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SPACE AND TIME QUANTIZATION. PARTICLES WITH SPIN 0

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Abstract. The dynamical state of a particle with the spin 0 is described by a (4-dimensional) scalar wave function in the momentum representation. The time and the coordinates are associated with Hermitian linear operators acting on the wave functions. The eigenvalues and the eigenfunctions of the time and of the coordinates are determined by the wave equation. With its aid, we define the time-coordinate 4-tensor and the "current density" 4-vector, which satisfy some "conservation laws" and allow to write the time, the coordinates and the "charge" as integrals of the components of these quantities in the momentum space. The expression of wave function in the second quantization leads to two kinds of particles: particles proper and antiparticles. We express the time, the coordinates and the "charge" by creation and annihilation operators and determine their eigenvalues. Because the momentum is limited in value, the space and the time become discrete. Each position is populated with particles and antiparticles living a finite time. The changes in numbers of particles and antiparticles in different positions by creation and annihilation lead to displacement of the center of the particle system. This is interpreted as the macroscopic motion of a particle. Particularly, the strictly neutral particles are discussed.

1. Introduction

The space and the time are the reference frame of particle motion. The quantization of particle motion involves the discrete character of space and time.

The idea of space and time quantization dates as early as antiquity. Alexander of Aphrodisias wrote on Epicureans: "Affirming that both the space and the motion, and the time consist of indivisible particles, they affirm that the moving body moves onto the entire expanse of space, which consists of indivisible parts, and each of indivisible parts that belong to it has not motion and is only the result of motion." The sentence "the motion does not exist, only the result of motion exists" can mean that a particle does not move from a space or time cell to another, but disappears in one of them and appears in a neighboring cell.

The relation between the idea of space and time discontinuity and the idea of disappearance and appearance of a particle kept during the entire evolution of these ideas. But a purely logical intuition of the natural philosophy concerning the relation between the discrete space and time and the disappearance and the appearance of a particle could be transformed into a physical conception only after the foundation of the relativistic quantum theory.

In the Heisenberg S-matrix theory [1, 2], the conception of space and time discrete character takes a new form. The S matrix permits to describe the state of a particle system after scattering if one knows its state before scattering. But the words "before scattering" and "after scattering" signify "long before" and "long after" as compared with the duration of scattering process itself. As to the time interval and the space domain in which the scattering takes place, Heisenberg considers that it is impossible to attribute a

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well-determined space and time localization of the particle in this process. He introduces a minimum space length and a minimum time interval which form a minimum 4-dimensional cell. Inside this cell, the space-time localization loses its sense.

I. E. Tamm [3] establishes the following connection between the discrete character of space-time and the creation and the annihilation of particles. Measuring the position of a particle with a maximum accuracy signifies finding the minimum volume of space where the particle is situated. Such a measurement is performed launching a beam of photons, electrons, and other particles and determining the flying directions of these particles scattered on the particle whose position is subjected to measurement. The experiment does not permit to measure the coordinate of the particle with an uncertainty lower than the wavelengths of scattered particles. The wavelength is inversely proportional to the momentum of the scattered particle. As the energy increases, it is more and more probable the appearance of new particles in the scattering process. These new particles, in their turn, decay rapidly. The products of this decay cannot be distinguished from the particles appeared during the initial collisions of high-energy particles. However, they do not fly off the scattering point, but off the neighborhood of this point. In this way, the transformation processes hinder the precise localization of the scattered particle.

A slightly different interpretation of the discrete character of space-time is given by Ya. I. Frenkel [4]. It supposes that a particle of a kind is transformed into a particle of another kind and the latter is again transformed into a particle of initial kind in the neighboring space-time cell. If one identifies the regenerated particle with the initial particle, there results a displacement of the particle identical with itself with the light velocity on a distance equal to the elementary length. In this way, we come to a discrete space-time on the light cone. Inside the light cone, the space-time is continuous.

At present, many other models of space and time quantization are known: bicrossproduct model [5, 6, 7, 8], q-deformed space-time [9, 10, 11], fuzzy or spin model [12, 13], Heisenberg model [14, 15, 16], noncommutative extensions to space-time [17], etc.

In this paper we present a theory of the space and time quantization based on the creation and annihilation of particles.

2. Theory

2.1. **Wave equation.** A particle with the spin 0 is described, in the frame of reference in which it is at rest, by a 3-scalar wave function [18]. Four-dimensionally, it can be a 4-scalar Ψ or the fourth component of a (time-like) 4-vector $\Psi_\mu (\mu = 0, 1, 2, 3)$, whose single component Ψ_0 is different of zero in the proper frame of reference.

For a free particle the only operator which can figure in the wave equation is the radius 4-vector operator $\hat{\mathrm{x}}^\mu$. In the momentum representation, its components are operators of differentiating with respect to the components of the momentum 4-vector $p^\mu = (\varepsilon/c, \mathbf{p})$:

$$\hat{\mathrm{x}}^\mu = -i\hbar \frac{\partial}{\partial p_\mu} = \left(-i\hbar c \frac{\partial}{\partial \varepsilon}, i\hbar \nabla_{\mathbf{p}} \right), \tag{1}$$

where ε is energy, $\mathbf{p}$ is momentum, c is light velocity in vacuum, and $\hbar$ is Planck's constant divided by 2π.

The wave equation must be a differential relation between Ψ and Ψ_μ, realized with the aid of the operator $\hat{\mathrm{x}}^\mu$. It has to admit the "plane wave"

$$\text{const.}e^{\frac{i}{\hbar} x_\mu p^\mu}, \; x_\mu p^\mu = t\varepsilon - \mathbf{r}\mathbf{p}, \tag{2}$$

as particular solutions for every 4-vector x^μ which satisfies the condition

$$x_\mu x^\mu = s^2, \tag{3}$$

where $s = c\tau$ is the "length" of the radius 4-vector and τ is the proper lifetime. According to the superposition principle, the wave equation must be linear: every linear combination of functions (2) also describes a possible state of the particle. It is necessary that it should have the least possible order because a high order would introduce useless solutions. Finally, it must be covariant under the Lorentz transformation.

These conditions are satisfied by the relations

$$\hat{\mathrm{x}}_\mu \Psi = s\Psi_\mu,\ \hat{\mathrm{x}}^\mu \Psi_\mu = s\Psi, \tag{4}$$

with s a quantity characterizing the particle.

Eliminating Ψ_μ between the equations (4), we obtain

$$\left(\hat{\mathrm{x}}_\mu \hat{\mathrm{x}}^\mu - s^2\right)\Psi = 0 \tag{5}$$

or

$$-\hbar^2 \frac{\partial^2 \Psi}{\partial p^\mu \partial p_\mu} \equiv -\hbar^2 \left(c^2 \frac{\partial^2}{\partial \varepsilon^2} - \Delta_{\mathbf{p}}\right)\Psi = s^2 \Psi. \tag{6}$$

With Ψ of the form of a "plane wave" (2), the equations (5) and (6) become $x_\mu x^\mu = s^2$, whence one sees that s is the "length" of the radius 4-vector. The form of the equation (5) follows immediately from the fact that $\hat{\mathrm{x}}_\mu \cdot \hat{\mathrm{x}}^\mu$ is the only scalar operator which can be constituted with the aid of $\hat{\mathrm{x}}^\mu$.

In fact, a particle with the spin 0 is described by a single (4-dimensional) scalar Ψ which verifies the second order equation (5). As respects the first order equations (4), the role of the wave function is played by the functions Ψ and Ψ_μ together, and Ψ_μ is reduced to the "4-gradient" of the scalar Ψ in momentum space. In the frame of reference in which the particle is at rest, the wave function does not depend on the momentum and the space components of the 4-vector Ψ_μ vanish rightly.

The equation (5) can be derived from a variational principle. This principle states that there exists a 4-scalar real function F (analogous to the Lagrangian density) such that the functional

$$\mathcal{F} = \int F \mathrm{d}^4 p \tag{7}$$

(analogous to the action integral) has an extremal value for the actual motion in the momentum space. It is easy to see that the "Lagrange equations"

$$\frac{\partial}{\partial p^\mu} \frac{\partial F}{\partial q_{,\mu}} = \frac{\partial F}{\partial q}, \tag{8}$$

where $q \equiv \Psi, \Psi^*$ are the "generalized coordinates" and $q_{,\mu} = \partial q / \partial p^\mu$ are the "generalized velocities" of the field of particles, coincide with the equation (5) for Ψ and Ψ^* if F is chosen in the form

$$F = \hbar^2 \frac{\partial \Psi^*}{\partial p^\mu} \frac{\partial \Psi}{\partial p_\mu} - s^2 \Psi^* \Psi. \tag{9}$$

For an isolated system of particles, the function F does not explicitly depend on p^μ. Then,

$$\frac{\partial F}{\partial p^\mu} = \sum_q \frac{\partial F}{\partial q} \frac{\partial q}{\partial p^\mu} + \sum_q \frac{\partial F}{\partial q_{,\nu}} \frac{\partial q_{,\nu}}{\partial p^\mu}. \tag{10}$$

Substituting (9) into (10) and taking into account that $q_{,\mu,\nu} = q_{,\nu,\mu}$, we find

$$\frac{\partial F}{\partial p^\mu} = \sum_q \frac{\partial}{\partial p^\nu}\left(\frac{\partial F}{\partial q_{,\nu}}\right) q_{,\mu} + \sum_q \frac{\partial F}{\partial q_{,\mu}} \frac{\partial q_{,\mu}}{\partial p^\nu} = \frac{\partial}{\partial p^\nu}\left(\sum_q q_{,\mu} \frac{\partial F}{\partial p^\nu}\right). \tag{11}$$

With the substitution

$$\frac{\partial F}{\partial p^{\mu}} = \delta_{\mu}^{\nu} \frac{\partial F}{\partial p^{\nu}} \tag{12}$$

and the notation

$$X_{\mu}^{\nu} = \sum_{q} q_{,\mu} \frac{\partial F}{\partial p^{\nu}} - \delta_{\mu}^{\nu} F, \tag{13}$$

the relation (11) becomes

$$\frac{\partial X_{\mu}^{\nu}}{\partial p^{\nu}} = 0. \tag{14}$$

This equation has a form analogous to a conservation law in the momentum world.

The substitution of (9) into (13) yields

$$X_{\mu\nu} = \hbar^2 \left(\frac{\partial \Psi^*}{\partial p^{\mu}} \frac{\partial \Psi}{\partial p^{\nu}} + \frac{\partial \Psi^*}{\partial p^{\nu}} \frac{\partial \Psi}{\partial p^{\mu}} \right) - g_{\mu\nu} \left(\hbar^2 \frac{\partial \Psi^*}{\partial p^{\rho}} \frac{\partial \Psi}{\partial p_{\rho}} - s^2 \Psi^* \Psi \right), \tag{15}$$

where $g_{\mu\nu}$ is the metric tensor, defined as: $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$, $g_{\mu \neq \nu} = 0$.

Particularly,

$$X_{00} = \hbar^2 c^2 \frac{\partial \Psi^*}{\partial \varepsilon} \frac{\partial \Psi}{\partial \varepsilon} + \hbar^2 \nabla_{\mathbf{p}} \Psi^* \nabla_{\mathbf{p}} \Psi + s^2 \Psi^* \Psi, \tag{16}$$

$$X_{\alpha 0} = \hbar^2 c \left(\frac{\partial \Psi^*}{\partial p^{\alpha}} \frac{\partial \Psi}{\partial \varepsilon} + \frac{\partial \Psi^*}{\partial \varepsilon} \frac{\partial \Psi}{\partial p^{\alpha}} \right), \alpha = 1, 2, 3. \tag{17}$$

As we will show later, the 4-vector

$$X_{\mu} = \int X_{\mu 0} \mathrm{d}^3 p \tag{18}$$

represents the sum of radius 4-vectors of system particles. For this reason, it is naturally to call $X^{\mu\nu}$ the time-coordinate 4-tensor. X_{00} is positively defined.

The formula (16) can serve for normalizing the wave function. The "plane wave" normalized to "one particle in the momentum volume $V_{\mathbf{p}} = 1$" is written as

$$\Psi_{\mathrm{r}} = \frac{1}{\sqrt{2ct}} e^{\frac{i}{\hbar} x_{\mu} p^{\mu}}. \tag{19}$$

Indeed, for this function we have $X_{00} = ct$.

The equations (4) admit another "conservation law". It follows from these equations (and the analogous equations for Ψ^*) that

$$\frac{\partial j^{\mu}}{\partial p^{\mu}} = 0, \tag{20}$$

where

$$j_{\mu} = cs \left(\Psi^* \Psi_{\mu} + \Psi_{\mu}^* \Psi \right) = -i\hbar c \left(\Psi^* \frac{\partial \Psi}{\partial p^{\mu}} - \Psi \frac{\partial \Psi^*}{\partial p^{\mu}} \right). \tag{21}$$

j^{μ} plays the role of the "current density" 4-vector. Then (21) is the "continuity equation", which expresses the "conservation law" of

$$Q = \frac{1}{c} \int j_0 \mathrm{d}^3 p, \tag{22}$$

with

$$j_0 = -i\hbar c^2 \left(\Psi^* \frac{\partial \Psi}{\partial \varepsilon} - \Psi \frac{\partial \Psi^*}{\partial \varepsilon} \right). \tag{23}$$

j_0/c is not positively defined. Therefore, it cannot be interpreted as the probability density of localizing the particle in the momentum space.

2.2. **Particles and antiparticles.** In the second quantization method, an arbitrary wave function Ψ must be expanded in a complete set of eigenfunctions of the free particle, e.g., in "plane waves" Ψ_{r}:

$$\Psi = \sum_{\mathrm{r}} a_{\mathrm{r}}\Psi_{\mathrm{r}},\ \Psi^* = \sum_{\mathrm{r}} a_{\mathrm{r}}^*\Psi_{\mathrm{r}}^*. \tag{24}$$

Then, the coefficients a_{r} and a_{r}^* must be interpreted as the annihilation and creation operators $\hat{\mathrm{a}}_{\mathrm{r}}$ and $\hat{\mathrm{a}}_{\mathrm{r}}^+$ of some particles in the corresponding states.

For a "plane wave", the solution of the equation (5), the time t, must satisfy the condition

$$c^2t^2 = \mathbf{r}^2 + s^2 \tag{25}$$

or

$$t = \pm\frac{1}{c}\sqrt{\mathbf{r}^2 + s^2}. \tag{26}$$

But only positive values of the time have physical sense. However, we cannot simply omit negative values: only the superposition of all independent particular solutions of the wave equation gives its general solution. Hence, it appears the necessity to reinterpret the coefficients of the expansion of Ψ and Ψ^* in a complete set of eigenfunctions in the second quantization. We write this expansion in the form

$$\hat{\Psi} = \sum_{\mathrm{r}} \frac{1}{\sqrt{2ct}}\hat{\mathrm{a}}_{\mathrm{r}}^{(+)}e^{\frac{i}{\hbar}(t\varepsilon - \mathbf{rp})} + \sum_{\mathrm{r}} \frac{1}{\sqrt{2ct}}\hat{\mathrm{a}}_{\mathrm{r}}^{(-)}e^{-\frac{i}{\hbar}(t\varepsilon + \mathbf{rp})}, \tag{27}$$

where t is the positive quantity (26).

In the second quantization method we substitute the annihilation operators $\hat{\mathrm{a}}_{\mathrm{r}}$ of some particles for $\hat{\mathrm{a}}_{\mathrm{r}}^{(+)}$ and the creation operators $\hat{\mathrm{b}}_{-\mathrm{r}}^+$ of other particles for $\hat{\mathrm{a}}_{\mathrm{r}}^{(-)}$. In the second sum, one also substitutes $-\mathbf{r}$ for the summation variable $\mathbf{r}$ so that the exponential factor may take the form $\exp[-(i/\hbar)(t\varepsilon - \mathbf{rp})]$. Then, the operators $\hat{\Psi}$ and $\hat{\Psi}^+$ become

$$\begin{aligned}\hat{\Psi} &= \textstyle\sum_{\mathrm{r}} \frac{1}{\sqrt{2ct}}\hat{\mathrm{a}}_{\mathrm{r}}e^{\frac{i}{\hbar}(t\varepsilon - \mathbf{rp})} + \sum_{\mathrm{r}} \frac{1}{\sqrt{2ct}}\hat{\mathrm{b}}_{\mathrm{r}}^+e^{-\frac{i}{\hbar}(t\varepsilon - \mathbf{rp})},\\ \hat{\Psi}^+ &= \textstyle\sum_{\mathrm{r}} \frac{1}{\sqrt{2ct}}\hat{\mathrm{a}}_{\mathrm{r}}^+e^{-\frac{i}{\hbar}(t\varepsilon - \mathbf{rp})} + \sum_{\mathrm{r}} \frac{1}{\sqrt{2ct}}\hat{\mathrm{b}}_{\mathrm{r}}e^{\frac{i}{\hbar}(t\varepsilon - \mathbf{rp})}.\end{aligned} \tag{28}$$

Thus, it must distinguish two kinds of particles. They are called particles and antiparticles.

Substituting $\hat{\Psi}$ and $\hat{\Psi}^+$ for Ψ and Ψ^*, respectively, into (18) and (22), we get

$$\begin{aligned}\hat{\mathrm{T}} &= \tfrac{1}{c}\hat{\mathrm{X}}_0 = \tfrac{1}{c}\textstyle\int \hat{\mathrm{X}}_{00}\mathrm{d}^3p = \sum_{\mathrm{r}} t\left(\hat{\mathrm{a}}_{\mathrm{r}}^+\hat{\mathrm{a}}_{\mathrm{r}} + \hat{\mathrm{b}}_{\mathrm{r}}\hat{\mathrm{b}}_{\mathrm{r}}^+\right),\\ \hat{\mathrm{X}}_\alpha &= \textstyle\int \hat{\mathrm{X}}_{\alpha 0}\mathrm{d}^3p = \sum_{\mathrm{r}} x_\alpha\left(\hat{\mathrm{a}}_{\mathrm{r}}^+\hat{\mathrm{a}}_{\mathrm{r}} + \hat{\mathrm{b}}_{\mathrm{r}}\hat{\mathrm{b}}_{\mathrm{r}}^+\right),\\ \hat{\mathrm{Q}} &= \tfrac{1}{c}\textstyle\int \hat{\mathrm{j}}_0\mathrm{d}^3p = \sum_{\mathrm{r}}\left(\hat{\mathrm{a}}_{\mathrm{r}}^+\hat{\mathrm{a}}_{\mathrm{r}} - \hat{\mathrm{b}}_{\mathrm{r}}\hat{\mathrm{b}}_{\mathrm{r}}^+\right).\end{aligned} \tag{29}$$

The Bose commutation relation

$$\hat{\mathrm{b}}_{\mathrm{r}}\hat{\mathrm{b}}_{\mathrm{r}}^+ - \hat{\mathrm{b}}_{\mathrm{r}}^+\hat{\mathrm{b}}_{\mathrm{r}} = 1 \tag{30}$$

leads to

$$\hat{\mathrm{b}}_{\mathrm{r}}\hat{\mathrm{b}}_{\mathrm{r}}^+ = \hat{\mathrm{b}}_{\mathrm{r}}^+\hat{\mathrm{b}}_{\mathrm{r}} + 1. \tag{31}$$

The eigenvalues of the operators $\hat{\mathrm{a}}_{\mathrm{r}}^+\hat{\mathrm{a}}_{\mathrm{r}}$ and $\hat{\mathrm{b}}_{\mathrm{r}}^+\hat{\mathrm{b}}_{\mathrm{r}}$ are equal to the number of particles N_{r} and the number of antiparticles $\bar{N}_{\mathrm{r}}$ respectively. Omitting the non-essential, infinite, additive constants, one obtains

$$\begin{aligned}T &= \textstyle\sum_{\mathrm{r}} t\left(N_{\mathrm{r}} + \bar{N}_{\mathrm{r}}\right),\\ \mathbf{R} &= \textstyle\sum_{\mathrm{r}} \mathbf{r}\left(N_{\mathrm{r}} + \bar{N}_{\mathrm{r}}\right),\\ Q &= \textstyle\sum_{\mathrm{r}}\left(N_{\mathrm{r}} - \bar{N}_{\mathrm{r}}\right).\end{aligned} \tag{32}$$

Q is called the "charge" of the particle field. It is sometimes a conservative quantity. For free particles, not only Q, but also $N_{\rm r}$ and $\bar{N}_{\rm r}$ separately are conserved. In the case of interacting particles, if Q is conserved, its expression shows that only particle-antiparticle pairs can appear and disappear. If a particle is electrically charged, its antiparticle must have an opposite charge, according to the immutable law of charge conservation. For charged particle, Q also defines the total electric charge of the system, measured in units of the elementary charge. Particles and antiparticles can be electrically neutral.

The processes of creation and annihilation of particles and antiparticles take place when the momentum components attain values of the order of m_0c, where m_0 is the rest mass. Therefore, the values of momentum components are restricted to the intervals $[-p_{0x}, +p_{0x}]$, $[-p_{0y}, +p_{0y}]$, and $[-p_{0z}, +p_{0z}]$. For this reason, the eigenvalue spectrum of coordinates is discrete:

$$x = n_x \frac{\pi\hbar}{2p_{0x}} = n_x \frac{\lambda_{0x}}{4},\ y = n_y \frac{\pi\hbar}{2p_{0y}} = n_y \frac{\lambda_{0y}}{4},\ z = n_z \frac{\pi\hbar}{2p_{0z}} = n_z \frac{\lambda_{0z}}{4}, \quad n_x, n_y, n_z = 1, 2, 3, \ldots \tag{33}$$

Here λ_{0x}, λ_{0y}, and λ_{0x} are the de Broglie wavelengths corresponding to p_{0x}, p_{0y}, and p_{0z}, respectively, of the order of the Compton wavelength $\lambda_{\rm C} = 2\pi\hbar/(m_0c)$.

The eigenvalues of the time are

$$t = \frac{1}{c}\sqrt{\mathbf{r}^2 + s^2} = \sqrt{n_x^2 \left(\frac{\lambda_{0x}}{4c}\right)^2 + n_y^2 \left(\frac{\lambda_{0y}}{4c}\right)^2 + n_z^2 \left(\frac{\lambda_{0z}}{4c}\right)^2 + \left(\frac{s}{c}\right)^2} = \sqrt{n_x^2 \left(\frac{T_{0x}}{4}\right)^2 + n_y^2 \left(\frac{T_{0y}}{4}\right)^2 + n_z^2 \left(\frac{T_{0z}}{4}\right)^2 + \tau^2}, \tag{34}$$

where T_{0x}, T_{0y}, and T_{0z} are the periods of electromagnetic waves associated with photons having the momentum components p_{0x}, p_{0y}, and p_{0z}.

Therefore, the space and the time become discrete. Each position is populated with $N_{\rm r}$ particles and $\bar{N}_{\rm r}$ antiparticles. In a position, particle-antiparticle pairs can appear and disappear by annihilation and creation of photons. A particle-antiparticle pair "lives" in a position a certain time, then it is annihilated by emission of photons. The photons move with the light velocity to neighboring positions, where they are annihilated by conversion into particle-antiparticle pairs. Identifying the regenerated particles with the initial particles, it follows a displacement of the particle system. The expression

$$\mathbf{R}_{\rm c} = \frac{\sum_{\rm r} \mathbf{r}\left(N_{\rm r} + \bar{N}_{\rm r}\right)}{\sum_{\rm r}\left(N_{\rm r} + \bar{N}_{\rm r}\right)} \tag{35}$$

gives the position of the center of the particle system at any instant of time. The mean time interval between the creation of a particle-antiparticle pair in a position and its regeneration in a neighboring position is

$$T_{\rm c} = \frac{\sum_{\rm r} t\left(N_{\rm r} + \bar{N}_{\rm r}\right)}{\sum_{\rm r}\left(N_{\rm r} + \bar{N}_{\rm r}\right)}. \tag{36}$$

The velocity of the center of the particle system can be identified with the macroscopic velocity of the particle subjected to measurement. For isotropic creation-annihilation processes, this velocity is equal to zero. In the case of anisotropic creation-annihilation processes, it is different of zero.

2.3. **Strictly neutral particles.** The coefficients $\hat{a}_{\rm r}^{(+)}$ and $\hat{a}_{\rm r}^{(-)}$ of the wave function $\hat{\Psi}$ (27) in the second quantization have been considered as operators referring to different particles. But this is not obligatory. Particularly, the creation and annihilation operators which figure in $\hat{\Psi}$ can refer to the same particles. Denoting, in this case, the annihilation and creation operators by $\hat{c}_r$ and $\hat{c}_r^+$, respectively, we will write the operator $\hat{\Psi}$ in the form

$$\hat{\Psi} = \sum_{r} \frac{1}{\sqrt{2ct}} \hat{c}_r e^{\frac{i}{\hbar}(t\varepsilon - \mathbf{rp})} + \sum_{r} \frac{1}{\sqrt{2ct}} \hat{c}_r^+ e^{-\frac{i}{\hbar}(t\varepsilon - \mathbf{rp})}. \tag{37}$$

The field described by such an operator corresponds to a system of identical particles which "coincide with their antiparticles".

The operator (37) is Hermitian $\left(\hat{\Psi}^+ = \hat{\Psi}\right)$; the field is real. The "number of degrees of freedom" for a real field is half the one for a complex field. Because of this, the function F, expressed with the help of the real operator $\hat{\Psi}$, must contain the factor 1/2 in addition to (9):

$$\hat{F} = \frac{1}{2}\left(\hbar^2 \frac{\partial\hat{\Psi}}{\partial p^\mu}\frac{\partial\hat{\Psi}}{\partial p_\mu} - s^2\hat{\Psi}^2\right). \tag{38}$$

The corresponding time-coordinate 4-tensor is

$$\hat{X}_{\mu\nu} = \hbar^2 \frac{\partial\hat{\Psi}}{\partial p^\mu}\frac{\partial\hat{\Psi}}{\partial p^\nu} - g_{\mu\nu}\frac{1}{2}\left(\hbar^2 \frac{\partial\hat{\Psi}}{\partial p^\rho}\frac{\partial\hat{\Psi}}{\partial p_\rho} - s^2\hat{\Psi}^2\right). \tag{39}$$

In particular,

$$\hat{X}_{00} = \frac{1}{2}\left[\hbar^2 c^2 \left(\frac{\partial\hat{\Psi}}{\partial\varepsilon}\right)^2 + \hbar^2\left(\nabla_{\mathbf{p}}\hat{\Psi}\right)^2 + s^2\hat{\Psi}^2\right], \tag{40}$$

$$\hat{X}_{\alpha 0} = \hbar^2 c \frac{\partial\hat{\Psi}}{\partial p^\alpha}\frac{\partial\hat{\Psi}}{\partial\varepsilon}, \ \alpha = 1, 2, 3. \tag{41}$$

The radius 4-vector is given by

$$\begin{aligned} \hat{T} &= \tfrac{1}{2}\sum\nolimits_r t\left(\hat{c}_r^+\hat{c}_r + \hat{c}_r\hat{c}_r^+\right), \\ \hat{\mathbf{R}} &= \tfrac{1}{2}\sum\nolimits_r \mathbf{r}\left(\hat{c}_r^+\hat{c}_r + \hat{c}_r\hat{c}_r^+\right). \end{aligned} \tag{42}$$

Using the Bose commutation relation

$$\hat{c}_r\hat{c}_r^+ - \hat{c}_r^+\hat{c}_r = 1 \tag{43}$$

and neglecting some additive constants, the eigenvalues of the operators (42) are

$$\begin{aligned} T &= \sum\nolimits_r tN_r, \\ \mathbf{R} &= \sum\nolimits_r \mathbf{r}N_r. \end{aligned} \tag{44}$$

The "charge" Q of a real field is equal to zero. This is evident from the fact that Q must change its sign when antiparticles are substituted for particles; but, in this case, particles and antiparticles coincide. Indeed, the quantity

$$\hat{j}_\mu = -i\hbar c\left(\hat{\Psi}^+\frac{\partial\hat{\Psi}}{\partial p^\mu} - \hat{\Psi}\frac{\partial\hat{\Psi}^+}{\partial p^\mu}\right) \tag{45}$$

vanishes for $\hat{\Psi}^+ = \hat{\Psi}$. This property, in its turn, expresses the absence of a particular conservation law which would limit the possible changes in the number of particles. Evidently, such particles are electrically neutral. They are called strictly neutral particles. The electrically neutral particles can be annihilated only as particle-antiparticle pairs, while the strictly neutral particles can be annihilated individually.

3. Conclusions

Because the momentum of the particles is limited in value, the space and the time are quantized. Each position is populated with particles and antiparticles. Particles can appear or disappear as particle-antiparticle pairs or individually (for the strictly neutral particles), by annihilation or creation of photons, in a given position. The emitted photons move with the light velocity to neighboring positions, where they give rise to new particles. Identifying the generated particles with the initial particles, it follows their displacement from a position to another. The macroscopic motion of the particles represents an averaging of the elementary displacement-regeneration processes.

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