are associated with Hermitian linear operators acting on the wave functions. The eigenvalues and the eigenfunctions of the time and of the coordinates are determined by the wave equation expressed in diverse forms. With its aid, we define the time, the coordinates and the "charge" of the particle field. The expression of wave function in the second quantization leads to two kinds of particles: particles proper and antiparticles. We express the time, the coordinates and the "charge" by creation and annihilation operators and determine their eigenvalues. Because the momentum is limited in value, the space and the time become discrete. Each position is populated with particles and antiparticles living a finite time. The changes in numbers of particles and antiparticles in different positions by creation and annihilation lead to displacement of the center of the particle system. This is interpreted as the macroscopic motion of a particle.

1. Introduction

The idea of space and time quantization dates as early as antiquity. However, it could be transformed into a physical conception only after the foundation of the relativistic quantum theory.

The concept of quantum space-time is a generalization of the usual concept of space-time in which some variables that ordinarily commute are assumed not to commute and form a different Lie algebra. The choice of this algebra still varies from theory to theory. As a result of this change, some variables that are usually continuous may become discrete.

The idea of quantum space-time was proposed in the early days of quantum theory by W. Heisenberg [1, 2], V. Ambartsumian and D. Ivanenko [3] as a way to eliminate infinities from quantum field theory. The germ of the idea passed from W. Heisenberg to R. Peierls, who noted that electrons in a magnetic field can by regarded as moving in a quantum space-time, and to R. Oppenheimer, who carried it to H. P. Snyder [4], who published the first concrete example. The Snyder algebra is still compound, however. C. N. Yang [5] replaced it by a simple algebra. I. E. Segal [6] argued that physics is evolving towards simple Lie algebras.

At present, many other models of space and time quantization are known: bicrossproduct model [7, 8, 9, 10], q-deformed space-time [11, 12, 13], fuzzy or spin model [14, 15], Heisenberg model [16, 17], noncommutative extensions to space-time [18], etc.

In this paper, we present a theory of the space and time quantization based on the creation and the annihilation of particles as in [19].

2010 *Mathematics Subject Classification.* 81T70.
Key words and phrases. space and time quantization, particles with the spin 1/2.

2. Theory

2.1. **Wave equation.** A particle with the spin 1/2 is described, in the frame of reference in which it is at rest, by a 3-spinor wave function [20]. Four-dimensionally, it can be an undotted 4-spinor Ξ^α ($\alpha = 1, 2$) or a dotted 4-spinor $\mathrm{H}_{\dot\alpha}$ ($\dot\alpha = 1, 2$). The wave function of a particle with the spin 1/2 in an arbitrary frame of reference is a 4-bispinor whose components are the above-mentioned 4-spinors.

For a free particle, the only operator which can figure in the wave equation is the radius 4-vector operator $\hat{\mathrm{x}}^\mu$. In the momentum representation, its components are operators of differentiating with respect to the components of the momentum 4-vector $p^\mu = (\varepsilon/c, \mathbf{p})$:

$$\hat{\mathrm{x}}^\mu = -i\hbar\frac{\partial}{\partial p_\mu} = \left(-i\hbar c\frac{\partial}{\partial\varepsilon}, i\hbar\nabla_{\mathbf{p}}\right), \tag{1}$$

where ε is energy, $\mathbf{p}$ is momentum, c is light velocity in vacuum, and $\hbar$ is Planck's constant divided by 2π. In spinor notations, the spinor operator $\hat{\mathrm{x}}_{\alpha\dot\beta}$ with

$$\begin{aligned}\hat{\mathrm{x}}^{1\dot1} &= \hat{\mathrm{x}}_{2\dot2} = \hat{\mathrm{x}}_0 + \hat{\mathrm{x}}_3,\ \hat{\mathrm{x}}^{1\dot2} = -\hat{\mathrm{x}}_{2\dot1} = \hat{\mathrm{x}}_1 - i\hat{\mathrm{x}}_2,\\ \hat{\mathrm{x}}^{2\dot1} &= -\hat{\mathrm{x}}_{1\dot2} = \hat{\mathrm{x}}_1 + i\hat{\mathrm{x}}_2,\ \hat{\mathrm{x}}^{2\dot2} = \hat{\mathrm{x}}_{1\dot1} = \hat{\mathrm{x}}_0 - \hat{\mathrm{x}}_3\end{aligned} \tag{2}$$

corresponds to this vector operator.

The wave equation is a differential relation between the 4-spinors Ξ^α and $\mathrm{H}_{\dot\alpha}$, realized with the aid of the spinor operator $\hat{\mathrm{x}}_{\alpha\dot\beta}$. The condition of relativistic invariance fixes the following system of equations

$$\begin{aligned}\hat{\mathrm{x}}^{\alpha\dot\beta}\mathrm{H}_{\dot\beta} &= s\Xi^\alpha,\\ \hat{\mathrm{x}}_{\dot\beta\alpha}\Xi^\alpha &= s\mathrm{H}_{\dot\beta},\end{aligned} \tag{3}$$

where s is a dimensional constant.

Let us eliminate one of the two spinors from the equations (3) by substituting $\mathrm{H}_{\dot\beta}$ from the second equation into the first equation:

$$\hat{\mathrm{x}}^{\alpha\dot\beta}\mathrm{H}_{\dot\beta} = \frac{1}{s}\hat{\mathrm{x}}^{\alpha\dot\beta}\hat{\mathrm{x}}_{\gamma\dot\beta}\Xi^\gamma = s\Xi^\alpha.$$

Taking into account that

$$\hat{\mathrm{x}}^{\alpha\dot\beta}\hat{\mathrm{x}}_{\gamma\dot\beta} = \hat{\mathrm{x}}^\beta\hat{\mathrm{x}}_\beta\delta^\alpha_\gamma,$$

we have

$$\left(\hat{\mathrm{x}}^\beta\hat{\mathrm{x}}_\beta - s^2\right)\Xi^\gamma = 0, \tag{4}$$

which shows that $s = c\tau$ is the "length" of the radius 4-vector and τ is the proper lifetime.

The relativistic wave equation represented by the system (3) can be transcribed in diverse forms.

With the aid of the formula

$$\hat{\overleftrightarrow{\mathrm{x}}} = \hat{\mathrm{x}}^0 + \hat{\mathbf{r}}\hat{\boldsymbol{\sigma}},$$

let us rewrite the equations (3) in the form

$$\begin{aligned}(\hat{\mathrm{x}}_0 + \hat{\mathbf{r}}\hat{\boldsymbol{\sigma}})\,\mathrm{H} &= s\Xi,\\ (\hat{\mathrm{x}}_0 - \hat{\mathbf{r}}\hat{\boldsymbol{\sigma}})\,\Xi &= s\mathrm{H},\end{aligned} \tag{5}$$

where $\hat{\overleftrightarrow{\mathrm{x}}}$ is the spinor operator $\hat{\mathrm{x}}^{\alpha\dot\beta}$, $\hat{\boldsymbol{\sigma}}$ is a vector operator represented by the vector matrix $\boldsymbol{\sigma}$, having the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix} \tag{6}$$

as components, and Ξ and H are the spinors

$$\Xi = \begin{pmatrix} \Xi^1 \\ \Xi^2 \end{pmatrix}, \mathrm{H} = \begin{pmatrix} \mathrm{H}_{\dot{1}} \\ \mathrm{H}_{\dot{2}} \end{pmatrix}. \tag{7}$$

Having in mind that $\hat{\mathrm{x}}_\mu^+ = -\hat{\mathrm{x}}_\mu$, because all the operators $\hat{\mathrm{x}}_\mu$ contain the factor i, and that $(\hat{\boldsymbol{\sigma}}F)^+ = F^+\hat{\boldsymbol{\sigma}}^+ = F^+\hat{\boldsymbol{\sigma}}$, because all the Pauli operators are Hermitian $\left(\hat{\boldsymbol{\sigma}}^+ = \hat{\boldsymbol{\sigma}}\right)$, the adjoints of the equations (5) are

$$\begin{aligned} \mathrm{H}^+ (\hat{\mathrm{x}}_0 + \hat{\mathbf{r}}\hat{\boldsymbol{\sigma}}) &= -s\Xi^+, \\ \Xi^+ (\hat{\mathrm{x}}_0 - \hat{\mathbf{r}}\hat{\boldsymbol{\sigma}}) &= -s\mathrm{H}^+, \end{aligned} \tag{8}$$

where

$$\Xi^+ = \begin{pmatrix} \Xi^{1*} & \Xi^{2*} \end{pmatrix}, \mathrm{H}^+ = \begin{pmatrix} \mathrm{H}_{\dot{1}}^* & \mathrm{H}_{\dot{2}}^* \end{pmatrix} \tag{9}$$

are the adjoints of the spinors Ξ and H. In the equations (8), the operators $\hat{\mathrm{x}}^\mu$ act on the function situated on the left of them.

The equations (5) can be rewritten as

$$\begin{pmatrix} \hat{O} & \hat{\mathrm{x}}_0 + \hat{\mathbf{r}}\hat{\boldsymbol{\sigma}} \\ \hat{\mathrm{x}}_0 - \hat{\mathbf{r}}\hat{\boldsymbol{\sigma}} & \hat{O} \end{pmatrix} \begin{pmatrix} \Xi \\ \mathrm{H} \end{pmatrix} = s \begin{pmatrix} \Xi \\ \mathrm{H} \end{pmatrix}$$

or

$$(\hat{\gamma}\hat{\mathrm{x}} - s)\,\Psi = 0, \tag{10}$$

with

$$\hat{\gamma}\hat{\mathrm{x}} \equiv \hat{\gamma}^\mu \hat{\mathrm{x}}_\mu = \hat{\gamma}^0 \hat{\mathrm{x}}_0 - \hat{\boldsymbol{\gamma}}\hat{\mathbf{r}},$$

where the operators $\hat{\gamma}^0$and $\hat{\boldsymbol{\gamma}}$ are represented by the matrices

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} O_2 & I_2 \\ I_2 & O_2 \end{pmatrix}, \boldsymbol{\gamma} = \begin{pmatrix} \mathbf{O_2} & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{O_2} \end{pmatrix}, \\ I_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{O_2} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \end{aligned} \tag{11}$$

and with

$$\Psi = \begin{pmatrix} \Xi \\ \mathrm{H} \end{pmatrix}. \tag{12}$$

Similarly, the equations (8) can be transcribed in the form

$$\begin{pmatrix} \Xi^+ & \mathrm{H}^+ \end{pmatrix} \begin{pmatrix} \hat{O} & \hat{\mathrm{x}}_0 - \hat{\mathbf{r}}\hat{\boldsymbol{\sigma}} \\ \hat{\mathrm{x}}_0 + \hat{\mathbf{r}}\hat{\boldsymbol{\sigma}} & \hat{O} \end{pmatrix} = -s \begin{pmatrix} \Xi^+ & \mathrm{H}^+ \end{pmatrix}$$

or

$$\Psi^+ \left(\hat{\gamma}^0 \hat{\mathrm{x}}_0 + \hat{\boldsymbol{\gamma}}\hat{\mathbf{r}} + s\right) = 0,$$

in which

$$\Psi^+ = \begin{pmatrix} \Xi^+ & \mathrm{H}^+ \end{pmatrix}. \tag{13}$$

Multiplying the both members of the latter equation on the right by $\hat{\gamma}^0$ and taking into account that the operators $\hat{\gamma}^0$ and $\hat{\boldsymbol{\gamma}}$ anticommute, we have

$$\bar{\Psi}\,(\hat{\gamma}\hat{\mathrm{x}} + s) = 0, \tag{14}$$

where

$$\bar{\Psi} = \Psi^+ \hat{\gamma}^0 \tag{15}$$

is the Dirac or relativistic conjugate of the wave function Ψ.

The representation (12) of the wave function is called spinor representation.

In the limiting case in which the velocity of the particle is small, the particle must be described by a single spinor with two components. Indeed, in the limit $\mathbf{r} \to \mathbf{0}$, $ct \to s$, the equations (5) give $\Xi = \mathrm{H}$, that is, the two spinors which form the bispinor coincide. This property reflects the insufficiency of the spinor form of the wave equation.

Adding and subtracting the equations (5), one obtain

$$\begin{aligned} \hat{\mathrm{x}}_0\Phi - \hat{\mathbf{r}}\hat{\boldsymbol{\sigma}}\mathrm{X} &= s\Phi, \\ -\hat{\mathrm{x}}_0\mathrm{X} + \hat{\mathbf{r}}\hat{\boldsymbol{\sigma}}\Phi &= s\mathrm{X}, \end{aligned} \tag{16}$$

with

$$\Phi = \frac{\Xi + \mathrm{H}}{\sqrt{2}},\ \mathrm{X} = \frac{\Xi - \mathrm{H}}{\sqrt{2}}. \tag{17}$$

The equations (16) can be rewritten in the form (10) if we introduce the notations

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} I_2 & O_2 \\ O_2 & -I_2 \end{pmatrix},\ \boldsymbol{\gamma} = \begin{pmatrix} \mathbf{O_2} & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & \mathbf{O_2} \end{pmatrix}, \\ I_2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{O_2} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \end{aligned} \tag{18}$$

and

$$\Psi = \begin{pmatrix} \Phi \\ \mathrm{X} \end{pmatrix}. \tag{19}$$

The Dirac conjugate (15) of the wave function (19) with

$$\Psi^+ = \begin{pmatrix} \Phi^+ & \mathrm{X}^+ \end{pmatrix} \tag{20}$$

and γ^0 given by the first equation (18) satisfies a wave equation of the form (14).

The representation (19) of the wave function is named standard representation.

It removes the insufficiency of the spinor representation: in the nonrelativistic limit, the components X of the wave function vanish.

Besides spinor and standard representations, many other representations of the wave function are possible. They can be obtained by choosing any linear combinations of the spinors Ξ and H as independent components of the wave functions. It is convenient to restrict the admissible linear transformations to unitary transformations. Generally, the operators $\hat{\gamma}$ must satisfy only the conditions that ensure the fulfillment of the equation $x^2 = s^2$. For finding these conditions, we will multiply the equation (10) on the left by $\hat{\gamma}\hat{\mathrm{x}}$:

$$(\hat{\gamma}^\mu \hat{\mathrm{x}}_\mu)(\hat{\gamma}^\nu \hat{\mathrm{x}}_\nu)\Psi = s(\hat{\gamma}^\mu \hat{\mathrm{x}}_\mu)\Psi = s^2\psi.$$

Because all the operators $\hat{\mathrm{x}}_\mu$ commute with each other, the tensor $\hat{\mathrm{x}}_\mu\hat{\mathrm{x}}_\nu$ is symmetric, so that we can write

$$\frac{1}{2}(\hat{\gamma}^\mu\hat{\gamma}^\nu + \hat{\gamma}^\nu\hat{\gamma}^\mu)\hat{\mathrm{x}}_\mu\hat{\mathrm{x}}_\nu\Psi = s^2\Psi.$$

It follows from this equation that the matrices γ^μ representing the operators $\hat{\gamma}^\mu$ satisfy the anticommutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I_4, \tag{21}$$

where

$$I_4 = \begin{pmatrix} I_2 & O_2 \\ O_2 & I_2 \end{pmatrix} \tag{22}$$

is the 4×4 unity matrix. These relations show that the matrices γ^μ have the squares

$$\left(\gamma^0\right)^2 = I_4,\ \left(\gamma^1\right)^2 = \left(\gamma^2\right)^2 = \left(\gamma^3\right)^2 = -I_4 \tag{23}$$

and the different matrices γ^μ anticommute with each other. The matrix γ^0 is Hermitian, whereas the matrix $\boldsymbol{\gamma}$ is anti-Hermitian:

$$\gamma^{0+} = \gamma^0,\ \boldsymbol{\gamma}^+ = -\boldsymbol{\gamma}. \tag{24}$$

The wave equation (10) can be solved in the derivative of the wave function with respect to the energy. To this end, it is sufficient to multiply this equation on the left by $\hat{\gamma}^0$. Finally, we have

$$-i\hbar\frac{\partial\Psi}{\partial\varepsilon} = \hat{\mathrm{t}}\Psi, \tag{25}$$

where $\hat{\mathrm{t}}$ is the time operator, having the expression

$$\hat{\mathrm{t}} = \frac{1}{c}\left(\hat{\boldsymbol{\alpha}}\hat{\mathbf{r}} + \hat{\beta}s\right), \tag{26}$$

with

$$\hat{\boldsymbol{\alpha}} = \hat{\gamma}^0\hat{\boldsymbol{\gamma}},\ \hat{\beta} = \hat{\gamma}^0. \tag{27}$$

The matrices $\boldsymbol{\alpha}$ and β representing the operators $\hat{\boldsymbol{\alpha}}$ and $\hat{\beta}$ satisfy the relations

$$\alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = 2\delta_{\mu\nu}I_4,\ \boldsymbol{\alpha}\beta + \beta\boldsymbol{\alpha} = \mathbf{O}_4,\ \beta^2 = I_4, \tag{28}$$

in which

$$\mathbf{O}_4 = \begin{pmatrix} \mathbf{O}_2 & \mathbf{O}_2 \\ \mathbf{O}_2 & \mathbf{O}_2 \end{pmatrix} \tag{29}$$

is the 4×4 null vector matrix. All these matrices are Hermitian:

$$\boldsymbol{\alpha}^+ = \boldsymbol{\alpha},\ \beta^+ = \beta. \tag{30}$$

They have the expressions

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{O}_2 \\ \mathbf{O}_2 & -\boldsymbol{\sigma} \end{pmatrix},\ \beta = \begin{pmatrix} O_2 & I_2 \\ I_2 & O_2 \end{pmatrix} \tag{31}$$

in the spinor representation and the expressions

$$\boldsymbol{\alpha} = \begin{pmatrix} \mathbf{O}_2 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & \mathbf{O}_2 \end{pmatrix},\ \beta = \begin{pmatrix} I_2 & O_2 \\ O_2 & -I_2 \end{pmatrix} \tag{32}$$

in the standard representation.

As in the case of particles with the spin 0 [19], the wave equation (10) admits two conservation laws in the momentum space: the conservation law of the time-coordinate 4-tensor $X^{\mu\nu}$ and the conservation law of the "current density" 4-vector j^μ.

The time-coordinate 4-tensor $X^{\mu\nu}$ allows us to determine the sum of radius 4-vectors of system particles:

$$X_\mu = \int X_{\mu 0}\mathrm{d}^3p. \tag{33}$$

For particles with the spin 1/2, it is easier to find the expressions of the quantities X_μ starting from the average values of the time

$$T = \int \Psi^+\hat{\mathrm{t}}\Psi\mathrm{d}^3p = -i\hbar\int \Psi^+\frac{\partial\Psi}{\partial\varepsilon}\mathrm{d}^3p = -i\hbar\int \bar{\Psi}\gamma^0\frac{\partial\Psi}{\partial\varepsilon}\mathrm{d}^3p \tag{34}$$

and of the radius vector

$$\mathbf{R} = \int \Psi^+\hat{\mathbf{r}}\Psi\mathrm{d}^3p = i\hbar\int \Psi^+\nabla_{\mathbf{p}}\Psi\mathrm{d}^3p = i\hbar\int \bar{\Psi}\gamma^0\nabla_{\mathbf{p}}\Psi\mathrm{d}^3p. \tag{35}$$

The expression of the "current density" 4-vector j^μ is obtained as follows. Multiplying the equation (10) on the left by $\bar{\Psi}$ and the equation (14) on the right by Ψ and adding the resulting equations, we get

$$\bar{\Psi}\gamma^\mu\left(\hat{\mathrm{x}}_\mu\Psi\right) + \left(\hat{\mathrm{x}}_\mu\bar{\Psi}\right)\gamma^\mu\Psi = \hat{\mathrm{x}}_\mu\left(\bar{\Psi}\gamma^\mu\Psi\right) = -i\frac{\hbar}{c}\frac{\partial}{\partial p^\mu}\left(c\bar{\Psi}\gamma^\mu\Psi\right) = 0.$$

This equation has the form of the "continuity equation" in the momentum space

$$\frac{\partial j^\mu}{\partial p^\mu} = 0,$$

where the expression of the "current density" 4-vector j^μ is

$$j^\mu = c\bar{\Psi}\gamma^\mu\Psi = \left(c\Psi^+\Psi,\, c\Psi^+\gamma^0\boldsymbol{\gamma}\Psi\right). \tag{36}$$

This formula gives the possibility to determine the "charge"

$$Q = \frac{1}{c}\int j_0 \mathrm{d}^3p = \int \Psi^+\Psi \mathrm{d}^3p = \int \bar{\Psi}\gamma^0\Psi \mathrm{d}^3p. \tag{37}$$

2.2. **Plane waves.** The particular solutions of the wave equation can be chosen in the form of plane waves. The values of the time which figure in the expressions of these waves can be both positive $\left[+t = +(1/c)\sqrt{\mathbf{r}^2 + s^2}\right]$ and negative $\left[-t = -(1/c)\sqrt{\mathbf{r}^2 + s^2}\right]$.

The particular solutions with positive values of the time have the expressions

$$\Psi_{\mathbf{r}} = \frac{1}{\sqrt{2ct}} u_x e^{\frac{i}{\hbar}x_\mu p^\mu}. \tag{38}$$

Changing the sign of $\mathbf{r}$, the particular solutions with negative values of time can be written in the form

$$\Psi_{-\mathbf{r}} = \frac{1}{\sqrt{2ct}} u_{-x} e^{-\frac{i}{\hbar}x_\mu p^\mu}. \tag{39}$$

The components of the bispinor amplitudes u_x and u_{-x} satisfy the system of algebraic equations

$$\begin{aligned} (\gamma x - sI_4)\, u_x &= O_{4\times 1}, \\ (\gamma x + sI_4)\, u_{-x} &= O_{4\times 1}, \end{aligned} \tag{40}$$

in which $O_{4\times 1}$ is the 4×1 null matrix. We will normalize the bispinor amplitudes by the invariant conditions

$$\begin{aligned} \bar{u}_x u_x &= 2s, \\ \bar{u}_{-x} u_{-x} &= -2s, \end{aligned} \tag{41}$$

where $\bar{u}_{\pm x} = u^+_{\pm x}\gamma^0$. Multiplying the equations (40) on the left by $\bar{u}_{\pm x}$, one obtains

$$(\bar{u}_{\pm x}\gamma u_{\pm x})\, x = 2s^2 = 2x^2,$$

whence one see that

$$\bar{u}_{\pm x}\gamma u_{\pm x} = 2x. \tag{42}$$

The current 4-density vector is

$$j = c\bar{\Psi}\gamma\Psi = c\frac{1}{2ct}\bar{u}_{\pm x}\gamma u_{\pm x} = \frac{x}{t}, \tag{43}$$

i.e., $j^\mu = (c, \mathbf{v})$, where $\mathbf{v} = \mathbf{r}/t$ is the velocity of the particle. This shows that the functions $\Psi_{\pm x}$ are normalized to "one particle in the momentum volume $V_{\mathbf{p}} = 1$".

The general solutions of the wave equation are linear combinations of the particular solutions (38) and (39) of the same equation.

2.3. **Particles and antiparticles.** In the second quantization method, the wave functions Ψ and $\bar{\Psi}$ are replaced by the operators

$$\begin{aligned} \hat{\Psi} &= \sum_{\mathbf{r}\sigma} \frac{1}{\sqrt{2ct}} \left(\hat{\mathrm{a}}_{\mathbf{r}\sigma} u_{x\sigma} e^{\frac{i}{\hbar}px} + \hat{\mathrm{b}}^{+}_{\mathbf{r}\sigma} u_{-x-\sigma} e^{-\frac{i}{\hbar}px} \right), \\ \hat{\bar{\Psi}} &= \hat{\Psi}^{+}\gamma^{0} = \sum_{\mathbf{r}\sigma} \frac{1}{\sqrt{2ct}} \left(\hat{\mathrm{a}}^{+}_{\mathbf{r}\sigma} \bar{u}_{x\sigma} e^{-\frac{i}{\hbar}px} + \hat{\mathrm{b}}_{\mathbf{r}\sigma} \bar{u}_{-x-\sigma} e^{\frac{i}{\hbar}px} \right), \end{aligned} \tag{44}$$

where the summation is performed over all the values of the radius vector $\mathbf{r}$ and over the two values $\pm 1/2$ of the spin magnetic quantum number σ, $(\hat{\mathrm{a}}_{\mathbf{r}\sigma}, \hat{\mathrm{a}}^{+}_{\mathbf{r}\sigma})$ are the annihilation and creation operators of particles and $\left(\hat{\mathrm{b}}_{\mathbf{r}\sigma}, \hat{\mathrm{b}}^{+}_{\mathbf{r}\sigma}\right)$ are the annihilation and creation operators of antiparticles.

Replacing the wave functions Ψ and $\bar{\Psi}$ by the operators $\hat{\Psi}$ and $\hat{\bar{\Psi}}$, taking into account the orthogonality of wave functions with different $\mathbf{r}$ or σ and having in mind the relation $\bar{u}_{\pm x}\gamma^{0}u_{\pm x} = 2ct$, one obtain the expressions of the operators representing the time (34), the radius vector (35) and the "charge" (37) of the particle field:

$$\begin{aligned} \hat{\mathrm{T}} &= \sum_{\mathbf{r}\sigma} t \left(\hat{\mathrm{a}}^{+}_{\mathbf{r}\sigma} \hat{\mathrm{a}}_{\mathbf{r}\sigma} - \hat{\mathrm{b}}_{\mathbf{r}\sigma} \hat{\mathrm{b}}^{+}_{\mathbf{r}\sigma} \right), \\ \hat{\mathbf{R}} &= \sum_{\mathbf{r}\sigma} \mathbf{r} \left(\hat{\mathrm{a}}^{+}_{\mathbf{r}\sigma} \hat{\mathrm{a}}_{\mathbf{r}\sigma} - \hat{\mathrm{b}}_{\mathbf{r}\sigma} \hat{\mathrm{b}}^{+}_{\mathbf{r}\sigma} \right), \\ \hat{\mathrm{Q}} &= \sum_{\mathbf{r}\sigma} \left(\hat{\mathrm{a}}^{+}_{\mathbf{r}\sigma} \hat{\mathrm{a}}_{\mathbf{r}\sigma} + \hat{\mathrm{b}}_{\mathbf{r}\sigma} \hat{\mathrm{b}}^{+}_{\mathbf{r}\sigma} \right). \end{aligned} \tag{45}$$

The Fermi anticommutation relation

$$\hat{\mathrm{b}}_{\mathbf{r}\sigma} \hat{\mathrm{b}}^{+}_{\mathbf{r}\sigma} + \hat{\mathrm{b}}^{+}_{\mathbf{r}\sigma} \hat{\mathrm{b}}_{\mathbf{r}\sigma} = 1 \tag{46}$$

gives

$$\hat{\mathrm{b}}_{\mathbf{r}\sigma} \hat{\mathrm{b}}^{+}_{\mathbf{r}\sigma} = -\hat{\mathrm{b}}^{+}_{\mathbf{r}\sigma} \hat{\mathrm{b}}_{\mathbf{r}\sigma} + 1. \tag{47}$$

The eigenvalues of the operators $\hat{\mathrm{a}}^{+}_{\mathbf{r}\sigma}\hat{\mathrm{a}}_{\mathbf{r}\sigma}$ and $\hat{\mathrm{b}}^{+}_{\mathbf{r}\sigma}\hat{\mathrm{b}}_{\mathbf{r}\sigma}$ are equal to the number of particles $N_{\mathbf{r}\sigma}$ and to the number of antiparticles $\bar{N}_{\mathbf{r}\sigma}$, respectively. Omitting the non-essential, infinite, additive constants, one obtains

$$\begin{aligned} T &= \sum_{\mathbf{r}\sigma} t \left(N_{\mathbf{r}\sigma} + \bar{N}_{\mathbf{r}\sigma} \right), \\ \mathbf{R} &= \sum_{\mathbf{r}\sigma} \mathbf{r} \left(N_{\mathbf{r}\sigma} + \bar{N}_{\mathbf{r}\sigma} \right), \\ Q &= \sum_{\mathbf{r}\sigma} \left(N_{\mathbf{r}\sigma} - \bar{N}_{\mathbf{r}\sigma} \right). \end{aligned} \tag{48}$$

According to the Pauli exclusion principle, the numbers $N_{\mathbf{r}\sigma}$ and $\bar{N}_{\mathbf{r}\sigma}$ can have only two values: 0 and 1.

In the case of interacting particles, processes of creation and annihilation of particles and antiparticles take place when the momentum components attain values of the order of m_0c, where m_0 is the rest mass. If Q is conserved, its expression shows that only particle-antiparticle pairs can appear and disappear.

Then, the space and the time become discrete. Each position is populated with $N_{\mathbf{r}\sigma}$ particles and $\bar{N}_{\mathbf{r}\sigma}$ antiparticles. In a position, particle-antiparticle pairs can appear and disappear by annihilation and creation of photons. A particle-antiparticle pair "lives" in a position a certain time, then it is annihilated by emission of photons. The photons move with the light velocity to neighboring positions, where they are annihilated by conversion into particle-antiparticle pairs. Identifying the regenerated particles with the initial particles, it follows a displacement of the particle system. The expression

$$\mathbf{R}_{\mathrm{c}} = \frac{\sum_{\mathbf{r}\sigma} \mathbf{r} \left(N_{\mathbf{r}\sigma} + \bar{N}_{\mathbf{r}\sigma} \right)}{\sum_{\mathbf{r}\sigma} \left(N_{\mathbf{r}\sigma} + \bar{N}_{\mathbf{r}\sigma} \right)} \tag{49}$$

gives the position of the center of the particle system at any instant of time. The mean time interval between the creation of a particle-antiparticle pair in a position and its regeneration in a neighboring position is

$$T_{\mathrm{c}} = \frac{\sum_{\mathbf{r}\sigma} t\left(N_{\mathbf{r}\sigma} + \bar{N}_{\mathbf{r}\sigma}\right)}{\sum_{\mathbf{r}\sigma}\left(N_{\mathbf{r}\sigma} + \bar{N}_{\mathbf{r}\sigma}\right)}. \quad (50)$$

The velocity of the center of the particle system can be identified with the macroscopic velocity of the particle subjected to measurement. For isotropic creation-annihilation processes, this velocity is equal to zero. In the case of anisotropic creation-annihilation processes, it is different of zero.

3. Conclusions

Because the momentum of the particles is limited in value, the space and the time are quantized. Each position is occupied with particles and antiparticles. Particle-antiparticle pairs can appear or disappear by annihilation or creation of photons, in a given position. The emitted photons move with the light velocity to neighboring positions, where they generate new particles. If we identify the generated particles with the initial particles, it follows their displacement from a position to another. The macroscopic motion of the particles represents an averaging of the elementary displacement-regeneration processes.

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