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INTEGRABLE SOLUTIONS FOR IMPLICIT FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper we study the existence of integrable solutions for initial value problem for fractional order implicit differential equations. Our results are based on Schauder's fixed point theorem and the Banach contraction principle fixed point theorem.

1. Introduction

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [5, 13, 16, 17, 18, 20]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [3, 4], Kilbas *et al.* [14], Lakshmikantham *et al.* [15], and the papers by Agarwal *et al* [1, 2], Belarbi *et al.* [6], Benchohra *et al.* [7], and the references therein.

Our our knowledge, the literature on integral solutions for fractional differential equations is very limited. El-Sayed and Hashem [12] studies the existence of integral and continuous solutions for quadratic integral equations. El-Sayed and Abd El Salam considered L^p-solutions for a weighted Cauchy problem for differential equations involving the Riemann-Liouville fractional derivative.

Motivated by the above papers, in this paper we deal with the existence of solutions for initial value problem (IVP for short), for fractional order implicit differential equation

$$^cD^\alpha y(t) = f(t, y(t), ^cD^\alpha y(t)),\ t \in J := [0, T] \tag{1}$$

$$y(0) = y_0, \tag{2}$$

where $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, $y_0 \in \mathbb{R}$, and $^cD^\alpha$ is the Caputo fractional derivative.

This paper is organized as follows. In Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following section. In Section 3, we give two results, the first one is based on Schauder's fixed point theorem (Theorem 3) and the second one on the Banach contraction principle (Theorem 4). An example is given in Section 4 to demonstrate the application of our main results. These results can be considered as a contribution to this emerging field.

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2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Let $L^1(J)$ denotes the class of Lebesgue integrable functions on the interval $J=[0,T]$, with the norm $\|u\|_{L_1}=\int_J |u(t)|dt$.

Definition 1. *([14, 19]). The fractional (arbitrary) order integral of the function $h\in L^1([a,b],\mathbb{R}_+)$ of order $\alpha\in\mathbb{R}_+$ is defined by*

$$I_a^\alpha h(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)ds,$$

where $\Gamma(.)$ is the gamma function. When $a=0$, we write $I^\alpha h(t)=h(t)*\varphi_\alpha(t)$, where $\varphi_\alpha(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t>0$, and $\varphi_\alpha(t)=0$ for $t\leq 0$, and $\varphi_\alpha\to\delta(t)$ as $\alpha\to 0$, where δ is the delta function.

Definition 2. *([14, 19]). The Riemann-Liouville fractional derivative of order $\alpha>0$ of function $h\in L^1([a,b],\mathbb{R}_+)$, is given by*

$$(D_{a+}^\alpha h)(t)=\frac{1}{\Gamma(n-\alpha)}\Big(\frac{d}{dt}\Big)^n\int_a^t (t-s)^{n-\alpha-1}h(s)ds,$$

Here $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of α. If $\alpha\in(0,1]$, then

$$(D_{a+}^\alpha h)(t)=\frac{d}{dt}I_{a+}^{1-\alpha}h(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{ds}\int_a^t (t-s)^{-\alpha}h(s)ds.$$

Definition 3. *([14]). The Caputo fractional derivative of order $\alpha>0$ of function $h\in L^1([a,b],\mathbb{R}_+)$ is given by*

$$({}^cD_{a+}^\alpha h)(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)ds,$$

where $n=[\alpha]+1$. If $\alpha\in(0,1]$, then

$$({}^cD_{a+}^\alpha h)(t)=I_{a+}^{1-\alpha}\frac{d}{dt}h(t)=\int_a^t\frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\frac{d}{ds}h(s)ds.$$

The following properties are some of the main ones of the fractional derivatives and integrals.

Proposition 1. [14] *Let $\alpha,\beta>0$. Then we have*

(i) $I^\alpha:\ L^1(J,\mathbb{R}_+)\to L^1(J,\mathbb{R}_+)$, and if $f\in L^1(J,\mathbb{R}_+)$, then

$$I^\alpha I^\beta f(t)=I^\beta I^\alpha f(t)=I^{\alpha+\beta}f(t).$$

(ii) If $f\in L^p(J,\mathbb{R}_+)$, $1\leq p\leq+\infty$, then $\|I^\alpha f\|_{L_p}\leq\frac{T^\alpha}{\Gamma(\alpha+1)}\|f\|_{L_p}$

(iii) The fractional integration operator I^α is linear

(v) The fractional order integral operator I^α maps L^1 into itself continuously.

The following theorems will be needed.

Theorem 1. *(Schauder fixed point theorem [10]) Let E a Banach space and Q be a convex subset of E and $F:Q\longrightarrow Q$ is compact, and continuous map. Then F has at least one fixed point in Q.*

Theorem 2. *(Kolmogorov compactness criterion [10]) Let $\Omega\subseteq L^p(J,\mathbb{R})$, $1\leq p\leq\infty$. If*

(i) Ω is bounded in $L^p(J,\mathbb{R})$, and

(ii) $u_h\longrightarrow u$ as $h\longrightarrow 0$ uniformly with respect to $u\in\Omega$,

then Ω is relatively compact in $L^p(J,\mathbb{R})$, where

$$u_h(t) = \frac{1}{h}\int_t^{t+h} u(s)ds.$$

3. Existence of solutions

Let us start by defining what we mean by an integrable solution of the problem (1)–(2).

Definition 4. *A function $y \in L^1(J,\mathbb{R})$ is said to be a solution of IVP (1)–(2) if y satisfies (1) and (2).*

For the existence of solutions for the problem (1)–(2), we need the following auxiliary lemma.

Lemma 1. *The solution of the IVP (1)–(2) can be expressed by the integral equation*

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds, \tag{3}$$

where x is the solution of the functional integral equation

$$x(t) = f\left(t, y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds, x(t)\right). \tag{4}$$

Proof. Let ${}^cD^\alpha y(t) = x(t))$ in equation (1), then

$$x(t) = f(t, y(t), x(t)) \tag{5}$$

and

$$\begin{aligned} y(t) &= y(0) + I^\alpha x(t)) \\ &= y(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds. \end{aligned} \tag{6}$$

□

Let us introduce the following assumptions:

- **(H1)** $f : J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is measurable in $t \in J$, for any $(u_1, u_2) \in \mathbb{R}^2$ and continuous in $(u_1, u_2) \in \mathbb{R}^2$, for almost all $t \in J$.
- **(H2)** There exist a positive function $a \in L^1(J,\mathbb{R})$ and constants, $b_i > 0; i = 1, 2$ such that:

$$|f(t, u_1, u_2)| \leq |a(t)| + b_1|u_1| + b_2|u_2|, \forall (t, u_1, u_2) \in J \times \mathbb{R}^2.$$

Our first result is based on Schauder fixed point theorem.

Theorem 3. *Assume that the assumptions $(H1) - (H2)$ are satisfied. If*

$$\frac{b_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{b_2 T^\alpha}{\Gamma(\alpha+1)} < 1, \tag{7}$$

then the IVP (1)–(2) has at least one solution $y \in L^1(J,\mathbb{R})$.

Proof. Transform the problem (1)–(2) into a fixed point problem. Consider the operator

$$H : L^1(J,\mathbb{R}) \longrightarrow L^1(J,\mathbb{R})$$

defined by:

$$(Hx)(t) = y_0 + I^\alpha x(t), \tag{8}$$

where

$$x(t) = f(t, y_0 + I^\alpha x(t), x(t)).$$

The operator H is well defined, indeed, for each $x \in L^1(J,\mathbb{R})$, from assumptions **(H1)** and **(H2)**, we obtain

$$\begin{aligned}
\|Hx\|_{L_1} &= \int_0^T |Hx(t)|dt \\
&= \int_0^T |y_0 + I^\alpha x(t))|dt \\
&\leq T|y_0| + \int_0^T \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|x(s))|ds\Big)dt \\
&\leq T|y_0| + \int_0^T \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y_0+I^\alpha x(s),x(s))|ds\Big)dt \\
&\leq T|y_0| + \int_0^T \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|a(s)+b_1(y_0+I^\alpha x(s))+b_2(x(s)|ds\Big)dt \\
&\leq T|y_0| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|a\|_{L_1} + \frac{b_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{b_2T^\alpha}{\Gamma(\alpha+1)}\|x\|_{L_1} \\
&\quad +b_1\int_0^T \Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}I^\alpha|x(s)|ds\Big)dt \\
&\leq T|y_0| + \frac{T^\alpha}{\Gamma(\alpha+1)}\|a\|_{L_1} + \frac{b_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)} + \frac{b_2T^\alpha}{\Gamma(\alpha+1)}\|x\|_{L_1} \\
&\quad +\frac{b_1T^{2\alpha}}{\Gamma(2\alpha+1)}\|x\|_{L_1} < +\infty. \qquad (9)
\end{aligned}$$

Let

$$r = \frac{T|y_0| + \left(\frac{T^\alpha\|a\|_{L_1}+b_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)}\right)}{1-\left(\frac{b_1T^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{b_2T^\alpha}{\Gamma(\alpha+1)}\right)},$$

and consider the set

$$B_r = \{x \in L^1(J,\mathbb{R}) : \|x\|_{L_1} \leq r.\}.$$

Clearly B_r is nonempty, bounded, convex and closed.

Now, we will show that $HB_r \subset B_r$, indeed, for each $x \in B_r$, from (7) and (9) we get

$$\begin{aligned}
\|Hx\|_{L_1} &\leq T|y_0| + \left(\frac{T^\alpha\|a\|_{L_1}+b_1|y_0|T^{\alpha+1}}{\Gamma(\alpha+1)}\right) \\
&+ \left(\frac{b_1T^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{b_2T^\alpha}{\Gamma(\alpha+1)}\right)\|x\|_{L_1} \\
&\leq r.
\end{aligned}$$

Then $HB_r \subset B_r$. Assumption **(H1)** implies that H is continuous. Now, we will show that H is compact, this is HB_r is relatively compact. Clearly HB_r is bounded in $L^1(J,\mathbb{R})$, i.e condition **(i)** of Kolmogorov compactness criterion is satisfied. It remains to show $(Hx)_h \longrightarrow (Hx)$ in $L^1(J,\mathbb{R})$ for each $x \in B_r$.

Let $x \in B_r$, then we have

$$\begin{aligned}
& \|(Hx)_h - (Hx)\|_{L^1} \\
= & \int_0^T |(Hx)_h(t) - (Hx)(t)| dt \\
= & \int_0^T \left| \frac{1}{h} \int_t^{t+h} (Hx)(s) ds - (Hx)(t) \right| dt \\
\leq & \int_0^T \left(\frac{1}{h} \int_t^{t+h} |(Hx)(s) - (Hx)(t)| ds \right) dt \\
\leq & \int_0^T \left(\frac{1}{h} \int_t^{t+h} |I^\alpha x(s) - I^\alpha x(t)| ds \right) dt \\
\leq & \int_0^T \frac{1}{h} \int_t^{t+h} |I^\alpha f(s, y_0 + I^\alpha x(s), x(s)) - I^\alpha f(t, y_0 + I^\alpha x(t), x(t))| ds dt.
\end{aligned}$$

Since $x \in B_r \subset L^1(J,\mathbb{R})$ and assumption **(H2)** that implies $f \in L^1(J,\mathbb{R})$ and by Proposition 1 (v), it follows that $I^\alpha f \in L^1(J,\mathbb{R})$, then we have

$$\frac{1}{h} \int_t^{t+h} |I^\alpha f(s, y_0 + I^\alpha x(s), x(s)) - I^\alpha f(t, y_0 + I^\alpha x(t), x(t)| ds \longrightarrow 0 \ as \ h \longrightarrow 0, \ t \in J.$$

Hence

$$(Hx)_h \longrightarrow (Hx) \ \text{ uniformly as } \ h \longrightarrow 0.$$

Then by Kolmogorov compactness criterion, HB_r is relatively compact. As a consequence of Schauder's fixed point theorem the IVP (1)–(2) has at least one solution in B_r. □

The following result is based on the Banach contraction principle.

Theorem 4. *Assume that **(H1)** and the following condition hold.*

(H3) *There exist constants $k_1, k_2 > 0$ such that*

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq k_1|x_1 - x_2| + k_2|y_1 - y_2|, \ t \in J, \ x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

If

$$\frac{k_1 T^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{k_2 T^\alpha}{\Gamma(\alpha+1)} < 1, \tag{10}$$

then the IVP (1)–(2) *has a unique solution $y \in L^1(J,\mathbb{R})$.*

Proof. We shall use the Banach contraction principle to prove that H defined by (8) has a fixed point. Let $x, y \in L^1(J,\mathbb{R})$, and $t \in J$. Then we have,

$$\begin{aligned}
|(Hx)(t) - (Hy)(t)| &= |I^\alpha [f(t, y_0 + I^\alpha x(t), x(t)) - f(t, y_0 + I^\alpha y(t), y(t))]| \\
&\leq k_1 I^{2\alpha}|x(t) - y(t)| + k_2 I^\alpha |x(t) - y(t)| \\
&\leq \frac{k_1}{\Gamma(2\alpha)} \int_0^t (t-s)^{2\alpha-1}|x(s) - y(s)| ds \\
&\quad + \frac{k_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|x(s) - y(s)| ds.
\end{aligned}$$

Thus

$$\begin{aligned}\|(Hx)-(Hy)\|_{L_1} &\leq \frac{k_1T^{2\alpha}}{\Gamma(2\alpha+1)}\|x-y\|_{L_1}+\frac{k_2T^{\alpha}}{\Gamma(\alpha+1)}\|x-y\|_{L_1}\\ &\leq \left(\frac{k_1T^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{k_2T^{\alpha}}{\Gamma(\alpha+1)}\right)\|x-y\|_{L_1}.\end{aligned}$$

Consequently by (10) H is a contraction. As a consequence of the Banach contraction principle, we deduce that H has a fixed point which is a solution of the problem (1)–(2). □

4. Example

Let us consider the following fractional initial value problem,

$$^cD^{\alpha}y(t)=\frac{e^{-t}}{(e^t+8)(1+|y(t)|+|^cD^{\alpha}y(t)|)},\ t\in J:=[0,1],\ \alpha\in(0,1], \tag{11}$$

$$y(0)=1. \tag{12}$$

Set

$$f(t,y,z)=\frac{e^{-t}}{(e^t+8)(1+y+z)},\ (t,y,z)\in J\times[0,+\infty)\times[0,+\infty).$$

Let $y,z\in[0,+\infty)$ and $t\in J$. Then we have

$$\begin{aligned}|f(t,y_1,z_1)-f(t,y_2,z_2)| &= \left|\frac{e^{-t}}{e^t+8}\left(\frac{1}{1+y_1+z_1}-\frac{1}{1+y_2+z_2}\right)\right|\\ &\leq \frac{e^{-t}(|y_1-y_2|+|z_1-z_2|)}{(e^t+8)(1+y_1+z_1)(1+y_2+z_2)}\\ &\leq \frac{e^{-t}}{(e^t+8)}(|y_1-y_2|+|z_1-z_2|)\\ &\leq \frac{1}{9}|y_1-y_2|+\frac{1}{9}|z_1-z_2|.\end{aligned}$$

Hence the condition **(H3)** holds with $k_1=k_2=\frac{1}{9}$. We shall check that condition (10) is satisfied with $T=1$. Indeed

$$\frac{k_1T^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{k_2T^{\alpha}}{\Gamma(\alpha+1)}=\frac{1}{9\Gamma(2\alpha+1)}+\frac{1}{9\Gamma(\alpha+1)}<1. \tag{13}$$

Then by Theorem 4, the problem (11)–(12) has a unique integrable solution on $[0,1]$.

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