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HAHN'S PROBLEM WITH RESPECT TO A THIRD-ORDER DIFFERENTIAL OPERATOR

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ABSTRACT. In the present work, we are interested to the lowering operator $\mathscr{O}_{c;1,3,2}$, given by a Linear combination of three successive Laguerre derivatives

$$\mathscr{O}_{c;1,3,2} := \mathcal{L}_{2,c} + 3\mathcal{L}_{1,c} + 2\mathcal{L}_{0,c},$$

where $\mathcal{L}_{i,c} := D(x-c)D\dots(x-c)D,\ 0 \le i \le 2$, (i.e. containing $(i+1)$ ordinary derivatives with respect to the x variable), and c is an arbitrary complex number. Then, we establish an intertwining relation between the operators $\mathscr{O}_{c;1,3,2}$ and the standard derivative D. Besides, an analogue to the Hahn problem for the operator $\mathscr{O}_{c;1,3,2}$ is studied. As a consequence, some integral relations between the corresponding polynomials are deduced. Finally, some expansions in series of Laguerre polynomials are presented.

1. INTRODUCTION

In [2], the authors have exhaustively described the $\hat{\Omega}_{1,c}$-classical orthogonal polynomials where $\hat{\Omega}_{1,c}$ is the second-order linear differential operator $\hat{\Omega}_{1,c} = \hat{\Omega}_1 - cD^2$, where c is an arbitrary complex number and $\hat{\Omega}_1$ is the generalized Laguerre operator introduced by Dattoli and Ricci. More precisely, the $\hat{\Omega}_{1,c}$-classical polynomial sequences are, after suitable shifting, the Laguerre polynomial sequence $\{L_n^{(1)}\}_{n\ge 0}$, when $c = 0$, or the Jacobi polynomial sequence $\{P_n^{(\alpha-2,1)}\}_{n\ge 0}$ with parameter $\alpha \ne -n+2$, $n \ge 1$, when $c = 1$. The two following relations are given

$$L_n^{(1)}(x) = \frac{\hat{\Omega}_1\ L_{n+1}^{(1)}(x)}{(n+1)(n+2)},\ n \ge 0,$$

$$P_n^{(\alpha,1)}(x) = \frac{\hat{\Omega}_{1,1}\ P_{n+1}^{(\alpha-2,1)}(x)}{(n+1)(n+2)},\ n \ge 0.$$

As a consequence of these results, the authors have deduced some integral relations between the corresponding polynomials and they have studied some expansions in series of Laguerre polynomials with parameter $\alpha = 1$.

The aim of this paper is to describe the $\mathscr{O}_{c;1,3,2}$-classical orthogonal polynomials where $\mathscr{O}_{c;1,3,2}$ is the third-order linear differential operator

$$\mathscr{O}_{c;1,3,2} := \mathcal{L}_{2,c} + 3\mathcal{L}_{1,c} + 2\mathcal{L}_{0,c},\ c \in \mathbb{C},$$

where $\mathcal{L}_{i,c} := D(x-c)D\dots(x-c)D$, $0 \le i \le 2$, (i.e. containing $(i+1)$ ordinary derivatives with respect to the x variable). So, we give some new results.

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Let us recall that sequence of Laguerre monic polynomials, (with parameter $\alpha \neq -n$, $n \geq 1$), is orthogonal with respect to the weight function $x^{\alpha}e^{-x}$ on the interval $(0, +\infty)$, and given by the following explicit Taylor expansion formula [15]:

$$L_n^{(\alpha)}(x) = \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{n}{\nu} \frac{\Gamma(n+\alpha+1)}{\Gamma(\nu+\alpha+1)}\, x^{\nu}. \tag{1}$$

Taking the first, second and third derivative of (1), with n replaced by $n+1$ and $\alpha = 2$, and by multiplying by x^i, $0 \leq i \leq 2$, respectively, we obtain

$$\begin{aligned} L_{n+1}^{(2)'}(x) &= \sum_{\nu=0}^{n} (-1)^{n-\nu} \frac{(n+1)!}{\nu!(n-\nu)!} \frac{(n+3)!}{(\nu+3)!}\, x^{\nu}, \\ xL_{n+1}^{(2)''}(x) &= \sum_{\nu=1}^{n} (-1)^{n-\nu} \frac{(n+1)!}{(\nu-1)!(n-\nu)!} \frac{(n+3)!}{(\nu+3)!}\, x^{\nu}, \\ x^2 L_{n+1}^{(2)^{(3)}}(x) &= \sum_{\nu=2}^{n} (-1)^{n-\nu} \frac{(n+1)!}{(\nu-2)!(n-\nu)!} \frac{(n+3)!}{(\nu+3)!}\, x^{\nu}. \end{aligned}$$

Combination of three last equations yields

$$\Big(x^2D^3 + 6xD^2 + 6D\Big) L_{n+1}^{(2)}(x) = (n+1)(n+2)(n+3) L_n^{(1)}(x), \quad n \geq 0.$$

Equivalently

$$L_n^{(1)}(x) = \frac{\mathscr{O}_{1,3,2}\; L_{n+1}^{(2)}(x)}{(n+1)(n+2)(n+3)},\ n \geq 0, \tag{2}$$

where $\mathscr{O}_{1,3,2} := DxDxD + 3DxD + 2D$. Notice that the operator $DxDx...DxD$, is the generalized Laguerre derivative. For more details about this operator see [8].

In view of (2), we can say that $\{L_n^{(2)}\}_{n\geq 0}$ is an $\mathscr{O}_{1,3,2}$-classical polynomial sequence, since it fulfills Hahn property relatively to the lowering operator $\mathscr{O}_{1,3,2}$, i.e., it is an orthogonal polynomial sequence whose sequence of $\mathscr{O}_{1,3,2}$-derivatives is also orthogonal, [10, 11]. Hahn's property can be considered for other differential operators (see, for example, [1, 16, 17]).

For a given $c \in \mathbb{C}$, let consider $\mathscr{O}_{c;1,3,2} : \mathbb{P} \to \mathbb{P}$, the third-order linear differential operator defined, in the linear space $\mathbb{P}$ of polynomials with complex coefficients, by

$$\mathscr{O}_{c;1,3,2} := \mathcal{L}_{2,c} + 3\mathcal{L}_{1,c} + 2\mathcal{L}_{0,c},$$

where $\mathcal{L}_{i,c} := D(x-c)D\ldots(x-c)D,\ 0 \leq i \leq 2$, (i.e. containing $(i+1)$ ordinary derivatives with respect to the x variable).

Obviously, $\mathscr{O}_{0;1,3,2} := \mathscr{O}_{1,3,2} = DxDxD + 3DxD + 2D$.

Notice that the above operator reduce by exactly one the degree of any polynomial (*lowering operator*). As examples of lowering operators considered are the q-derivative H_q (see [13]); the Hahn's operator D_w of finite differences [1]; the Dunkl operator $\mathcal{D} := D + \theta H_{-1}$ for $\theta \in \mathbb{C}\backslash\{0\}$ introduced by Dunkl [9].

In this contribution we will analyze the $\mathscr{O}_{c;1,3,2}$-classical polynomial sequence and then we will provide a full description of them. We will emphasize two basic facts.

(i) The $\mathscr{O}_{c;1,3,2}$-classical character is preserved up to a linear change of variable (by shifting).

(ii) The $\mathscr{O}_{c;1,3,2}$-classical polynomial sequences constitute a subfamily of the well-known D-classical ones (Hermite, Laguerre, Bessel and Jacobi). More precisely, the $\mathscr{O}_{c;1,3,2}$-classical polynomial sequences are, after suitable shifting, the Laguerre polynomial sequence $\{L_n^{(2)}\}_{n\geq 0}$, when $c=0$, or the Jacobi polynomial sequence $\{P_n^{(\alpha-3,2)}\}_{n\geq 0}$ with parameter $\alpha \neq -n+3$, $n\geq 1$, when $c=1$.

The structure of the manuscript is as follows. The second section deals with the basic definitions and the theoretical background we will need in the sequel. In the third section, we exhaustively describe the $\mathscr{O}_{c;1,3,2}$-classical sequences. As a consequence, we establish a new integral relation between the monic Bessel and Jacobi polynomials:

$$B_n^{(\frac{\alpha+1}{2})}(x) = \frac{1}{(n+2)!}\int_0^{+\infty} t^2 e^{-t} P_n^{(\alpha-3,2)}\big(tx+1\big)\,\mathrm{d}t,\; n\geq 0, \quad (\alpha\neq -n+3,\; n\geq 1).$$

In the last section, as an application of the results obtained, we give some expansions in series of Laguerre polynomials with parameter $\alpha=2$.

2. Main results

2.1. **Basic definitions.** Let $\mathbb{P}$ be the linear space of polynomials in one variable with complex coefficients and $\mathbb{P}'$ its algebraic dual space. We denote by $\langle u,p\rangle$ the action of $u\in\mathbb{P}'$ on $p\in\mathbb{P}$ and by $(u)_n := \langle u, x^n\rangle$, $n\geq 0$, the sequence of moments of u with respect to the polynomial sequence $\{x^n\}_{n\geq 0}$. Let us define the following operations in $\mathbb{P}'$. For linear functionals u and v, any polynomial q, and any $(a,b,c)\in\mathbb{C}\backslash\{0\}\times\mathbb{C}^2$, let $Du=u'$, qu, $(x-c)^{-1}u$, $\tau_{-b}u$ and $h_a u$ be the linear functionals defined by duality [19],

$$\begin{aligned}
\langle u',p\rangle &:= -\langle u,p'\rangle, \quad \langle qu,p\rangle := \langle u,qp\rangle,\\
\langle (x-c)^{-1}u,\; p\rangle &:= \langle u,\; \theta_c p\rangle = \Big\langle u,\; \frac{p(x)-p(c)}{x-c}\Big\rangle,\\
\langle \tau_{-b}u,\; p\rangle &:= \langle u,\; \tau_b p\rangle = \langle u,\; p(x-b)\rangle,\\
\langle h_a u,\; p\rangle &:= \langle u,\; h_a p\rangle = \langle u,\; p(ax)\rangle, \quad p\in\mathbb{P}.
\end{aligned}$$

Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials (MPS) with $\deg P_n = n$ and let $\{u_n\}_{n\geq 0}$ be its dual sequence, $u_n\in\mathbb{P}'$, defined by $\langle u_n, P_m\rangle = \delta_{n,m}$, $n,\ m\geq 0$. Notice that u_0 is said to be the canonical functional associated with the MPS $\{P_n\}_{n\geq 0}$.
Recall that any $u\in\mathbb{P}'$ can be represented as $u=\sum_{n=0}^{+\infty}\langle u,P_n\rangle u_n$. So, if $\{u_n^{[1]}\}_{n\geq 0}$ denotes the dual sequence of the MPS $\{P_n^{[1]}\}_{n\geq 0}$ where $P_n^{[1]}(x) := (n+1)^{-1}P'_{n+1}(x)$, $n\geq 0$, then $Du_n^{[1]} = -(n+1)u_{n+1}$, $n\geq 0$. Likewise, the dual sequence $\{\tilde{u}_n\}_{n\geq 0}$ of the shifted MPS $\{\widetilde{P}_n\}_{n\geq 0}$, where $\widetilde{P}_n(x) := a^{-n}P_n(ax+b)$ with $(a,b)\in\mathbb{C}\backslash\{0\}\times\mathbb{C}$, is given by $\tilde{u}_n = a^n(h_{a^{-1}}\circ\tau_{-b})u_n$, $n\geq 0$.

An MPS $\{P_n\}_{n\geq 0}$ is said to be orthogonal (MOPS) with respect to $u\in\mathbb{P}'$ if $\langle u, P_nP_m\rangle = 0$, $n\neq m$, and $\langle u, P_n^2\rangle\neq 0$, $n\geq 0$. In this case, u is said to be quasi-definite (regular) [6]. Notice that $u=(u)_0 u_0$, with $(u)_0\neq 0$. For any quasi-definite linear functional u and any polynomial φ such that $\varphi u = 0$, then $\varphi = 0$ [18, 20].

Proposition 1. [19]. *An MPS $\{P_n\}_{n\geq 0}$ with dual sequence $\{u_n\}_{n\geq 0}$, is orthogonal with respect to u_0 if and only if one of the following statements hold.*

(i) $u_n = \langle u_0, P_n^2\rangle^{-1}P_n u_0$, $n\geq 0$.
(ii) $\{P_n\}_{n\geq 0}$ *satisfies a Three-Term Recurrence Relation*

$$\text{(TTRR)}\quad \begin{cases} P_0(x)=1, \quad P_1(x)=x-\beta_0,\\ P_{n+2}(x) = (x-\beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x),\; n\geq 0, \end{cases} \tag{3}$$

where $\beta_n = \langle u_0, xP_n^2\rangle/\langle u_0, P_n^2\rangle\in\mathbb{C}$ and $\gamma_{n+1} = \langle u_0, P_{n+1}^2\rangle/\langle u_0,P_n^2\rangle\in\mathbb{C}\backslash\{0\}$.

When $\{P_n\}_{n\geq 0}$ is an MOPS with respect to u_0, then $\{\widetilde{P}_n\}_{n\geq 0}$ is also orthogonal with respect to $\tilde{u}_0 = (h_{a^{-1}} \circ \tau_{-b})u_0$ and satisfies [6]

$$\text{(TTRR)} \quad \begin{cases} \widetilde{P}_0(x) = 1, \quad \widetilde{P}_1(x) = x - \tilde{\beta}_0, \\ \widetilde{P}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\widetilde{P}_{n+1}(x) - \tilde{\gamma}_{n+1}\widetilde{P}_n(x), \ n \geq 0, \end{cases}$$

where $\tilde{\beta}_n = a^{-1}(\beta_n - b)$ and $\tilde{\gamma}_{n+1} = a^{-2}\gamma_{n+1}$.

A linear functional u is said to be D-classical when it is quasi-definite and there exist two polynomials Φ and Ψ, Φ monic, $\deg \Phi = t \leq 2$, and $\deg \Psi = 1$, such that u satisfies a Pearson equation (see [5, 19, 20])

$$\text{(PE)} \quad (\Phi u)' + \Psi u = 0. \tag{4}$$

In such a case, the corresponding MOPS $\{P_n\}_{n\geq 0}$ is said to be D-classical.

The D-classical character of a linear functional is invariant under a shift (see [19]). Indeed, if u is a D-classical linear functional satisfying (4), then the shifted linear functional $\tilde{u} = (h_{a^{-1}} \circ \tau_{-b})u$ is also D-classical and satisfies

$$(\widetilde{\Phi}\tilde{u})' + \widetilde{\Psi}\tilde{u} = 0,$$

where $\widetilde{\Phi}(x) = a^{-t}\Phi(ax + b)$ and $\widetilde{\Psi}(x) = a^{1-t}\Psi(ax + b)$.

It is well-known that any D-classical polynomial sequence $\{P_n\}_{n\geq 0}$ can be characterized taking into account its orthogonality as well as a First Structure Relation (FSR), or a Second Structure Relation (SSR), or Second-Order Differential Equation (SODE) as follows.

$$\text{(FSR)} \qquad \Phi(x)P'_{n+1}(x) = r(x;n)P_{n+1}(x) + s_nP_n(x), \ n \geq 0, \tag{5}$$

$$\text{(SSR)} \qquad P_n(x) = P_n^{[1]}(x) + a_nP_{n-1}^{[1]}(x) + b_nP_{n-2}^{[1]}(x), \ n \geq 0, \tag{6}$$

$$\text{(SODE)} \quad \Phi(x)P''_{n+1}(x) - \Psi(x)P'_{n+1}(x) = \omega_nP_{n+1}(x), \ n \geq 0, \tag{7}$$

where for the fourth canonical situations, the explicit expression of the parameters involved in (3)–(7) are given in Table 1, (for more details, see [3, 5, 19, 20]).

Table 1. Some basic characteristics of classical orthogonal polynomials.

(C_1) Hermite: $P_n(x) = H_n(x)$, $n \geq 0$.

$\beta_n = 0$, $n \geq 0$, $\gamma_{n+1} = \frac{n+1}{2}$, $n \geq 0$,
$\Phi(x) = 1$, $\Psi(x) = 2x$.
$r(x;n) = 0$, $s_n = n + 1$, $n \geq 0$.
$a_n = b_n = 0$, $n \geq 0$.
$\omega_n = -2(n + 1)$, $n \geq 0$.

(C_2) Laguerre: $P_n(x) = L_n^{(\alpha)}(x)$, $n \geq 0$, $(\alpha \neq -n, \ n \geq 1)$.

$\beta_n = 2n + \alpha + 1$, $n \geq 0$, $\gamma_{n+1} = (n + 1)(n + \alpha + 1)$, $n \geq 0$.
$\Phi(x) = x$, $\Psi(x) = x - \alpha - 1$,
$r(x;n) = n + 1$, $s_n = \gamma_{n+1}$, $n \geq 0$.
$a_n = n$, $b_n = 0$, $n \geq 0$.
$\omega_n = -(n + 1)$, $n \geq 0$.

(C_3) Bessel $P_n(x) = B_n^{(\alpha)}(x)$, $n \geq 0$, $(\alpha \neq -\frac{n}{2}, \ n \geq 0)$.

$\beta_0 = -\frac{1}{\alpha}$, $\beta_n = \frac{1-\alpha}{(n+\alpha-1)(n+\alpha)}$, $n \geq 0$, $\gamma_n = -\frac{n(n+2\alpha-2)}{(2n+2\alpha-3)(n+\alpha-1)^2(2n+2\alpha-1)}$, $n \geq 1$.
$\Phi(x) = x^2$, $\Psi(x) = -2(\alpha x + 1)$.
$r(x;n) = (n + 1)(x - \frac{1}{n+\alpha})$, $s_n = -(2n + 2\alpha + 1)\gamma_{n+1}$, $n \geq 0$.
$a_n = \frac{n}{(n+\alpha-1)(n+\alpha)}$, $n \geq 1$, $a_0 = 0$, $b_n = \frac{(n-1)n}{(2n+2\alpha-3)(n+\alpha-1)^2(2n+2\alpha-1)}$, $n \geq 2$, $b_0 = b_1 = 0$.
$\omega_n = (n + 1)(n + 2\alpha)$, $n \geq 0$.

(C_4) Jacobi $P_n(x) = P_n^{(\alpha,\beta)}(x)$, $n \geq 0$, $(\alpha,\ \beta \neq -n,\ \alpha+\beta \neq -n-1,\ n \geq 1)$.
$\beta_0 = \frac{\alpha-\beta}{\alpha+\beta+2}$, $\beta_n = \frac{\alpha^2-\beta^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$, $\gamma_n = \frac{4n(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}$, $n \geq 1$.
$\Phi(x) = x^2 - 1$, $\Psi(x) = -(\alpha+\beta+2)x + \alpha - \beta$.
$r(x;n) = (n+1)(x - \frac{\alpha-\beta}{2n+\alpha+\beta+2})$, $s_n = -(2n+\alpha+\beta+3)\gamma_{n+1}$, $n \geq 0$.
$a_n = -\frac{2n(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$, $n \geq 1$, $a_0 = 0$, $b_n = -\frac{4(n-1)n(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}$, $n \geq 2$, $b_0 = b_1 = 0$.
$\omega_n = (n+1)(n+\alpha+\beta+2)$, $n \geq 0$.

Furthermore, we need the following properties of the monic Bessel and the monic Jacobi polynomials [6, 12]. For any integer $n \geq 0$, we have

$$B_n^{(\alpha)}(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{2^{n-\nu}\Gamma(n+2\alpha+\nu-1)}{\Gamma(2n+2\alpha-1)} x^{\nu}, \tag{8}$$

$$P_n^{(\alpha,\beta)}(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{2^{n-\nu}\Gamma(n+\alpha+\beta+\nu+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+1)\Gamma(\nu+\beta+1)} (x-1)^{\nu}, \tag{9}$$

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x), \quad \text{(symmetry)}. \tag{10}$$

2.2. **Some properties of the operator $\mathscr{O}_{c;1,3,2}$.** In the sequel, we will denote by "$\circ$" the composition law between linear operators on the linear space of polynomials. The following formulas are a straightforward consequence of the definition of the operators.

Lemma 1. *For any c in $\mathbb{C}$, we have*

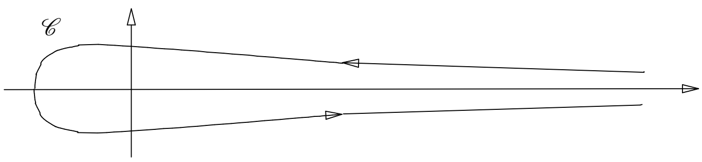
(i) $\mathscr{O}_{c;1,3,2}(x-c)^{n+1} = (n+1)(n+2)(n+3)(x-c)^n$, $n \geq 0$, *and* $\mathscr{O}_{c;1,3,2}(1) = 0$.
(ii) $\mathscr{O}_{c;1,3,2} \circ \Pi_{a,b} = a\Pi_{a,b} \circ \mathscr{O}_{ac+b;1,3,2}$ *and* $\Pi_{a,b} \circ \mathscr{O}_{c;1,3,2} = a^{-1}\mathscr{O}_{(c-b)/a;1,3,2} \circ \Pi_{a,b}$, *where* $\Pi_{a,b}f(x) = f(ax+b)$, *for every* $f \in \mathbb{P}$ *and where* $(a,b) \in \mathbb{C}\backslash\{0\} \times \mathbb{C}$.
(iii) $\mathscr{O}_{c;1,3,2}(pq) = q\mathscr{O}_{c;1,3,2}(p) + p\mathscr{O}_{c;1,3,2}(q) + 3(x-c)\big\{(x-c)(p''q' + p'q'') + 4p'q'\big\}$, *for every p and q in $\mathbb{P}$.*

Through an appropriate linear isomorphism called intertwining operator, we can establish a relationship between the operators $\mathscr{O}_{c;1,3,2}$ and the standard derivative D. We first recall the following formulas [12]:

$$n! = \int_0^{+\infty} t^n e^{-t}\, \mathrm{d}t, \quad n = 0, 1, 2, ..., \tag{11}$$

$$\frac{1}{n!} = \frac{-1}{2\pi i} \int_{\mathscr{C}} (-z)^{-n-1} e^{-z}\, \mathrm{d}z, \quad n = 0, 1, 2, ..., \tag{12}$$

where $\mathscr{C}$ is the following contour in the complex plane:



The following result is a straightforward consequence of Lemma 1 (i), (11) and (12).

Proposition 2. *For any $c \in \mathbb{C}$ and any $p \in \mathbb{P}$ with $\deg p = n \in \mathbb{N}$, the following relation holds*

$$\mathfrak{S}_c \circ \mathscr{O}_{c;1,3,2}(p) = (n+2)D \circ \mathfrak{S}_c(p),$$

where the operator $\mathfrak{S}_c : \mathbb{P} \to \mathbb{P}$ *and its reciprocal operator* $\mathfrak{S}_c^{-1}$ *are linear isomorphisms given by*

$$\begin{aligned} \mathfrak{S}_c(p)(x) &= \int_0^{+\infty} te^{-t} p\big(t(x-c)+c\big)\ \mathrm{d}t, \quad p \in \mathbb{P}, \\ \mathfrak{S}_c^{-1}(p)(x) &= \int_{\mathscr{C}} z^{-2} e^{-z} p\big(-z^{-1}(x-c)+c\big)\ \mathrm{d}z, \quad p \in \mathbb{P}, \end{aligned}$$

and $\mathscr{C}$ *is the same contour as above.*

Notice that the operator $\mathfrak{S}_c$ can be characterized taking into account its linearity as well as the fact

$$\mathfrak{S}_c\big((x-c)^n\big) = (n+1)!(x-c)^n, \quad n \geq 0. \tag{13}$$

By transposition of the operator $\mathscr{O}_{c;1,3,2}$, we get

$${}^t\mathscr{O}_{c;1,3,2} = -\mathcal{L}_{2,c} + 3\mathcal{L}_{1,c} - 2\mathcal{L}_{0,c}.$$

So, $\mathscr{O}_{c;-1,3,-2} : \mathbb{P}' \to \mathbb{P}'$ is the operator defined by

$$\mathscr{O}_{c;-1,3,-2} = -\mathcal{L}_{2,c} + 3\mathcal{L}_{1,c} - 2\mathcal{L}_{0,c}. \tag{14}$$

For any MPS $\{P_n\}_{n\geq 0}$ we define

$$Q_n(x;c) := \frac{\mathscr{O}_{c;1,3,2}\ P_{n+1}(x)}{(n+1)(n+2)(n+3)}, \quad n \geq 0. \tag{15}$$

Clearly, $\{Q_n(\ ;c)\}_{n\geq 0}$ is an MPS, $\deg Q_n(\ ;c) = n$. If $\{v_n(c)\}_{n\geq 0}$ denotes the dual sequence of $\{Q_n(\ ;c)\}_{n\geq 0}$, then we have

$$\mathscr{O}_{c;-1,3,-2}(v_n(c)) = (n+1)(n+2)(n+3)u_{n+1}, \quad n \geq 0. \tag{16}$$

3. The $\mathscr{O}_{c;1,3,2}$-classical orthogonal polynomials

Definition 1. *An MOPS* $\{P_n\}_{n\geq 0}$ *(orthogonal with respect to* u_0*) is said to be* $\mathscr{O}_{c;1,3,2}$*-classical, when it satisfies the Hahn's property with respect to the operator* $\mathscr{O}_{c;1,3,2}$*, i.e., the MPS* $\{Q_n(\ ;c)\}_{n\geq 0}$ *given by* (15) *is also orthogonal. In this case* u_0 *is also said to be an* $\mathscr{O}_{c;1,3,2}$*-classical linear functional.*

Any shift leaves invariant the $\mathscr{O}_{c;1,3,2}$-classical character.

Lemma 2. *When* $\{P_n\}_{n\geq 0}$ *is* $\mathscr{O}_{c;1,3,2}$*-classical, then for any* $(a,b) \in \mathbb{C}\backslash\{0\}\times\mathbb{C}$ *the shifted polynomial sequence* $\{\widetilde{P}_n\}_{n\geq 0}$ *given by* $\widetilde{P}_n(x) = a^{-n}P_n(ax+b)$*,* $n \geq 0$*, is* $\mathscr{O}_{\widetilde{c};1,3,2}$*−classical, where* $\widetilde{c} = a^{-1}(c-b)$*.*

Proof. Assume that $\{P_n\}_{n\geq 0}$ is $\mathscr{O}_{c;1,3,2}$-classical. By Definition 1, the MPS $\{Q_n(\ ;c)\}_{n\geq 0}$ given by (15), i.e.,

$$(n+1)(n+2)(n+3)Q_n(x;c) = (x-c)^2 P^{(3)}_{n+1}(x) + 6(x-c)P''_{n+1}(x) + 6P'_{n+1}(x),\ n \geq 0, \tag{17}$$

is orthogonal.
For any fixed $(a,b) \in \mathbb{C}\backslash\{0\}\times\mathbb{C}$, let $\{\widetilde{P}_n\}_{n\geq 0}$ and $\{\widetilde{Q}_n\}_{n\geq 0}$ be the shifted MOPS given by $\widetilde{P}_n(x) = a^{-n}P_n(ax+b)$ and $\widetilde{Q}_n(x) = a^{-n}Q_n(ax+b;c)$. Replacing x by $ax+b$ in (17), we get

$$(n+1)(n+2)(n+3)\widetilde{Q}_n(x) = (x-\widetilde{c})^2 \widetilde{P}^{(3)}_{n+1}(x) + 6(x-\widetilde{c})\widetilde{P}''_{n+1}(x) + 6\widetilde{P}'_{n+1}(x),$$

where $\widetilde{c} = a^{-1}(c-b)$ or, equivalently, $(n+1)(n+2)(n+3)\widetilde{Q}_n(x) = \mathscr{O}_{\widetilde{c};1,3,2}\widetilde{P}_{n+1}(x)$, $n \geq 0$. Hence, $\{\widetilde{P}_n\}_{n\geq 0}$ is $\mathscr{O}_{\widetilde{c};1,3,2}$-classical. □

Our purpose here is to describe all the $\mathscr{O}_{c;1,3,2}$-classical polynomial sequences. In what follows, we denote by $Q_n(x) := Q_n(x;c)$, $n \geq 0$, if there is no ambiguity.
Assume that $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ are MOPS satisfying

$$\begin{cases} P_0(x) = 1, \; P_1(x) = x - \beta_0, \\ P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \; \gamma_{n+1} \neq 0, \; n \geq 0, \end{cases} \tag{18}$$

$$\begin{cases} Q_0(x) = 1, \; Q_1(x) = x - \xi_0, \\ Q_{n+2}(x) = (x - \xi_{n+1})Q_{n+1}(x) - \lambda_{n+1}Q_n(x), \; \lambda_{n+1} \neq 0, \; n \geq 0. \end{cases} \tag{19}$$

The dual sequences of $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ will be denoted by $\{u_n\}_{n\geq 0}$ and $\{v_n\}_{n\geq 0}$, respectively. By Proposition 1 (i), we get

$$u_n = \frac{P_n}{\langle u_0, P_n^2\rangle}u_0, \; n \geq 0 \quad ; \quad v_n = \frac{Q_n}{\langle v_0, Q_n^2\rangle}v_0, \; n \geq 0. \tag{20}$$

Let us start with some auxiliary results.

Lemma 3. *The MOPS $\{Q_n\}_{n\geq 0}$ is classical and the following properties hold.*

(i) (SSR) $Q_n(x) = Q_n^{[1]}(x) + c_nQ_{n-1}^{[1]}(x) + d_nQ_{n-2}^{[1]}(x)$, $n \geq 0$, *where*

$$c_n = \frac{n}{3}\big(\beta_{n+1} - \xi_n\big), \; n \geq 0, \quad d_n = \frac{n-1}{3}\Big(\frac{n}{n+3}\gamma_{n+1} - \lambda_n\Big), \; n \geq 1, \; d_0 = 0.$$

(ii) (PE) $(\Phi v_0)' + \Psi v_0 = 0$, *where*

$$\begin{aligned} \kappa\Phi(x) &= d_2(\lambda_1\lambda_2)^{-1}Q_2(x) + c_1\lambda_1^{-1}Q_1(x) + 1, \\ \Psi(x) &= (\kappa\lambda_1)^{-1}Q_1(x). \end{aligned}$$

Here, κ is a normalization factor.

Proof. Let us introduce the MPS $\{Z_n\}_{n\geq 0}$ given by

$$(n+1)(n+2)Z_n(x) := (x-c)^2P_n''(x) + 4(x-c)P_n'(x) + 2P_n(x), \; n \geq 0. \tag{21}$$

Taking derivatives in both hand sides of (21), where n is replaced by $n+1$, and taking into account (15), we get

$$Z_n^{[1]}(x) = Q_n(x), \; n \geq 0. \tag{22}$$

Notice that $Z_n(x)$ is a monic primitive of $Q_n(x)$.
From (18) and (21), we obtain

$$\begin{gathered}(n+3)(n+4)Z_{n+2} = \\ (n+2)(n+3)(x-\beta_{n+1})Z_{n+1} - (n+1)(n+2)\gamma_{n+1}Z_n + 2\Big((x-c)^2P_{n+1}\Big)', n \geq 0.\end{gathered}$$

Differentiating in both hand sides of the previous identity and inserting (21) and (22), we obtain

$$\begin{aligned}(n+2)(n+3)(n+4)Q_{n+1} = {}& (n+1)(n+2)(n+3)(x-\beta_{n+1})Q_n - \\ & n(n+1)(n+2)\gamma_{n+1}Q_{n-1} + 3(n+2)(n+3)Z_{n+1}.\end{aligned}$$

On account of (19), it follows that

$$Z_n(x) = Q_n(x) + a_nQ_{n-1}(x) + b_nQ_{n-2}(x), \; n \geq 0, \tag{23}$$

where $a_n = \frac{n}{3}(\beta_n - \xi_{n-1})$, $n \geq 1$, $a_0 = 0$, $b_n = \frac{n}{3}(\frac{n-1}{n+2}\gamma_n - \lambda_{n-1})$, $n \geq 2$, $b_0 = b_1 = 0$.
By differentiating both hand sides of (23) and using (22), (i) holds.

Now, let $\{v_n^{[1]}\}_{n\geq 0}$ be the dual sequence of $\{Q_n^{[1]}\}_{n\geq 0}$. From (i), we have $\langle v_0^{[1]}, Q_n\rangle = 0$, $n \geq 3$, $\langle v_0^{[1]}, Q_2\rangle = d_2$, $\langle v_0^{[1]}, Q_1\rangle = c_1$, and $\langle v_0^{[1]}, Q_0\rangle = 1$. So, $v_0^{[1]} = d_2v_2 + c_1v_1 + v_0$, and by (20), we get $v_0^{[1]} = \kappa\Phi(x)v_0$, where $\kappa\Phi(x) = d_2\lambda_1^{-1}\lambda_2^{-1}Q_2(x) + c_1\lambda_1^{-1}Q_1(x) + 1$ and κ

is a normalization factor. Since $(v_0^{[1]})' = -v_1 = -\lambda_1^{-1} Q_1 v_0$, then $(\Phi v_0)' + \Psi v_0 = 0$, where $\Psi(x) = (\kappa\lambda_1)^{-1} Q_1(x)$. Hence, (ii) holds. □

Lemma 4. *There exist three non-zero polynomials E, F and G, with $\deg E \leq 4$, $\deg F \leq 3$ and $\deg G \leq 2$ such that*

(i) $(x-c)^2 v_0 = E u_0$.

(ii) $E(x)Q_n^{(3)}(x) + F(x)Q_n''(x) + G(x)Q_n'(x) + \rho_0 P_1(x) Q_n(x) = \rho_n P_{n+1}(x)$, $n \geq 0$, *where*

$$\begin{aligned}
E(x) &= (-1/6)\Big(\rho_3 P_4(x) - \rho_0 P_1(x) Q_3(x) - G(x)Q_3'(x) - F(x)Q_3''(x)\Big),\\
F(x) &= (1/2)\Big(\rho_2 P_3(x) - \rho_0 P_1(x) Q_2(x) - G(x) Q_2'(x)\Big),\\
G(x) &= \rho_1 P_2(x) - \rho_0 P_1(x) Q_1(x),\\
\rho_n &= (n+1)(n+2)(n+3)\frac{\langle v_0, Q_n^2\rangle}{\langle u_0, P_{n+1}^2\rangle}\\
&= \frac{E^{(4)}(0)}{24} n(n-1)(n-2) + \frac{F^{(3)}(0)}{6} n(n-1) + \frac{G^{(2)}(0)}{2} n + \rho_0.
\end{aligned}$$

(iii) *The following relations hold*

(a) $B(x)E(x) = -\rho_0 \Phi^3(x) P_1(x)$,

(b) $3A(x)E(x) = -\Phi^2(x) G(x)$,

(c) $3\big(\Phi'(x) + \Psi(x)\big)E(x) = \Phi(x)F(x)$,

(d) $\gamma_1 \Phi^3(x) = \Big(3\rho_1^{-1}\Phi(x)A(x) + \big((\rho_1^{-1} - \rho_0^{-1})x + \beta_1\rho_0^{-1} - \xi_0\rho_1^{-1}\big)B(x)\Big)E(x)$,

where,

$$\begin{aligned}
A(x) &= \Big(2\Phi'(x) + \Psi(x)\Big)\Big(\Phi'(x) + \Psi(x)\Big) - \Big(\Phi''(x) + \Psi'(x)\Big)\Phi(x),\\
B(x) &= -\Big(3\Phi'(x) + \Psi(x)\Big)A(x) + \Phi(x)A'(x).
\end{aligned}$$

Proof. From (14), (16) and (20), we get

$$-(x-c)^2\Big(Q_n v_0^{(3)} + 3Q_n' v_0'' + 3Q_n'' v_0' + Q_n^{(3)} v_0\Big) = \rho_n P_{n+1} u_0,\ n \geq 0, \tag{24}$$

where $\rho_n = (n+1)(n+2)(n+3)\langle v_0, Q_n^2\rangle\langle u_0, P_{n+1}^2\rangle^{-1}$, $n \geq 0$.
From (24), with $n = 0$, we obtain

$$-(x-c)^2 v_0^{(3)} = \rho_0 P_1 u_0. \tag{25}$$

Using (24) and (25), it follows that

$$-(x-c)^2\Big(3Q_n' v_0'' + 3Q_n'' v_0' + Q_n^{(3)} v_0\Big) = (\rho_n P_{n+1} - \rho_0 P_1 Q_n) u_0. \tag{26}$$

Taking $n = 1$ in (26), we get

$$-3(x-c)^2 v_0'' = G u_0, \tag{27}$$

where $G(x) = \rho_1 P_2(x) - \rho_0 P_1(x) Q_1(x)$.
Combining (27) and (26), we obtain

$$-3(x-c)^2\Big(3Q_n'' v_0' + Q_n^{(3)} v_0\Big) = (\rho_n P_{n+1} - \rho_0 P_1 Q_n - G Q_n') u_0. \tag{28}$$

For $n = 2$ in (28), we get

$$-3(x-c)^2 v_0' = F u_0, \tag{29}$$

where $F(x) = (1/2)\Big(\rho_2 P_3(x) - \rho_0 P_1(x)Q_2(x) - G(x)Q_2'(x)\Big)$.
Insertion of (29) into (28) yields

$$-(x-c)^2 Q_n^{(3)} v_0 = \Big(\rho_n P_{n+1} - \rho_0 P_1 Q_n - G Q_n' - F Q_n''\Big) u_0. \tag{30}$$

Hence, taking $n = 3$ in (30), (i) holds.
On the other hand, by substituting $(x-c)^2 v_0 = E u_0$ in (30) and taking into account the quasi-definiteness of u_0, we deduce (ii).
According to Lemma 3 (ii), we can write

$$\Phi v_0' = -(\Phi' + \Psi) v_0, \quad \Phi^2 v_0'' = A v_0, \quad \Phi^3 v_0^{(3)} = B v_0, \tag{31}$$

where $A(x) = \Big(2\Phi'(x) + \Psi(x)\Big)\Big(\Phi'(x) + \Psi(x)\Big) - \Big(\Phi''(x) + \Psi'(x)\Big)\Phi(x)$ and $B(x) = -\Big(3\Phi'(x) + \Psi(x)\Big)A(x) + \Phi(x)A'(x)$.
Now, if we we multiply (25), (27) and (29) by Φ^3, Φ^2 and Φ respectively, and we take into account (31), (i) as well as the quasi-definiteness of u_0, we get, respectively, (a), (b) and (c).
Finally, multiplying (24) by Φ^3 and using (31), (i) and the quasi-definiteness of u_0, we obtain

$$\Big(\Phi^3 Q_n^{(3)} + C Q_n'' + 3A\Phi Q_n' + B Q_n\Big) E = -\rho_n \Phi^3 P_{n+1},\ n \geq 0, \tag{32}$$

where $C(x) = -3\Phi^2(x)\Big(\Phi'(x) + \Psi(x)\Big)$.
For $n = 1$ in (32), $n = 0$ in (18), $n = 0$ in (32), and (19), we get

$$\begin{aligned} \Big(3A(x)\Phi(x) + (x - \xi_0)B(x)\Big)E(x) &= -\rho_1 \Phi^3(x)\Big((x-\beta_1)P_1(x) - \gamma_1\Big), \\ &= \rho_1 \rho_0^{-1}(x - \beta_1)B(x)E(x) + \rho_1\gamma_1\Phi^3(x). \end{aligned}$$

Thus (d) holds. □

Now we will describe all the $\mathscr{O}_{c;1,3,2}$-classical polynomial sequences.

Theorem 1. *The $\mathscr{O}_{c;1,3,2}$-classical polynomial sequences are, up to a suitable affine transformation in the variable, one of the following D-classical polynomial sequences*

(i) $P_n(x) = L_n^{(2)}(x)$, $n \geq 0$ *and* $Q_n(x) = L_n^{(1)}(x)$, $n \geq 0$, *with* $c = 0$.
(ii) $P_n(x) = P_n^{(\alpha-3,2)}(x)$, $n \geq 0$, *where* $\alpha \neq -n+3$, $n \geq 2$, *with* $Q_n(x) = P_n^{(\alpha,1)}(x)$, $n \geq 0$, *and* $c = 1$.

Proof. From Lemma 3, $\{Q_n\}_{n\geq 0}$ is D-classical. According to Lemma 2, we will analyze the four canonical situations given in Table 1.

($\mathbf{C}_1$). $\{Q_n\}_{n\geq 0}$ is the Hermite MOPS. From Table 1 **($\mathbf{C}_1$)**, $A(x) = 2(2x^2-1)$ and $B(x) = 4x(-2x^2+3)$. Since Lemma 4 (iii) (a), $4x(-2x^2+3)E(x) = \rho_0 P_1(x)$. This yields a contradiction, since $\deg P_1 = 1$ and $\rho_0 \neq 0$.

($\mathbf{C}_2$). $\{Q_n\}_{n\geq 0}$ is the Laguerre MOPS. From Table 1 **($\mathbf{C}_2$)**, we obtain $A(x) = x^2 - 2\alpha x + \alpha(\alpha-1)$ and $B(x) = -x^3 + 3\alpha x^2 - 3\alpha(\alpha-1)x + \alpha(\alpha-1)(\alpha+2)$. By Lemma 4, (iii), we get

$$\Big(x^3 - 3\alpha x^2 + 3\alpha(\alpha-1)x - \alpha(\alpha-1)(\alpha-2)\Big)E = \rho_0 x^3 P_1, \tag{33}$$

$$3\Big(-x^2 + 2\alpha x - \alpha(\alpha-1)\Big)E = x^2 G, \tag{34}$$

$$3(x-\alpha)E = xF, \tag{35}$$

$$\gamma_1 x^3 = \Big(3\rho_1^{-1} x A + ((\rho_1^{-1} - \rho_0^{-1})x + \beta_1\rho_0^{-1} - \xi_0\rho_1^{-1})B\Big)E, \tag{36}$$

From (33), $\deg E = 1$. So, $E(x) = \rho_0 x$, by (33) and (36)). Hence, (33) becomes $x^3 - 3\alpha x^2 + 3\alpha(\alpha-1)x - \alpha(\alpha-1)(\alpha-2) = x^2 P_1$. Thus, $\alpha(\alpha-1) = 0$. Necessarily, $\alpha = 1$. Otherwise, if $\alpha = 0$ then $F(x) = 3\rho_0 x$, $G(x) = -3\rho_0 x$ and $P_1(x) = x$. But, by evaluating at $x = 0$ the equation given by Lemma 4 (ii), we obtain $P_{n+1}(0) = 0$, $n \geq 0$, which contradicts the orthogonality of $\{P_n\}_{n\geq 0}$.
From (33) with $\alpha = 1$ and $E(x) = \rho_0 x$, we get $P_1(x) = x - 3$ and, as a consequence, $\beta_0 = 3$. Thus,

$$A(x) = x(x-2),\ B(x) = -x^2(x-3),\ F(x) = 3\rho_0(x-1),\ G(x) = -3\rho_0(x-2). \quad (37)$$

Since $\xi_0 = 2$ (see Table 1 **($\mathbf{C}_2$)** with $\alpha = 1$) and by (37), (36) yields $\rho_1 = \rho_0$, $\beta_1 = 5$, and $\gamma_1 = 3$. Then, $\rho_0 = 6\langle v_0, Q_0^2\rangle\langle u_0, P_1^2\rangle^{-1} = 6\gamma_1^{-1} = 2$. From Table 1 **($\mathbf{C}_2$)**, with $\alpha = 1$, and by Lemma 3 (i), $\beta_{n+1} = 2n+5$, $n \geq 1$, and $\gamma_{n+1} = (n+1)(n+3)$, $n \geq 2$. By Lemma 4 (ii), $2xQ_n^{(3)}(x) + 6(x-1)Q_n''(x) - 6(x-2)Q_n'(x) + 2(x-3)Q_n(x) = \rho_n P_{n+1}(x)$, $n \geq 0$. Thus $\rho_n = 2$, $n \geq 0$. Therefore,

$$xQ_n^{(3)}(x) + 3(x-1)Q_n''(x) - 3(x-2)Q_n'(x) + (x-3)Q_n(x) = P_{n+1}(x),\ n \geq 0.$$

Since $2 = \rho_1 = 24\langle v_0, Q_1^2\rangle\langle u_0, P_2^2\rangle^{-1} = 24\lambda_1(\gamma_1\gamma_2)^{-1}$, then $\gamma_2 = 8$, $\beta_n = 2n+3$ and $\gamma_{n+1} = (n+1)(n+3)$, $n \geq 0$. Hence, $P_n = L_n^{(2)}$, $n \geq 0$, and $Q_n = L_n^{(1)}$, $n \geq 0$. In addition, by (17) with $n = 1$, (18) and (19), we get $c = 2\xi_0 - (1/2)(\beta_0 + \beta_1) = 0$. Also, by Lemma 4 (i), we get $x^2 v_0 = 2xu_0$, i.e., $xv_0 = 2u_0$. Thus, we have

$$L_n^{(1)}(x) = \frac{\mathscr{O}_{1,3,2}\ L_{n+1}^{(2)}(x)}{(n+1)(n+2)(n+3)}, \quad n \geq 0, \quad (38)$$

$$xL_n^{(1)(3)}(x) + 3(x-1)L_n^{(1)''}(x) - 3(x-2)L_n^{(1)'}(x) + (x-3)L_n^{(1)}(x) = L_{n+1}^{(2)}(x),\ n \geq 0. \quad (39)$$

($\mathbf{C}_3$). $\{Q_n\}_{n\geq 0}$ is the Bessel MOPS with parameter $\alpha \neq -n/2$, $n \geq 0$. From Table 1 **($\mathbf{C}_3$)**, we get $A(x) = 2(1-\alpha)(3-2\alpha)x^2 + 4(2\alpha-3)x + 4$ and $B(x) = 4\big[(\alpha-1)(2\alpha-3)(\alpha-2)x^3 + 3(2\alpha-3)(\alpha-2)x^2 + 6(\alpha-2)x + 2\big]$. By Lemma 4 (iii) (a), one has $B(x)E(x) = -\rho_0 P_1(x)x^6$. This requires that $\deg B = 3$ and $\deg E = 4$, since $\deg B \leq 3$, $\deg E \leq 4$, and $\deg B + \deg E = 7$. But, from the previous equation, we must have $B(0) = 0$, that contradicts the fact that $B(0) = 8$.

($\mathbf{C}_4$). $\{Q_n\}_{n\geq 0}$ is the Jacobi MOPS with parameters α and β satisfying α, $\beta \neq -n$, $\alpha + \beta \neq -n-1$, $n \geq 1$. From Table 1 **($\mathbf{C}_4$)**, we have

$$A(x) = (\alpha+\beta-1)\Big((\alpha+\beta)x^2 + 2(\beta-\alpha)x\Big) + (\alpha-\beta)^2 - (\alpha+\beta), \quad (40)$$

$$B(x) = (\alpha+\beta-2)\Big[(\alpha+\beta-1)\Big((\alpha+\beta)x^3 + 3(\beta-\alpha)x^2\Big) + 3\Big((\alpha-\beta)^2 - (\alpha+\beta)\Big)x\Big] + (\beta-\alpha)\Big((\alpha-\beta)^2 - 3\alpha - 3\beta + 2\Big). \quad (41)$$

By Lemma 4 (iii) (a), we have $B(x)E(x) = -\rho_0(x^2-1)^3 P_1(x)$. The fact that $\deg B + \deg E = 7$, $\deg E \leq 4$ and $\deg B \leq 3$, yields $\deg B = 3$ and $\deg E = 4$. By Lemma 4 (iii) (c), $E(x)$ divides $\Phi^3(x) = (x^2-1)^3$ and hence there are three situations to be considered. Either $E(x) = \mu(x-1)(x+1)^3$, $E(x) = \mu(x+1)(x-1)^3$ or $E(x) = \mu(x+1)^2(x-1)^2$, where μ is a non-zero real number.

($\mathbf{C}_{4,1}$). $E(x) = \mu(x-1)(x+1)^3$, $\mu \neq 0$.
According to Lemma 4 (iii), we have

$$\begin{aligned} \mu B &= -\rho_0(x-1)^2 P_1, & (42)\\ (x-1)G &= -3\mu(x+1)A, & (43)\\ F &= 3\mu\Big(-(\alpha+\beta)x+\alpha-\beta\Big)(x+1)^2, & (44)\\ \mu^{-1}\gamma_1(x-1)^2 &= 3\rho_1^{-1}(x^2-1)A+\Big((\rho_1^{-1}-\rho_0^{-1})x+\beta_1\rho_0^{-1}-\xi_0\rho_1^{-1}\Big)B, & (45) \end{aligned}$$

where $\xi_0 = (\alpha-\beta)(\alpha+\beta+2)^{-1}$.
From (41) and (42), $\rho_0 = -\mu(\alpha+\beta-2)(\alpha+\beta-1)(\alpha+\beta)$, $\beta_0 = (\alpha-5\beta)(\alpha+\beta)^{-1}$, and $\beta(\beta-1) = 0$. We must have $\beta = 1$. Otherwise, if $\beta = 0$, then $A(x) = \alpha(\alpha-1)(x-1)^2$, $B(x) = \alpha(\alpha-1)(\alpha-2)(x-1)^3$ and $P_1(1) = G(1) = F(1) = 0$. But, by evaluating at $x = 1$ the equation given by Lemma 4 (ii) we get $P_{n+1}(1) = 0$, $n \geq 0$, which contradicts the orthogonality of $\{P_n\}_{n\geq 0}$.
For $\beta = 1$, we get $\rho_0 = -\mu\alpha(\alpha-1)(\alpha+1)$, $\beta_0 = (\alpha-5)(\alpha+1)^{-1}$, $F(x) = 3\mu\big(-(\alpha+1)x+\alpha-1\big)(x+1)^2$ and $G(x) = -3\mu\alpha\big((\alpha+1)x-\alpha+3\big)(x+1)$. This expression of $F(x)$ and $G(x)$ together with that given by Lemma 4 (ii), yields by identification of the coefficients $\beta_1 = \frac{(\alpha-1)(\alpha-5)}{(\alpha+1)(\alpha+3)}$ and $\gamma_1 = \frac{12(\alpha-2)}{(\alpha+1)^2(\alpha+2)}$. Moreover, since $\rho_0 = 6\gamma_1^{-1}$, we get $\mu = -\frac{(\alpha+1)(\alpha+2)}{2\alpha(\alpha-1)(\alpha-2)}$.
From Table 1 **($\mathbf{C}_4$)** with $\beta = 1$ and by Lemma 3 (i),

$$\begin{aligned} \beta_n &= \frac{(\alpha-3)^2-2}{(2n+\alpha-1)(2n+\alpha+1)},\ n \geq 2,\\ \gamma_{n+1} &= \frac{4(n+1)(n+3)(n+\alpha)(n+\alpha-2)}{(2n+\alpha)(2n+\alpha+1)^2(2n+\alpha+2)},\ n \geq 2. \end{aligned}$$

By Lemma 4 (ii), $(x-1)(x+1)^3Q_n^{(3)}(x)+3\Big(-(\alpha+1)x+\alpha-1\Big)(x+1)^2Q_n''(x)-3\alpha\Big((\alpha+1)x-\alpha+3\Big)(x+1)Q_n'(x)-\alpha(\alpha-1)(\alpha+1)(x-\beta_0)Q_n(x) = \mu^{-1}\rho_nP_{n+1}(x)$, $n \geq 0$. This allows us to deduce that

$$\rho_n = \Big(n^3-3(\alpha+2)n^2+(-3\alpha^2+5)n-\alpha(\alpha-1)(\alpha+1)\Big)\mu, \quad n \geq 0.$$

Consequently,

$$\begin{aligned} &(x-1)(x+1)^3Q_n^{(3)}(x)+3\Big(-(\alpha+1)x+\alpha-1\Big)(x+1)^2Q_n''(x)-\\ &3\alpha\Big((\alpha+1)x-\alpha+3\Big)(x+1)Q_n'(x)-\alpha(\alpha-1)(\alpha+1)(x-\beta_0)Q_n(x) =\\ &\Big(n^3-3(\alpha+2)n^2+(-3\alpha^2+5)n-\alpha(\alpha-1)(\alpha+1)\Big)P_{n+1}(x), \quad n \geq 0. \end{aligned}$$

Since $\frac{(\alpha+1)^2(\alpha+2)^2}{2(\alpha-2)(\alpha-1)} = \rho_1 = 24\lambda_1(\gamma_1\gamma_2)^{-1}$, then $\gamma_2 = \frac{32(\alpha^2-1)}{(\alpha+2)(\alpha+3)^2(\alpha+4)}$. Therefore,

$$\begin{aligned} \beta_n &= \frac{(\alpha-3)^2-2^2}{(2n+\alpha-1)(2n+\alpha+1)},\ n \geq 0,\\ \gamma_{n+1} &= \frac{4(n+1)(n+3)(n+\alpha)(n+\alpha-2)}{(2n+\alpha)(2n+\alpha+1)^2(2n+\alpha+2)},\ n \geq 0. \end{aligned}$$

Thus, $P_n(x) = P_n^{(\alpha-3,2)}(x)$ and $Q_n(x) = P_n^{(\alpha,1)}(x)$, $n \geq 0$ with $\alpha \neq -n+3$, $n \geq 1$. In addition, by (17) with $n = 1$, (18) and (19), we get $c = 2\xi_0 - (1/2)(\beta_0+\beta_1) = 1$.

By Lemma 4 (i), $(x-1)^2 v_0 = \mu(x-1)(x+1)^3 u_0$, where $\mu = -\frac{(\alpha+1)(\alpha+2)}{2\alpha(\alpha-1)(\alpha-2)}$. Thus, $v_0 = \mu(x+1)^3 u_0$. As a consequence, we have

$$P_n^{(\alpha,1)}(x) = \frac{\mathscr{O}_{1;1,3,2}\, P_{n+1}^{(\alpha-3,2)}(x)}{(n+1)(n+2)(n+3)}, \quad n \geq 0, \tag{46}$$

and

$$\begin{aligned}(x-1)(x+1)^3 P_n^{(\alpha,1)^{(3)}}(x) + 3\Big(-(\alpha+1)x+\alpha-1\Big)(x+1)^2 P_n^{(\alpha,1)''}(x) - \\ 3\alpha\Big((\alpha+1)x-\alpha+3\Big)(x+1)P_n^{(\alpha,1)'}(x) - \alpha(\alpha-1)\Big((\alpha+1)x-\alpha+5\Big)P_n^{(\alpha,1)}(x) \\ = \varrho_n P_{n+1}^{(\alpha-3,2)}(x), \quad n \geq 0,\end{aligned} \tag{47}$$

where $\varrho_n = n^3 - 3(\alpha+2)n^2 + (-3\alpha^2+5)n - \alpha(\alpha-1)(\alpha+1)$, $n \geq 0$.

($\mathbf{C}_{4,2}$). $E(x) = \mu(x+1)(x-1)^3$, $\mu \neq 0$.
By a similar computation as in **($\mathbf{C}_{4,1}$)**, we get $P_n(x) = P_n^{(2,\beta-3)}(x)$, $n \geq 0$, where $\beta \neq -n+3$, $n \geq 1$, as well as $Q_n(x) = P_n^{(1,\beta)}(x)$, $n \geq 0$, and $c = -1$. Through a suitable shift, we get **($\mathbf{C}_{4,1}$)**. Indeed, from (10) and Lemma 2, the MOPS $\{\widetilde{P}_n\}_{n\geq 0}$ where $\widetilde{P}_n(x) = (-1)^n P_n(-x) = P_n^{(\beta-3,2)}(x)$, is $\mathscr{O}_{-1;1,3,2}$-classical.

($\mathbf{C}_{4,3}$). $E(x) = \mu(x+1)^2(x-1)^2$, $\mu \neq 0$.
According to Lemma 4 (iii), we have

$$\mu B = -\rho_0(x^2-1)P_1, \tag{48}$$

$$G = -3\mu A, \tag{49}$$

$$F = 3\mu\big(-(\alpha+\beta)x+\alpha-\beta\big)(x^2-1). \tag{50}$$

By identification, relation (48) becomes

$$\beta_0 = 3\frac{\alpha-\beta}{\alpha+\beta} \tag{51}$$

$$= \frac{(\beta-\alpha)\Big((\alpha-\beta)^2-3\alpha-3\beta+2\Big)}{(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2)} \tag{52}$$

$$\rho_0\mu^{-1} = -(\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2) \tag{53}$$

$$= 3(\alpha+\beta-2)\big((\alpha-\beta)^2-(\alpha+\beta)\big) \tag{54}$$

By using (51) and (52), we get

$$\alpha^2+\beta^2+\alpha\beta-3\alpha-3\beta+2=0, \tag{55}$$

and by using (53) and (54), we get

$$\alpha^2+\beta^2+\alpha\beta-\alpha-\beta=0. \tag{56}$$

Then, (55) and (56) give $(\alpha-1)(\beta-1)=0$, hence $\alpha=1$ or $\beta=1$. For example, if $\beta=1$, we have $A(1)=B(1)=B'(1)=0$. Then, by using (48)−(50), we get $P_1(1) = G(1) = F(1) = 0$. But, by evaluating at $x=1$ the equation given by Lemma 4 (ii) we get $P_{n+1}(1)=0$, $n \geq 0$, which contradicts the orthogonality of $\{P_n\}_{n\geq 0}$. □

In the linear space $\mathbb{P}$ we can introduce two operators $\mathscr{L}$ and $\mathscr{P}_\alpha$, where $\alpha \neq -n+3$, $n \geq 1$, defined as follows: for any $p \in \mathbb{P}$,

$$\mathscr{L}(p) = xp^{(3)} + 3(x-1)p'' - 3(x-2)p' + (x-3)p,$$

$$\begin{aligned}\mathscr{P}_\alpha(p) = {} & (x-1)(x+1)^3p^{(3)} + 3\Big(-(\alpha+1)x+\alpha-1\Big)(x+1)^2p'' - \\ & 3\alpha\Big((\alpha+1)x-\alpha+3\Big)(x+1)p' - \alpha(\alpha-1)\Big((\alpha+1)x-\alpha+5\Big)p.\end{aligned}$$

According to (39) and (47) we, respectively, obtain

$$\mathscr{L}(L_n^{(1)}) = L_{n+1}^{(2)},\ n \geq 0, \tag{57}$$

$$\mathscr{P}_\alpha(P_n^{(\alpha,1)}) = \varrho_n P_{n+1}^{(\alpha-3,2)},\ n \geq 0, \tag{58}$$

where $\varrho_n = n^3 - 3(\alpha+2)n^2 + (-3\alpha^2+5)n - \alpha(\alpha-1)(\alpha+1)$, $n \geq 0$.
Notice that $\mathscr{L}$ and $\mathscr{P}_\alpha$ are *raising operators* [4].
If we apply the operator $\mathscr{O}_{1,3,2}$, (resp. $\mathscr{O}_{1;1,3,2}$) to both hand sides of (57) (resp. (58)) and taking into account (38) (resp. (46)), we obtain

$$\mathscr{O}_{1,3,2} \circ \mathscr{L}(L_n^{(1)}) = (n+1)(n+2)(n+3)L_n^{(1)}, \quad n \geq 0, \tag{59}$$

$$\mathscr{O}_{1;1,3,2} \circ \mathscr{P}_\alpha(P_n^{(\alpha,1)}) = (n+1)(n+2)(n+3)\varrho_n P_n^{(\alpha,1)}, \quad n \geq 0. \tag{60}$$

The operator $\mathscr{O}_{1,3,2} \circ \mathscr{L}$, (resp. $\mathscr{O}_{1;1,3,2} \circ \mathscr{P}_\alpha$) preserves the degree of polynomials and also the orthogonality of the MPS $\{L_n^{(1)}\}_{n\geq 0}$, (resp. $\{P_n^{(\alpha,1)}\}_{n\geq 0}$). Such operators are called *transfer operators* (see [4, 14]).
Secondly, by using (13) and (1), the intertwining operator $\mathfrak{S}_0$ satisfies

$$\mathfrak{S}_0(xL_n^{(2)}) = (n+2)!x(x-1)^n,\ n \geq 0. \tag{61}$$

Thus, the following integral relation holds

$$x^n = \frac{1}{(n+2)!}\int_0^{+\infty} t^2 e^{-t} L_n^{(2)}\big(t(x+1)\big)\mathrm{d}t, \quad n \geq 0.$$

From (8), (9), and (13), the intertwining operator $\mathfrak{S}_1$ satisfies

$$\mathfrak{S}_1\big((x-1)P_n^{(\alpha-3,2)}\big)(x) = (n+2)!(x-1)\ B_n^{(\frac{\alpha+1}{2})}(x-1),\ n \geq 0, \tag{62}$$

where $\{B_n^{(\frac{\alpha+1}{2})}\}_{n\geq 0}$ is the Bessel MOPS with parameter $\frac{\alpha+1}{2}$, (see Table 1 $(\mathbf{C}_3)$). Then, the following new integral relation holds

$$B_n^{(\frac{\alpha+1}{2})}(x) = \frac{1}{(n+2)!}\int_0^{+\infty} t^2 e^{-t} P_n^{(\alpha-3,2)}\big(tx+1\big)\ \mathrm{d}t,\ n \geq 0,\ (\alpha \neq -n+3,\ n \geq 1).$$

4. Some expansions in series of Laguerre polynomials

As an application of the previous results, our purpose is to give some new expansion in series of Laguerre polynomials.

First, let us establish the following result:

Theorem 2. *Let* $f(x) = \sum_{n\geq 0} a_n x^n$ *be a real function defined in the real line. The following statements are equivalent.*

(i) *The function* f *is* $1-$*periodic.*
(ii) *The following equality holds*

$$\sum_{n\geq 0} \frac{a_n}{(n+2)!}\ x^n = \sum_{n\geq 0} \frac{a_n}{(n+2)!}\ L_n^{(2)}(x).$$

(iii) *The sequence* $(a_n)_{n\geq 0}$ *satisfies*

$$a_\nu = \sum_{n\geq 0} (-1)^n \binom{n+\nu}{\nu} a_{n+\nu}, \quad \nu \geq 0.$$

Proof. (i)⇒(ii). Let $g(x) = \sum_{n\geq 0} \frac{a_n}{(n+1)!}\, x^n - \sum_{n\geq 0} \frac{a_n}{(n+1)!}\, L_n^{(2)}(x)$. By applying the operator $\mathfrak{S}_0$ to the function $xg(x)$ and taking into account (13) and (61), we get $\mathfrak{S}_0(xg(x)) = f(x) - f(x-1)$. Since f is 1-periodic, then $\mathfrak{S}_0(xg(x)) = 0$. This implies $xg(x) = 0$, since $\mathfrak{S}_0$ is one-to-one. Hence, we finally get $g = 0$.

(ii)⇒(iii). By using (1) with $\alpha = 2$ and taking into account (ii), we get

$$\begin{aligned}\sum_{n\geq 0} \frac{a_n}{(n+2)!} x^n &= \sum_{n\geq 0} a_n \sum_{\nu=0}^{n} \frac{(-1)^{n-\nu}}{(\nu+2)!}\binom{n}{\nu} x^\nu \\ &= \sum_{\nu\geq 0} \frac{1}{(\nu+2)!} \sum_{n\geq \nu} (-1)^{n-\nu} a_n \binom{n}{\nu} x^\nu \\ &= \sum_{\nu\geq 0} \frac{1}{(\nu+2)!} \sum_{n\geq 0} (-1)^{n} a_{n+\nu} \binom{n+\nu}{\nu} x^\nu.\end{aligned}$$

By identification, we find (iii).

(iii)⇒(i). Using (iii), we can write

$$\begin{aligned}f(x) &= \sum_{n\geq 0}\sum_{\kappa\geq 0} (-1)^{\kappa}\binom{\kappa+n}{n} a_{\kappa+n}\, x^n \\ &= \sum_{n\geq 0}\sum_{\kappa\geq 0} (-1)^{\kappa-n}\binom{\kappa}{n} a_{\kappa}\, x^n \\ &= \sum_{\kappa\geq 0} a_\kappa \sum_{n=0}^{\kappa} (-1)^{\kappa-n}\binom{\kappa}{n}\, x^n \\ &= \sum_{\kappa\geq 0} a_\kappa (x-1)^\kappa = f(x-1).\end{aligned}$$

Then, f is $1-$periodic. □

Examples. Let us consider the following 1-periodic functions.

• For every $x \in \mathbb{R}$,

$$\cos(2\pi x) = \sum_{n\geq 0} c_n x^n, \quad \text{with} \begin{cases} c_{2n+1} = 0, & n\geq 0, \\ c_{2n} = \frac{(-1)^n (2\pi)^{2n}}{(2n)!}, & n \geq 0.\end{cases}$$

• For every $x \in \mathbb{R}$,

$$\sin(2\pi x) = \sum_{n\geq 0} s_n x^n, \quad \text{with} \begin{cases} s_{2n} = 0, & n\geq 0, \\ s_{2n+1} = \frac{(-1)^n (2\pi)^{2n+1}}{(2n+1)!}, & n \geq 0.\end{cases}$$

• For every $x \in \mathbb{R}$, such that $x \neq \frac{\pi}{2} + l\pi$, $l \in \mathbb{Z}$,

$$\tan(\pi x) = \sum_{n\geq 0} t_n x^n, \ \text{with} \begin{cases} t_{2n} = 0, & n\geq 0, \\ t_{2n+1} = \frac{(-1)^n \pi^{2n+1} 2^{2n+2}(2^{2n+2}-1)}{(2n+2)!} B_{2n+2}, & n \geq 0,\end{cases}$$

where $\{B_k\}_{k\geq 0}$, are the Bernoulli numbers satisfying

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = \delta_{n,1}, \quad n \geq 1.$$

According to Theorem 2 (ii),

$$\Re e\big(\ {}_0F_1(2\pi i x, 2)\ \big) = \sum_{n\geq 0} \frac{(-1)^n (2\pi)^{2n}}{(2n)!(2n+2)!}\ x^{2n} = \sum_{n\geq 0} \frac{(-1)^n (2\pi)^{2n}}{(2n)!(2n+2)!}\ L^{(2)}_{2n}(x),$$

$$\Im m\big(\ {}_0F_1(2\pi i x, 2)\ \big) = \sum_{n\geq 0} \frac{(-1)^n (2\pi)^{2n+1}}{(2n+1)!(2n+3)!}\ x^{2n+1} = \sum_{n\geq 0} \frac{(-1)^n (2\pi)^{2n+1}}{(2n+1)!(2n+3)!}\ L^{(2)}_{2n+1}(x),$$

where ${}_0F_1(z, 2)$ is the confluent hypergeometric function given by

$${}_0F_1(z, 2) = \sum_{n\geq 0} \frac{z^n}{(n+1)!n!}.$$

In addition,

$$\sum_{n\geq 0} \frac{(-1)^n \pi^{2n+1} 2^{2n+2}(2^{2n+2}-1)}{(2n+2)!(2n+3)!}\ B_{2n+2}\ x^{2n+1} =$$

$$\sum_{n\geq 0} \frac{(-1)^n \pi^{2n+1} 2^{2n+2}(2^{2n+2}-1)}{(2n+2)!(2n+3)!}\ B_{2n+2}\ L^{(2)}_{2n+1}(x).$$

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