TJMM
6 (2014), No. 2, 109-120

SOME INTEGRAL OPERATORS OF ANALYTIC FUNCTIONS

RADU DIACONU

Abstract. In the present paper we define two integral operators $F^{m,n}_{\gamma_1,\dots,\lambda_l}$ and $G^{m,n}_{\gamma_1,\dots,\lambda_l}$, defined using the differential operator $SR^{m,n}$. We introduce some classes defined by these operators and we investigate properties of the integral operators on these classes. Also, are obtained subordination results for functions $f \in \mathcal{T}$ associated with the differential operator $SR^{m,n}$.

1. Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$ with $\mathcal{A}_1 = \mathcal{A}$. Denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting the functions f of the form $f(z) = z - \sum_{j=2}^{\infty} |a_j| z^j$, $z \in U$.

For the functions $f, g \in \mathcal{T}$, $f(z) = z - \sum_{j=2}^{\infty} |a_j| z^j$ and $g(z) = z - \sum_{j=2}^{\infty} |b_j| z^j$, $z \in U$, the Hadamard product or convolution of f and g is defined by $(f * g)(z) = z - \sum_{j=2}^{\infty} |a_j| \cdot |b_j| z^j$, $z \in U$.

Definition 1 (Sălăgean [9]). *For $f \in \mathcal{A}$, and $n \in \mathbb{N}$, the operator S^n is defined by $S^n : \mathcal{A} \to \mathcal{A}$,*

$$S^0 f(z) = f(z)$$
$$S^1 f(z) = zf'(z)$$
$$\dots$$
$$S^{n+1} f(z) = z\left(S^n f(z)\right)', \quad z \in U.$$

Remark 1. *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$, $z \in U$.*

If $f \in \mathcal{T}$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, then $S^n f(z) = z - \sum_{j=2}^{\infty} j^n a_j z^j$, $z \in U$.

Definition 2 (Ruscheweyh [8]). *For $f \in \mathcal{A}$ and $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,*

$$R^0 f(z) = f(z)$$
$$R^1 f(z) = zf'(z)$$
$$\dots$$
$$(n+1) R^{n+1} f(z) = z\left(R^n f(z)\right)' + nR^n f(z), \quad z \in U.$$

Remark 2. *If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$ $z \in U$.*

If $f \in \mathcal{T}$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z - \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$ $z \in U$.

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. analytic functions, Hadamard product, differential operator, integral operator, starlike and convex functions, differential subordination.

This work was suported by the strategic project POSDRU/159/1.5/S/138963 - "PERFORM"..

Following [1] we can define a new differential operator

Definition 3. *Let $n, m \in \mathbb{N}$. Denote by $SR_{\lambda}^{m,n} : \mathcal{A} \to \mathcal{A}$ the operator given by the Hadamard product of the generalized Sălăgean operator D_{λ}^{m} and the Ruscheweyh operator R^n,*

$$SR^{m,n} f(z) = (S^m * R^n) f(z), \tag{1}$$

for any $z \in U$ and each nonnegative integers m, n.

Remark 3. *If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $SR^{m,n} f(z) = z + \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} \times a_j^2 z^j$, $z \in U$.*

If $f \in \mathcal{T}$ and $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, then $SR^{m,n} f(z) = z - \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, $z \in U$.

Using simple computation one obtains the next result.

Proposition 1. *For $m, n \in \mathbb{N}$ we have*

$$SR^{m+1,n} f(z) = z \left(SR^{m,n} f(z)\right)' \tag{2}$$

and

$$z \left(SR^{m,n} f(z)\right)' = (n+1) SR^{m,n+1} f(z) - n SR^{m,n} f(z). \tag{3}$$

Proof. We have

$$\begin{gathered} SR_{\lambda}^{m+1,n} f(z) = z + \sum_{j=2}^{\infty} j^{m+1} \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j = \\ z + \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} j a_j^2 z^j = z \left(SR^{m,n} f(z)\right)', \end{gathered}$$

and

$$\begin{gathered} (n+1) SR^{m,n+1} f(z) - n SR^{m,n} f(z) = \\ (n+1) z + (n+1) \sum_{j=2}^{\infty} j^m \frac{(n+j)!}{(n+1)!(j-1)!} a_j^2 z^j - nz - n \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j = \\ z + (n+1) \sum_{j=2}^{\infty} j^m \frac{n+j}{n+1} \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j - n \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j = \\ z + \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} j a_j^2 z^j = z \left(SR^{m,n} f(z)\right)'. \end{gathered}$$

□

Breaz and Breaz [3] and Breaz, Owa and Breaz [4] introduced and investigated the following integral operators

$$F_{\gamma_1,\dots,\gamma_l}(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\gamma_1} \dots \left(\frac{f_l(t)}{t}\right)^{\gamma_l} dt,$$

$$G_{\gamma_1,\dots,\gamma_l}(z) = \int_0^z \left(f_1'(t)\right)^{\gamma_1} \dots \left(f_l'(t)\right)^{\gamma_l} dt,$$

where $f_i \in \mathcal{A}$, $\gamma_i \in \mathbb{R}$, $\gamma_i > 0$, $i \in \{1, 2, \dots, l\}$.

For functions $f_i \in \mathcal{T}$, $\gamma_i \in \mathbb{T}$, $i \in \{1, 2, \dots, l\}$, we define the integral operators $F_{\gamma_1,\dots,\gamma_l}^{m,n}$ and $G_{\gamma_1,\dots,\gamma_l}^{m,n}$ as follows

$$F_{\lambda,\gamma_1,\dots,\gamma_l}^{m,n}(z) = \int_0^z \left(\frac{SR^{m,n} f_1(t)}{t}\right)^{\gamma_1} \dots \left(\frac{SR^{m,n} f_l(t)}{t}\right)^{\gamma_l} dt, \tag{4}$$

$$G_{\gamma_1,\dots,\gamma_l}^{m,n}(z) = \int_0^z \left(\left(SR^{m,n} f_1(t)\right)'\right)^{\gamma_1} \dots \left(\left(SR^{m,n} f_l(t)\right)'\right)^{\gamma_l} dt, \tag{5}$$

for $z \in U$, $m, n \in \mathbb{N}$, $\lambda \geq 0$.

By using the differential operator $SR^{m,n} f$ and the integral operators $F_{\gamma_1,\dots,\gamma_l}^{m,n}$ and $G_{\gamma_1,\dots,\gamma_l}^{m,n}$, following [2], we introduce some subclasses of analytic functions $f \in \mathcal{T}$.

Definition 4. *A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{R}(\delta)$ if it satisfies the inequality*

$$\operatorname{Re}\left(\frac{z\left(SR^{m,n}f(z)\right)'}{SR^{m,n}f(z)}\right) < \delta, z \in U, \delta > 1.$$

Definition 5. *A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{D}(\delta)$ if it satisfies the inequality*

$$\operatorname{Re}\left(1 + \frac{z\left(SR^{m,n}f(z)\right)''}{\left(SR^{m,n}f(z)\right)'}\right) < \delta, z \in U, \delta > 1.$$

Definition 6. *A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{RA}(\beta, \mu)$, for $0 \le \beta < 1$ and $0 < \mu \le 1$, if it satisfies the inequality*

$$\left|\frac{z\left(SR^{m,n}f(z)\right)'}{SR^{m,n}f(z)} - 1\right| < \mu\left|\beta \cdot \frac{z\left(SR^{m,n}f(z)\right)'}{SR^{m,n}f(z)} - 1\right|, z \in U.$$

Definition 7. *A function $f \in \mathcal{T}$ is said to be in the class $\mathcal{DA}(\beta, \mu)$, for $0 \le \beta < 1$ and $0 < \mu \le 1$, if it satisfies the inequality*

$$\left|\frac{z\left(SR^{m,n}f(z)\right)''}{\left(SR^{m,n}f(z)\right)'}\right| < \mu\left|\beta\left(1 + \frac{z\left(SR^{m,n}f(z)\right)''}{\left(SR^{m,n}f(z)\right)'}\right) + 1\right|, z \in U.$$

Definition 8. *A family of functions f_i, $i \in \{1, 2, \ldots, l\}$ is said to be in the class $\mathcal{LAF}(\beta, \mu, \gamma_1, \ldots, \gamma_l)$ if it satisfies the inequality*

$$\operatorname{Re}\left(1 + \frac{z\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)''}{\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'}\right) \ge \beta\left|\frac{z\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)''}{\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'}\right| + \mu,$$

for some $\beta \ge 0$ and $-1 \le \mu \le 1$, where $F^{m,n}_{\gamma_1,\ldots,\gamma_l}$ is defined in (4).

Definition 9. *A family of functions f_i, $i \in \{1, 2, \ldots, l\}$ is said to be in the class $\mathcal{LAG}(\beta, \mu, \gamma_1, \ldots, \gamma_l)$ if it satisfies the inequality*

$$\operatorname{Re}\left(1 + \frac{z\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)''}{\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'}\right) \ge \beta\left|\frac{z\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)''}{\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'}\right| + \mu,$$

for some $\beta \ge 0$ and $-1 \le \mu \le 1$, where $G^{m,n}_{\gamma_1,\ldots,\gamma_l}$ is defined in (5).

Lemma 1. *For $f_i(z) = z - \sum_{j=2}^{\infty} a^i_j z^j \in \mathcal{T}$, $i \in \{1, 2, \ldots, l\}$, we have*

$$\frac{z\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)''}{\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'} = \sum_{i=1}^{l}\gamma_i\left[\frac{-\sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-2)!}\left(a^i_j\right)^2 z^{j-1}}{1 - \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!}\left(a^i_j\right)^2 z^{j-1}}\right],$$

where $F^{m,n}_{\gamma_1,\ldots,\gamma_l}$ is the integral operator given by (4).

Proof. Let $f_i(z) = z - \sum_{j=2}^{\infty} a^i_j z^j$, for $i \in \{1, 2, \ldots, l\}$, then

$$\left(SR^{m,n}f_i(z)\right)' = 1 - \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!\,(j-1)!} j\left(a^i_j\right)^2 z^{j-1}, z \in U.$$

We obtain

$$\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)' = \left(\frac{SR^{m,n}f_1(z)}{z}\right)^{\gamma_1} \cdots \left(\frac{SR^{m,n}f_l(z)}{z}\right)^{\gamma_l},$$

so

$$\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'' = E_1\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'\frac{z}{SR^{m,n}f_1(z)} + \ldots + E_l\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'\frac{z}{SR^{m,n}f_l(z)},$$

where

$$E_i = \gamma_i \frac{z\left(SR^{m,n} f_i\left(z\right)\right)' - SR^{m,n} f_i\left(z\right)}{z^2}, i \in \{1, 2, \ldots, l\}.$$

Next we calculate the expression $\frac{z\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)''}{\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)'}$.

$$\frac{z\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)''}{\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)'} = \sum_{i=1}^{l} \gamma_i \left[\frac{z\left(SR^{m,n} f_i\left(z\right)\right)'}{SR^{m,n} f_i\left(z\right)} - 1\right].$$

We find

$$\frac{z\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)''}{\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)'} = \sum_{i=1}^{l} \gamma_i \left[\frac{z - \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} j \left(a^i_j\right)^2 z^j}{z - \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} \left(a^i_j\right)^2 z^j} - 1\right] =$$

$$\sum_{i=1}^{l} \gamma_i \left[\frac{- \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-2)!} \left(a^i_j\right)^2 z^j}{z - \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} \left(a^i_j\right)^2 z^j}\right] =$$

$$\sum_{i=1}^{l} \gamma_i \left[\frac{- \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-2)!} \left(a^i_j\right)^2 z^{j-1}}{1 - \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} \left(a^i_j\right)^2 z^{j-1}}\right].$$

□

Lemma 2. *For* $f_i\left(z\right) = z - \sum_{j=2}^{\infty} a^i_j z^j \in \mathcal{T}$, $i \in \{1, 2, \ldots, l\}$, *we have*

$$\frac{z\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)''}{\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)'} = -\sum_{i=1}^{l} \gamma_i \left[\frac{\sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-2)!} j \left(a^i_j\right)^2 z^{j-1}}{1 - \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} j \left(a^i_j\right)^2 z^{j-1}}\right],$$

where $G^{m,n}_{\gamma_1,\ldots,\gamma_l}$ *is the integral operator given by* (5).

Proof. Let $f_i\left(z\right) = z - \sum_{j=2}^{\infty} a^i_j z^j$, for $i \in \{1, 2, \ldots, l\}$. It follows that

$$\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)' = \left(\left(SR^{m,n} f_1\left(z\right)\right)'\right)^{\gamma_1} \ldots \left(\left(SR^{m,n} f_l\left(z\right)\right)'\right)^{\gamma_l},$$

so

$$\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)'' = \sum_{i=1}^{l} \gamma_i \left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)' \frac{\left(SR^{m,n} f_i\left(z\right)\right)''}{\left(SR^{m,n} f_i\left(z\right)\right)'}.$$

Next, we calculate the expression $\frac{z\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)''}{\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)'}$.

$$\frac{z\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)''}{\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)'} = \sum_{i=1}^{l} \gamma_i \left[\frac{z\left(SR^{m,n} f_i\left(z\right)\right)''}{\left(SR^{m,n} f_i\left(z\right)\right)'}\right] =$$

$$\sum_{i=1}^{l} \gamma_i \left[\frac{- \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-2)!} j \left(a^i_j\right)^2 z^{j-1}}{1 - \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} j \left(a^i_j\right)^2 z^{j-1}}\right].$$

Hence

$$\frac{z\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)''}{\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}\left(z\right)\right)'} = -\sum_{i=1}^{l} \gamma_i \left[\frac{\sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-2)!} j \left(a^i_j\right)^2 z^{j-1}}{1 - \sum_{j=2}^{\infty} j^m \frac{(n+j-1)!}{n!(j-1)!} j \left(a^i_j\right)^2 z^{j-1}}\right].$$

□

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h an univalent function in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad \text{for} \quad z \in U, \tag{6}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (6).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (6) is said to be the best dominant of (6). The best dominant is unique up to a rotation of U.

Lemma 3 ([5])**.** *Let $q(z)$ be univalent in U and let $\phi(z)$ be analytic in a domain containing $q(U)$. If $\frac{zq'(z)}{\phi(q(z))}$ is starlike, then*

$$z\psi'(z)\,\phi(\psi(z)) \prec zq'(z)\,\phi(q(z)),\, z \in U,$$

then $\psi(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 4 ([7])**.** *If $p(z)$ and $q(z)$ are analytic in U, $q(z)$ is convex univalent, α, β and γ are complex and $\gamma \neq 0$. Further assume that*

$$\operatorname{Re}\left[\frac{\alpha}{\gamma} + \frac{2\beta}{\gamma}q(z) + \left(1 + \frac{zq''(z)}{q(z)}\right)\right] > 0.$$

If $p(z) = 1 + cz + \ldots$ is analytic in U and satisfies

$$\alpha p(z) + \beta p^2(z) + \gamma zp'(z) \prec \alpha q(z) + \beta q^2(z) + \gamma zq'(z),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Theorem 1 ([6])**.** *Let $q(z)$ be convex univalent and $0 < \beta \leq 1$,*

$$\operatorname{Re}\left[\frac{1-\beta}{\beta} + 2q(z) + \left(1 + \frac{zq''(z)}{q(z)}\right)\right] > 0.$$

If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{f(z)} + \beta z^2\frac{f''(z)}{f'(z)} \prec (1-\beta)\,q(z) + \beta q^2(z) + \beta zq'(z),$$

then $\frac{zf'(z)}{f(z)} \prec q(z)$ and $q(z)$ is the best dominant.

Theorem 2 ([6])**.** *Let $q(z)$ be analytic in U, $q(0) = 1$ and $h(z) = \frac{zq'(z)}{q(z)}$ be starlike univalent in U. If $f \in \mathcal{A}$ satisfies*

$$\frac{(zf(z))''}{f'(z)} - 2\frac{zf'(z)}{f(z)} \prec h(z),$$

then $\frac{z^2f'(z)}{f^2(z)} \prec q(z)$, and $q(z)$ is the best dominant.

2. The classes $\mathcal{LAF}(\beta,\mu,\gamma_1,\ldots,\gamma_l)$ and $\mathcal{LAG}(\beta,\mu,\gamma_1,\ldots,\gamma_l)$

In this section, we obtain sufficient conditions for a family of functions f_i to belongs to the classes $\mathcal{LAF}(\beta,\mu,\gamma_1,\ldots,\gamma_l)$ and $\mathcal{LAG}(\beta,\mu,\gamma_1,\ldots,\gamma_l)$.

Theorem 3. *Let the functions f_i belongs to the class $\mathcal{T}$, for $i\in\{1,2,\ldots,l\}$. Then $f_i\in\mathcal{LAF}(\beta,\mu,\gamma_1,\ldots,\gamma_l)$ for $i\in\{1,2,\ldots,l\}$ if and only if*

$$\sum_{i=1}^{l}\gamma_i(\beta+1)\left[\frac{\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-2)!}\left(a_j^i\right)^2}{1-\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-1)!}\left(a_j^i\right)^2}\right]\leq 1-\mu, \tag{7}$$

where $-1\leq\mu<1$, $\beta\geq 0$.

Proof. Suppose that (7) holds. It suffices to show that

$$\beta\left|\frac{z\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'}\right|-\operatorname{Re}\left(\frac{z\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'}\right)\leq 1-\mu,$$

where $-1\leq\mu<1$.

We have

$$\beta\left|\frac{z\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'}\right|-\operatorname{Re}\left(\frac{z\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'}\right)\leq(\beta+1)\left|\frac{z\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'}\right|.$$

Applying Lemma 1, we obtain

$$(\beta+1)\left|\frac{z\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'}\right|=$$

$$(\beta+1)\left|\sum_{i=1}^{l}\gamma_i\left[\frac{-\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-2)!}\left(a_j^i\right)^2z^{j-1}}{1-\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-1)!}\left(a_j^i\right)^2z^{j-1}}\right]\right|\leq$$

$$(\beta+1)\sum_{i=1}^{l}\gamma_i\left[\frac{\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-2)!}\left(a_j^i\right)^2|z|^{j-1}}{1-\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-1)!}\left(a_j^i\right)^2|z|^{j-1}}\right]\leq$$

$$(\beta+1)\sum_{i=1}^{l}\gamma_i\left[\frac{\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-2)!}\left(a_j^i\right)^2}{1-\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-1)!}\left(a_j^i\right)^2}\right].$$

The last expression is bounded above $1-\mu$ and hence we have

$$\beta\left|\frac{z\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'}\right|-\operatorname{Re}\left(\frac{z\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'}\right)\leq 1-\mu,$$

or equivalently

$$\operatorname{Re}\left(1+\frac{z\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'}\right)\geq\beta\left|\frac{z\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)''}{\left(F_{\gamma_1,\ldots,\gamma_l}^{m,n}(z)\right)'}\right|+\mu.$$

Hence $f_i\in\mathcal{LAF}(\beta,\mu,\gamma_1,\ldots,\gamma_l)$.

Conversely, suppose $f_i\in\mathcal{LAF}(\beta,\mu,\gamma_1,\ldots,\gamma_l)$, for $i\in\{1,2,\ldots,l\}$. From (7) and Lemma 1, we obtain that

$$1-\sum_{i=1}^{l}\gamma_i\left[\frac{\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-2)!}\left(a_j^i\right)^2|z|^{j-1}}{1-\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-1)!}\left(a_j^i\right)^2|z|^{j-1}}\right]\geq$$

$$\beta\left|\sum_{i=1}^{l}\gamma_i\left|\frac{\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-2)!}\left(a_j^i\right)^2z^{j-1}}{1-\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-1)!}\left(a_j^i\right)^2z^{j-1}}\right|\right|+\mu\geq$$

$$\beta\sum_{i=1}^{l}\gamma_i\left[\frac{\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-2)!}\left(a_j^i\right)^2z^{j-1}}{1-\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-1)!}\left(a_j^i\right)^2z^{j-1}}\right]+\mu,$$

which is equivalent to

$$\sum_{i=1}^{l}\gamma_i\beta\left[\frac{\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-2)!}\left(a_j^i\right)^2z^{j-1}}{1-\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-1)!}\left(a_j^i\right)^2z^{j-1}}\right]+$$

$$\sum_{i=1}^{l}\gamma_i\left[\frac{\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-2)!}\left(a_j^i\right)^2z^{j-1}}{1-\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-1)!}\left(a_j^i\right)^2z^{j-1}}\right]\leq 1-\mu,$$

which reduces to

$$\sum_{i=1}^{l}\gamma_i\left(\beta+1\right)\left[\frac{\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-2)!}\left(a_j^i\right)^2z^{j-1}}{1-\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-1)!}\left(a_j^i\right)^2z^{j-1}}\right]\leq 1-\mu,$$

when $z\to 1^-$ along the real axis, we obtain the inequality (7). □

Using the same technique as in the proof of Theorem 3 and applying Lemma 2, we obtain

Theorem 4. *Let f_i belongs to the class $\mathcal{T}$, for $i\in\{1,2,\ldots,l\}$. Then $f_i\in\mathcal{LAG}\left(\beta,\mu,\gamma_1,\ldots,\gamma_l\right)$ for $i\in\{1,2,\ldots,l\}$ if and only if*

$$\sum_{i=1}^{l}\gamma_i\left(\beta+1\right)\left[\frac{\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-2)!}j\left(a_j^i\right)^2}{1-\sum_{j=2}^{\infty}j^m\frac{(n+j-1)!}{n!(j-1)!}j\left(a_j^i\right)^2}\right]\leq 1-\mu, \qquad (8)$$

where $-1\leq\mu<1$, $\beta\geq 0$.

3. Integral operators $F^{m,n}_{\gamma_1,\ldots,\gamma_l}$ and $G^{m,n}_{\gamma_1,\ldots,\gamma_l}$ on the classes $\mathcal{D}(\delta)$, $\mathcal{R}(\delta)$, $\mathcal{RA}(\beta,\mu)$ and $\mathcal{DA}(\beta,\mu)$

In this section some of the properties of the integral operators $F^{m,n}_{\gamma_1,\ldots,\gamma_l}$ and $G^{m,n}_{\gamma_1,\ldots,\gamma_l}$ on the classes $\mathcal{D}(\delta)$, $\mathcal{R}(\delta)$, $\mathcal{RA}(\beta,\mu)$ and $\mathcal{DA}(\beta,\mu)$ are discussed.

Theorem 5. *Let $\gamma_i\in\mathbb{R}$ with $\gamma_i>0$, $i\in\{1,2,\ldots,l\}$, $f_i\in\mathcal{T}$ and $\left|\frac{(SR^{m,n}f_i(z))'}{SR^{m,n}f_i(z)}\right|<M_i$. If $f_i\in\mathcal{RA}(\beta_i,\mu_i)$, then $F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\in\mathcal{D}(\delta')$, where $\delta'=1+\sum_{i=1}^{l}\gamma_i\mu_i\left(\beta_iM_i+1\right)$.*

Proof. It is clear from (4) that $F^{m,n}_{\gamma_1,\ldots,\gamma_l}\in\mathcal{T}$.

On differentiating $F^{m,n}_{\gamma_1,\ldots,\gamma_l}$ given by (4), we obtain

$$\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'=\prod_{i=1}^{l}\left(\frac{SR^{m,n}f_i(z)}{z}\right)^{\gamma_i}. \qquad (9)$$

Differentiating (9) logarithmically and multiplying by z, we get

$$\frac{z\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)''}{\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'}=\sum_{i=1}^{l}\gamma_i\left[\frac{z\left(SR^{m,n}f_i(z)\right)'}{SR^{m,n}f_i(z)}-1\right],$$

equivalently

$$1+\frac{z\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)''}{\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'}=1+\sum_{i=1}^{l}\gamma_i\left[\frac{z\left(SR^{m,n}f_i(z)\right)'}{SR^{m,n}f_i(z)}-1\right]. \tag{10}$$

Taking real part of both sides of (10), we get

$$\operatorname{Re}\left[1+\frac{z\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)''}{\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'}\right]=1+\sum_{i=1}^{l}\gamma_i\left[\operatorname{Re}\left(\frac{z\left(SR^{m,n}f_i(z)\right)'}{SR^{m,n}f_i(z)}-1\right)\right]$$

$$\leq 1+\sum_{i=1}^{l}\gamma_i\left|\frac{z\left(SR^{m,n}f_i(z)\right)'}{SR^{m,n}f_i(z)}-1\right|.$$

Since $f_i\in\mathcal{RA}(\beta_i,\mu_i)$, for $i\in\{1,2,\ldots,l\}$, we have

$$\operatorname{Re}\left[1+\frac{z\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)''}{\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'}\right]<1+\sum_{i=1}^{l}\gamma_i\mu_i\left|\beta_i\frac{z\left(SR^{m,n}f_i(z)\right)'}{SR^{m,n}f_i(z)}+1\right|$$

$$<1+\sum_{i=1}^{l}\gamma_i\mu_i\beta_i\left|\frac{\left(SR^{m,n}f_i(z)\right)'}{SR^{m,n}f_i(z)}\right|+\sum_{i=1}^{l}\gamma_i\mu_i<1+\sum_{i=1}^{l}\gamma_i\mu_i\left(\beta_iM_i+1\right).$$

As $\sum_{i=1}^{l}\gamma_i\mu_i(\beta_iM_i+1)>0$, $F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\in\mathcal{D}(\delta')$, where $\delta'=1+\sum_{i=1}^{l}\gamma_i\mu_i(\beta_iM_i+1)$. □

By substituting $l=1$, $\gamma_1=\gamma$, $M_1=M$, $f_1=f$, in Theorem 5, we obtain

Corollary 1. *Suppose $\gamma\in\mathbb{R}$, with $\gamma>0$, $f\in\mathcal{T}$ and $\left|\frac{f'(z)}{f(z)}\right|<M$, where M is fixed. If $f\in\mathcal{RA}(\beta,\mu)$, then $\int_0^z\left(\frac{f(t)}{t}\right)^{\gamma}dt\in\mathcal{D}(\delta')$, where $\delta'=1+\gamma\mu(\beta M+1)$.*

Theorem 6. *Let $\gamma_i>0$ and $f_i\in\mathcal{T}$ for $i\in\{1,2,\ldots,l\}$, with $\delta_i>1$. Then $F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\in\mathcal{D}(\delta')$, where $\delta'=1+\sum_{i=1}^{l}\gamma_i(\delta_i-1)$.*

Proof. From (10), we have

$$\operatorname{Re}\left(1+\frac{z\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)''}{\left(F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'}\right)=\sum_{i=1}^{l}\gamma_i\operatorname{Re}\left(\frac{z\left(SR^{m,n}f_i(z)\right)'}{SR^{m,n}f_i(z)}\right)-\sum_{i=1}^{l}\gamma_i+1<$$

$$\sum_{i=1}^{l}\gamma_i\delta_i-\sum_{i=1}^{l}\gamma_i+1=1+\sum_{i=1}^{l}\gamma_i(\delta_i-1).$$

Since $\delta_i>1$, it is clear that $\sum_{i=1}^{l}\gamma_i(\delta_i-1)>0$ and hence $F^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\in\mathcal{D}(\delta')$, where $\delta'=1+\sum_{i=1}^{l}\gamma_i(\delta_i-1)$. □

Letting $l=1$, $\gamma_1=\gamma$, $\delta_1=\delta$ and $f_1=f$, in Theorem 6, we obtain

Corollary 2. *Suppose $\gamma>0$, $f\in\mathcal{R}(\delta)$ with $\delta>1$. Then $\int_0^z\left(\frac{f(t)}{t}\right)^{\gamma}dt\in\mathcal{D}(\delta')$, where $\delta'=1+\gamma(\delta-1)$.*

Theorem 7. *Let $\gamma_i>0$ and $f_i\in\mathcal{D}(\delta_i)$, for $i\in\{1,2,\ldots,l\}$, with $\delta_i>1$. Then $G^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\in\mathcal{D}(\delta')$, with $\delta'=1+\sum_{i=1}^{l}\gamma_i(\delta_i-1)$.*

Proof. From the definition of $G^{m,n}_{\gamma_1,\ldots,\gamma_l}$ given by (5), we have

$$\operatorname{Re}\left(1+\frac{z\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)''}{\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'}\right)=\sum_{i=1}^{l}\gamma_i\left\{\operatorname{Re}\left[1+\left(\frac{z\left(SR^{m,n}f_i(z)\right)''}{\left(SR^{m,n}f_i(z)\right)'}\right)\right]\right\}-\sum_{i=1}^{l}\gamma_i+1$$
$$<1+\sum_{i=1}^{l}\gamma_i\left(\delta_i-1\right).$$

As $\delta_i>1$, it is clear that $\sum_{i=1}^{l}\gamma_i\left(\delta_i-1\right)>0$ and hence we get that $G^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\in\mathcal{D}(\delta')$, where $\delta'=1+\sum_{i=1}^{l}\gamma_i\left(\delta_i-1\right)$. □

By substituting $l=1$, $\gamma_1=\gamma$, $\delta_1=\delta$ and $f_1=f$, in Theorem 7, we obtain the following

Corollary 3. *Let $\gamma>0$ and $f\in\mathcal{D}(\delta)$ with $\delta>1$. Then $\int_0^z\left(f'(t)\right)^\gamma dt\in\mathcal{D}(\delta')$, where $\delta'=1+\gamma\left(\delta-1\right)$.*

Theorem 8. *Let $f_i\in\mathcal{DA}(\beta_i,\mu_i)$, $\gamma_i\in R$ with $\gamma_i>0$ and $\left|\frac{(SR^{m,n}f_i(z))''}{(SR^{m,n}f_i(z))'}\right|<M_i$, $i\in\{1,2,\ldots,l\}$. Then $G^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\in\mathcal{D}(\delta')$, where $\delta'=1+\sum_{i=1}^{l}\gamma_i\mu_i\left[\beta_i\left(1+M_i\right)+1\right]$.*

Proof. From the definition of $G^{m,n}_{\gamma_1,\ldots,\gamma_l}$ given by (5), we have

$$\operatorname{Re}\left(1+\frac{z\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)''}{\left(G^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\right)'}\right)\le\sum_{i=1}^{l}\gamma_i\left|\frac{z\left(SR^{m,n}f_i(z)\right)''}{\left(SR^{m,n}f_i(z)\right)'}\right|<$$
$$1+\sum_{i=1}^{l}\gamma_i\mu_i\left|\beta_i\left(1+\frac{z\left(SR^{m,n}f_i(z)\right)''}{\left(SR^{m,n}f_i(z)\right)'}\right)+1\right|<$$
$$1+\sum_{i=1}^{l}\gamma_i\mu_i\beta_i\left(1+\left|\frac{z\left(SR^{m,n}f_i(z)\right)''}{\left(SR^{m,n}f_i(z)\right)'}\right|\right)+\sum_{i=1}^{l}\gamma_i\mu_i<1+\sum_{i=1}^{l}\gamma_i\mu_i\left[\beta_i\left(1+M_i\right)+1\right].$$

As $\sum_{i=1}^{l}\gamma_i\mu_i\left[\beta_i\left(1+M_i\right)+1\right]>0$, we conclude that $G^{m,n}_{\gamma_1,\ldots,\gamma_l}(z)\in\mathcal{D}(\delta')$, where $\delta'=1+\sum_{i=1}^{l}\gamma_i\mu_i\left[\beta_i\left(1+M_i\right)+1\right]$. □

By taking $l=1$, $\gamma_1=\gamma$, $M_1=1$, $f_1=f$, in Theorem 8, we obtain the following corollary

Corollary 4. *Let $\gamma\in\mathbb{R}$ with $\gamma>0$, $f\in\mathcal{DA}(\beta,\mu)$ and $\left|\frac{f''(z)}{f'(z)}\right|<M$, M is fixed. Then $\int_0^z\left(f'(t)\right)^\gamma dt\in\mathcal{D}(\delta')$, where $\delta'=1+\gamma\mu\left[\beta\left(1+M\right)+1\right]$.*

4. Subordination results for $f\in\mathcal{T}$ associated with the operator $SR^{m,n}$

In this section we extend Theorem 1 and Theorem 2 for functions $f\in\mathcal{T}$ defined through the operator given by (1).

Theorem 9. *Let $q(z)$ be convex and univalent, $\gamma\neq0$ and*

$$\operatorname{Re}\left\{\frac{(1-\gamma)(n+1)}{\gamma}+2(n+1)q(z)+\left(1+\frac{zq''(z)}{q'(z)}\right)\right\}>0.$$

If $f\in\mathcal{T}$ satisfies

$$\frac{SR^{m,n+1}f(z)}{SR^{m,n}f(z)}\left(1-\gamma-\frac{\gamma}{n+1}+\gamma\frac{n+2}{n+1}\frac{SR^{m,n+2}f(z)}{SR^{m,n+1}f(z)}\right)\prec(1-\gamma)q(z)+\gamma q^2(z)+\frac{\gamma}{n+1}zq'(z),\tag{11}$$

then

$$\frac{SR^{m,n+1}f(z)}{SR^{m,n}f(z)} \prec q(z) \tag{12}$$

and $q(z)$ is the best dominant.

Proof. Define the function

$$p(z) = \frac{SR^{m,n+1}f(z)}{SR^{m,n}f(z)}. \tag{13}$$

Logarithmic differential of (13) yields

$$\frac{p'(z)}{p(z)} = \frac{SR^{m,n}f(z)}{SR^{m,n+1}f(z)} \left[\frac{\left(SR^{m,n+1}f(z)\right)' SR^{m,n}f(z) - SR^{m,n+1}f(z)\left(SR^{m,n}f(z)\right)'}{\left(SR^{m,n}f(z)\right)^2}\right]$$

$$= \frac{\left(SR^{m,n+1}f(z)\right)'}{SR^{m,n+1}f(z)} - \frac{\left(SR^{m,n}f(z)\right)'}{SR^{m,n}f(z)}.$$

So,

$$\frac{zp'(z)}{p(z)} = \frac{z\left(SR^{m,n+1}f(z)\right)'}{SR^{m,n+1}f(z)} - \frac{z\left(SR^{m,n}f(z)\right)'}{SR^{m,n}f(z)}. \tag{14}$$

By using (2) in (14), we get

$$\frac{zp'(z)}{p(z)} = (n+2)\frac{SR^{m,n+2}f(z)}{SR^{m,n+1}f(z)} - (n+1) - (n+1)p(z) + n.$$

We obtain

$$\frac{zp'(z)}{p(z)} = (n+2)\frac{SR^{m,n+2}f(z)}{SR^{m,n+1}f(z)} - (n+1)p(z) - 1$$

and

$$\frac{SR^{m,n+2}f(z)}{SR^{m,n+1}f(z)} = \frac{1}{n+2}\left[\frac{zp'(z)}{p(z)} + (n+1)p(z) + 1\right]. \tag{15}$$

Hence from (15)

$$\frac{SR^{m,n+1}f(z)}{SR^{m,n}f(z)}\left[1-\gamma-\frac{\gamma}{n+1}+\gamma\frac{n+2}{n+1}\frac{SR^{m,n+2}f(z)}{SR^{m,n+1}f(z)}\right] =$$

$$p(z)\left[1-\gamma+\frac{\gamma}{n+1}\frac{zp'(z)}{p(z)}+\gamma p(z)\right] =$$

$$(1-\gamma)p(z) + \gamma p^2(z) + \frac{\gamma}{n+1}zp'(z). \tag{16}$$

In view of (16), the subordination (11) becomes

$$(1-\gamma)p(z) + \gamma p^2(z) + \frac{\gamma}{n+1}zp'(z) \prec (1-\gamma)q(z) + \gamma q^2(z) + \frac{\gamma}{n+1}zq'(z).$$

Applying Lemma 4, we obtain that

$$\frac{SR^{m,n+1}f(z)}{SR^{m,n}f(z)} \prec q(z)$$

and $q(z)$ is the best dominant. □

Theorem 10. *Let $q(z)$ be univalent in U, $q(0) \neq 0$, $\gamma \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike and univalent in U. If $f(z) \in \mathcal{T}$ satisfies*

$$\frac{SR^{m,n+2}f(z)}{SR^{m,n+1}f(z)} - \frac{\gamma(n+1)}{(n+2)}\frac{SR^{m,n+1}f(z)}{SR^{m,n}f(z)} \prec \frac{1}{n+2}\frac{zq'(z)}{q(z)} + 1 - \frac{\gamma(n-1)}{n+2}, \tag{17}$$

then

$$\frac{z^{\gamma-1}SR^{m,n+1}f(z)}{(SR^{m,n}f(z))^{\gamma}} \prec q(z) \tag{18}$$

and $q(z)$ is the best dominant.

Proof. Define the analytic function p in U by

$$p(z) = \frac{z^{\gamma-1}SR^{m,n+1}f(z)}{(SR^{m,n}f(z))^{\gamma}}. \tag{19}$$

Logarithmic differentiation of (19) yields

$$p'(z) = \frac{z^{\gamma-2}(\gamma-1)SR^{m,n+1}f(z) + z^{\gamma-1}\left(SR^{m,n+1}f(z)\right)'}{(SR^{m,n}f(z))^{\gamma}} - \frac{\gamma z^{\gamma-1}SR^{m,+1n}f(z)(SR^{m,n}f(z))'}{(SR^{m,n}f(z))^{\gamma}SR^{m,n}f(z)}.$$

So,

$$\frac{zp'(z)}{p(z)} = (\gamma-1) + \frac{z\left(SR^{m,n+1}f(z)\right)'}{SR^{m,n+1}f(z)} - \gamma\frac{z(SR^{m,n}f(z))'}{SR^{m,n}f(z)}. \tag{20}$$

Using (2) in (20) we get

$$\frac{zp'(z)}{p(z)} = \gamma - 1 + (n+2)\frac{SR^{m,n+2}f(z)}{SR^{m,n+1}f(z)} - (n+1) - \gamma\left[(n+1)p(z) - n\right],$$

which is equivalent to

$$\frac{SR^{m,n+2}f(z)}{SR^{m,n+1}f(z)} - \frac{\gamma(n+1)}{n+2}\frac{SR^{m,n+1}f(z)}{SR^{m,n}f(z)} = \frac{1}{n+2}\frac{zp'(z)}{p(z)} + \frac{\gamma n + n - \gamma + 2}{n+2}.$$

By hypothesis (17), we obtain $\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)}$. From Lemma 3 we obtain

$$\frac{z^{\gamma-1}SR^{m,n+1}f(z)}{(SR^{m,n}f(z))^{\gamma}} \prec q(z)$$

and $q(z)$ is the best dominant. □

References

[1] Andrei, L. *On some differential sandwich theorems using a generalized Sălăgean operator and Ruscheweyh operator*, Journal of Computational Analysis and Applications **18** (2015), (to appear).

[2] Andrei, L., Alb Lupaş, A., *Certain integral operators of analytic functions*, submitted 2014.

[3] Breaz, D., Breaz, N., *Two integral operators*, Studia Universitatis Babes-Bolyai **3** (2002), Cluj-Napoca, 13–21.

[4] Breaz, D., Owa, S., Breaz, N., *A new integral univalent operator*, Acta Univ. Apul. **16** (2008), 11–16.

[5] Miller, S.S., Mocanu, P.T., *Differential Subordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, No. 225, Marcel Dekker, New York and Basel, 2000.

[6] Ravichandran, V., *Certain applications of first order differential subordination*, Far East J. Math. Sci. **12** (2004), 41–51.

[7] Ravichandran, V., Darus, M., Hussain Khan, M., Subramanian, K.G., *Differential subordination associated with linear operators defined for multivalent functions*, Acta Mathematica Vietnamica **30** (2) (2005), 113–121.

[8] Ruscheweyh, St., *New criteria for univalent functions*, Proc. Amet. Math. Soc. **49** (1975), 109–115.

[9] Sălăgean, G.St., *Subclasses of univalent functions*, Lecture Notes in Math. **1013** (1983), Springer Verlag, Berlin, 362–372.

University of Pitești
Department of Mathematics
1 Târgul din Vale Street, 110040, Pitești, România
E-mail address: radudyaconu@yahoo.com
E-mail address: radu.diaconu@uamsibiu.ro