TJMM
6 (2014), No. 1, 69-84

POSITIVE SOLUTIONS FOR SECOND ORDER IMPULSIVE DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS ON TIME SCALES

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Abstract. In this paper, a class of second order impulsive differential equations with integral boundary conditions on time scales are considered. Under different combinations of superlineary and sublinearity of nonlinear term and the impulses, some existence, multiplicity, and nonexistence criteria of positive solutions are established based on Guo-Krasnosel'skii fixed point theorem, which are new and complement previously known results.

1. Introduction

Impulsive differential equations have been been extensively studied due to their potential applications in understanding mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology, economics and so on. see [1, 7, 8, 10, 15, 22, 18] and and the references therein. Lin and Jiang [17] studied the existence of multiple positive solutions for the second order impulse boundary value problem via the theory of fixed point index in cones.

On the other hand, the study of dynamic equations on time scales goes back to its founder Stefan Hilger [14] in order to unify continuous and discrete analysis and is a new area of still fairly theoretical exploration in mathematics. Motivating the subject is the notion that dynamic equations on time scales can build bridges between continuous and discrete equations. The study of differential equations with boundary conditions on time scales has received a lot of attention in the literature,see [2, 3, 4, 11, 21, 27] and the references therein. Li and Dong [16] we discussed the multiple positive solutions for the following fourth-order system of integral boundary value problem with a parameter on time scales.

Recently, Liu et al. [19] studied the existence of positive solutions for the singular second order integral boundary value problem by applying the fixed point index theorems

$$u''(t) + a(t)u'(t) + b(t)u(t) + \lambda c(t)f(t, u(t)) = 0, \quad t \in (0, 1),$$
$$u(0) = \int_0^1 g(s)u(s)ds, \quad u(1) = \int_0^1 h(s)u(s)ds,$$

where $c(t)$ is allowed to be singular at $t = 0, 1$ and $f(u)$ may be singular at $u = 0$.

2010 *Mathematics Subject Classification.* 34B15, 34B37.
Key words and phrases. Impulsive differential equations, integral boundary conditions, positive solution, fixed point theorem, time scales.

Hao et al. [13] presented the existence, multiplicity, and nonexistence of positive solutions for the following singular integral boundary value problem with impulse effects

$$\begin{aligned} &u''(t)+a(t)u'(t)+b(t)u(t)+\lambda c(t)f(t,u(t))=0, \quad t\neq t_k, \quad t\in(0,1),\\ &-\Delta u'|_{t=t_k}=\lambda I_k(u(t_k)), \quad t=t_k\in(0,1), \quad k=1,2,\dots,m,\\ &u(0)=\int_0^1 g(s)u(s)ds, \quad u(1)=\int_0^1 h(s)u(s)ds, \end{aligned} \tag{1}$$

where $\Delta u'|_{t=t_k}=u'(t_k^+)-u'(t_k^-)$, $u'(t_k^+)$ and $u'(t_k^-)$ denote the right and the left limit of $u'(t)$ at $t=t_k(k=1,2,\dots,m)$, respectively. For the case of $I_k=0$, $k=0,1,m$, $\lambda=1$, one of the special case of problem (1) with boundary value conditions $u(0)=0$ and $u(1)=\sum_{i=1}^n \alpha_i u(\xi_i)$, $0<\xi_1<\xi_2<\cdots<\xi_n<1$, and related problems have been extensively studied in many papers in recent years. The existence and multiplicity results of positive solutions are obtained by applying the Guo-Krasnosel'skii fixed point theorem in cones, Leggett-Williams fixed point theorem and fixed point index theory, see [9, 20, 26].

Yuan and Liu [25] studied the existence of multiple positive solutions for the following dynamic equation on time scales

$$\begin{aligned} &u^{\triangle\nabla}(t)+a(t)u^{\triangle}(t)+b(t)u(t)+\lambda q(t)f(t,u(t))=0, \quad t\in(0,1)_{\mathbb{T}},\\ &u(0)=0, \quad u(1)=\sum_{i=1}^n \alpha_i u(\xi_i). \end{aligned}$$

Motivated by the results mentioned above, in this paper, we consider the second order impulsive singular integral boundary value problem on time scales

$$\begin{aligned} &u^{\triangle\nabla}(t)+a(t)u^{\triangle}(t)+b(t)u(t)+\lambda c(t)f(t,u(t))=0, \quad t\neq t_k, \quad t\in(0,1)_{\mathbb{T}},\\ &-\Delta u^{\triangle}|_{t=t_k}=\lambda I_k(u(t_k)), \quad t=t_k\in(0,1)_{\mathbb{T}}, \quad k=1,2,\dots,m,\\ &u(\rho(0))=\int_{\rho(0)}^{\sigma(1)} g(s)u(s)\nabla s, \quad u(\sigma(1))=\int_{\rho(0)}^{\sigma(1)} h(s)u(s)\nabla s, \end{aligned} \tag{2}$$

where $\Delta u^{\triangle}|_{t=t_k}=u^{\triangle}(t_k^+)-u^{\triangle}(t_k^-)$, $u^{\triangle}(t_k^+)$ and $u^{\triangle}(t_k^-)$ denote the right and the left limit of $u^{\triangle}(t)$ at $t=t_k(k=1,2,\dots,m)$, respectively, $0=t_0<t_1<t_2<\cdots<t_m<t_{m+1}=1$, $a\in C([0,1],[0,+\infty))$, $b\in C([0,1],(-\infty,0])$, $c\in C([0,1],[0,+\infty))$, $c(t)\not\equiv 0$ is allowed to be singular $t=0,1$. $g,h\in L^1[0,1]_{\mathbb{T}}$ is nonnegative, λ is a positive parameter.

Under different combinations of superlineary and sublinearity of nonlinear term f and the impulses I_k, $k=1,2,\dots,m$, various existence, multiplicity, and nonexistence results for positive solutions are derived based on Guo-Krasnosel'skii fixed point theorem in terms of different values of λ. Some ideas of this paper are from [13] and [25]. Problem (2) is more general, it including multi-point boundary value, integral boundary value, nonlocal and impulsive problems as special cases. Hence, we generalize some known results in the literature to some degree, and so it is interesting and important to study the positive solutions for Problem (2).

The rest of this paper is organized as follows. In Section 2, we present some preliminaries and lemmas. In Section 3, we give the main result which state the sufficient conditions for the impulsive singular integral boundary value problem (2) to have existence, multiplicity, and nonexistence of positive solutions.

2. Preliminaries and lemmas

Some preliminary definitions and theorems on time scales can be found in [5, 6], which are excellent references for the calculus of time scales. We will recall some basic definitions and lemmas which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \qquad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided that it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be continuous function on $\mathbb{T}$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^{\Delta}(t)$ to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|[y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|, \qquad \forall s \in U.$$

If y is continuous, then y is right-dense continuous, and if y is delta differentiable at t, then y is continuous at t. A function $r : \mathbb{T} \to \mathbb{R}$ is called regressive if $1 + \mu(t)r(t) \neq 0$ for all $t \in \mathbb{T}^k$. A function r from $\mathbb{T}$ to $\mathbb{R}$ is positively regressive if $1 + \mu(t)r(t) > 0$ for every $t \in \mathbb{T}$. Denote $\mathfrak{R}^+$ is the set of positively regressive functions from $\mathbb{T}$ to $\mathbb{R}$, and $\mathbb{T}^+ = [0, +\infty) \cap \mathbb{T}$.

In this section, we nextly introduce some background definitions in Banach spaces, state fixed point theorems, and then present basic lemmas that are very important in the proof of the main results.

Define $PC([0,1]_{\mathbb{T}}, R) = \{x : [0,1]_{\mathbb{T}} \to R; x \in C((t_k, t_{k+1}]_{\mathbb{T}}, R), k = 0, 1, 2, \dots, m+1,$ and $x(t_k^+)$ and $x(t_k^-)$ exist with $x(t_k^-) = x(t_k), k = 1, 2, \dots, m\}$ with the norm $\|x\|_{PC} = \sup_{[0,1]_{\mathbb{T}}} |x(t)|$, and $PC^1([0,1]_{\mathbb{T}}, R) = \{x : x \in PC([0,1]_{\mathbb{T}}, R);\ x(t_k^+)$ and $x(t_k^-)$ exist and $x^{\triangle}$ is left continuous at t_k, for $k = 1, 2, \dots, m\}$. Let $\mathbb{C} = PC^1([0,1]_{\mathbb{T}}, R)$ be a Banach space with the norm $\|x\|_{\mathbb{C}} = \sup_{t \in [0,1]_{\mathbb{T}}}\{\|x\|_{PC}, \|x^{\triangle}\|_{PC}\}$.

The following well-known Guo-Krasnosel'skii fixed point theorem will play major role in our next analysis.

Lemma 1 ([12])**.** *Let X be a Banach space, and let $P \subset X$ be a cone in X. Let Ω_1 and Ω_2 be open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that, either*

(i) $\|Tw\| \leq \|w\|$, *$w \in P \cap \partial\Omega_1$, $\|Tw\| \geq \|w\|$, $w \in P \cap \partial\Omega_2$, or*
(ii) $\|Tw\| \geq \|w\|$, *$w \in P \cap \partial\Omega_1$, $\|Tw\| \leq \|w\|$, $w \in P \cap \partial\Omega_2$.*

Then T has a fixed point $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2 ([23])**.** *Assuming that $a \in C([0,1], [0, +\infty))$, $b \in C([0,1], (-\infty, 0])$. Then the equations*

$$\phi_1^{\triangle\nabla}(t) + a(t)\phi_1^{\triangle}(t) + b(t)\phi_1(t) = 0, \quad t \in (0,1)_{\mathbb{T}},$$
$$\phi_1(\rho(0)) = 0, \quad \phi_1(\sigma(1)) = 1,$$

and

$$\phi_2^{\triangle\nabla}(t) + a(t)\phi_2^{\triangle}(t) + b(t)\phi_2(t) = 0, \quad t \in (0,1)_{\mathbb{T}},$$
$$\phi_2(\rho(0)) = 1, \quad \phi_2(\sigma(1)) = 0,$$

have unique solutions ϕ_1 and ϕ_2, respectively, and

(*a*) ϕ_1 *is strictly increasing on* $[\rho(0), \sigma(1)]$,
(*b*) ϕ_2 *is strictly decreasing on* $[\rho(0), \sigma(1)]$.

For the sake of simplicity, we introduce some notations as follows.

(H_1) $a \in C([0,1],[0,+\infty))$, $b \in C([0,1],(-\infty,0])$.

(H_2) $g, h \in L^1[0,1]_{\mathbb{T}}$ is nonnegative, $k_1 > 0$, $k_4 > 0$ and $k_0 > 0$, where $k_0 = k_1k_4 - k_2k_3$,

$$k_1 = 1 - \int_{\rho(0)}^{\sigma(1)} \phi_2(s)g(s)\nabla s, \quad k_2 = \int_{\rho(0)}^{\sigma(1)} \phi_1(s)g(s)\nabla s,$$
$$k_3 = \int_{\rho(0)}^{\sigma(1)} \phi_2(s)h(s)\nabla s, \quad k_4 = 1 - \int_{\rho(0)}^{\sigma(1)} \phi_1(s)h(s)\nabla s.$$

(H_3) $c \in C([0,1],[0,+\infty))$, $c(t) \neq 0$ and $\int_{\rho(0)}^{\sigma(1)} H(s)c(s)\nabla s < +\infty$, where $H(s)$ in Lemma 4 as defined later, $f \in C([0,1]_{\mathbb{T}} \times [0,+\infty),[0,+\infty))$.

(H_4) $f(t,x) > 0$ for $t \in (0,1)_{\mathbb{T}}$, $x > 0$, and $I_k(x) > 0$ for $x > 0$, $k = 1,2,\ldots,m$.

In order to prove our main results, we present a useful lemma in this section. Consider the following impulsive boundary value problem

$$\begin{aligned} &u^{\triangle\nabla}(t) + a(t)u^{\triangle}(t) + b(t)u(t) + v(t) = 0, \ t \neq t_k, \ t \in (0,1)_{\mathbb{T}}, \\ &-\Delta u^{\triangle}|_{t=t_k} = \theta_k, \ t = t_k \in (0,1)_{\mathbb{T}}, \ k = 1,2,\ldots,m, \\ &u(\rho(0)) = \int_{\rho(0)}^{\sigma(1)} g(s)u(s)\nabla s, \ u(\sigma(1)) = \int_{\rho(0)}^{\sigma(1)} h(s)u(s)\nabla s, \end{aligned} \tag{3}$$

where φ_k and ψ_k are constants for $k = 1,2,\ldots,m$, $v(t) \in L^1[0,1]_T$.

Lemma 3. *Assume that (H_1) and (H_2) hold. For any $v(t) \in L^1[0,1]_T$, then the Problem (3) has the unique solution*

$$u(t) = \int_{\rho(0)}^{\sigma(1)} H(t,s)v(s)\nabla s + \sum_{k=1}^{m} H(t,t_k)\theta_k, \ t \in (0,1)_{\mathbb{T}}, \tag{4}$$

where

$$\begin{aligned} H(t,s) = G(t,s)p(s) &+ \frac{\phi_1(t)k_3 + \phi_2(t)k_4}{k_0} \int_{\rho(0)}^{\sigma(1)} G(\tau,s)p(s)g(\tau)\nabla\tau \\ &+ \frac{\phi_1(t)k_1 + \phi_2(t)k_2}{k_0} \int_{\rho(0)}^{\sigma(1)} G(\tau,s)p(s)h(\tau)\nabla\tau, \end{aligned}$$

$$p(t) = e_a(\rho(t),\rho(0)), \quad G(t,s) = \frac{1}{\phi_1^{\triangle}(\rho(0))} \begin{cases} \phi_1(s)\phi_2(t), & s \leq t, \\ \phi_1(t)\phi_2(s), & t \leq s. \end{cases}$$

Proof. First, we show that the unique solution of (3) can be represented by (4). In fact, we know from Lemma 2 that the equation

$$u^{\triangle\nabla}(t) + a(t)u^{\triangle}(t) + b(t)u(t) = 0, t \in (0,1)_{\mathbb{T}} \tag{5}$$

has known two linear independent solutions ϕ_1 and ϕ_2 since

$$\begin{vmatrix} \phi_1(\rho(0)) & \phi_1^{\triangle}(\rho(0)) \\ \phi_2(\rho(0)) & \phi_2^{\triangle}(\rho(0)) \end{vmatrix} = -\phi_1^{\triangle}(\rho(0)) \neq 0.$$

By the superposition principle, we can obtain that the unique solution of the Problem (4) can be represented by

$$\begin{aligned} u(t) = &\int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)v(s)\nabla s + \phi_1(t)\int_{\rho(0)}^{\sigma(1)} h(t)u(t)\nabla t \\ &+ \phi_2(t)\int_{\rho(0)}^{\sigma(1)} g(t)u(t)\nabla t + \sum_{k=1}^{m} G(t,t_k)p(t_k)\theta_k, \end{aligned} \tag{6}$$

thus

$$\begin{aligned} &\int_{\rho(0)}^{\sigma(1)} g(t)u(t)\nabla t = \\ &\quad = \int_{\rho(0)}^{\sigma(1)} g(t)\int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)v(s)\nabla s\nabla t + \int_{\rho(0)}^{\sigma(1)} g(t)\phi_1(t)\int_{\rho(0)}^{\sigma(1)} h(t)u(t)\nabla t\nabla t \\ &\qquad + \int_{\rho(0)}^{\sigma(1)} g(t)\phi_2(t)\int_{\rho(0)}^{\sigma(1)} g(t)u(t)\nabla t\nabla t + \int_{\rho(0)}^{\sigma(1)} g(t)\sum_{k=1}^{m} G(t,t_k)p(t_k)\theta_k\nabla t, \end{aligned}$$

$$\begin{aligned} &\int_{\rho(0)}^{\sigma(1)} h(t)u(t)\nabla t = \\ &\quad = \int_{\rho(0)}^{\sigma(1)} h(t)\int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)v(s)\nabla s\nabla t + \int_{\rho(0)}^{\sigma(1)} h(t)\phi_1(t)\int_{\rho(0)}^{\sigma(1)} h(t)u(t)\nabla t\nabla t \\ &\qquad + \int_{\rho(0)}^{\sigma(1)} h(t)\phi_2(t)\int_{\rho(0)}^{\sigma(1)} g(t)u(t)\nabla t\nabla t + \int_{\rho(0)}^{\sigma(1)} h(t)\sum_{k=1}^{m} G(t,t_k)p(t_k)\theta_k\nabla t, \end{aligned}$$

Solving the preceding two equations concerning $\kappa_1 = \int_{\rho(0)}^{\sigma(1)} g(t)u(t)\nabla t$ and $\kappa_2 = \int_{\rho(0)}^{\sigma(1)} h(t)u(t)\nabla t$, we have

$$\begin{pmatrix} k_1 & -k_2 \\ -k_3 & k_4 \end{pmatrix}\begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} = \begin{pmatrix} \int_{\rho(0)}^{\sigma(1)} g(t)\int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)v(s)\nabla s\nabla t \\ + \int_{\rho(0)}^{\sigma(1)} g(t)\sum_{k=1}^{m} G(t,t_k)p(t_k)\theta_k\nabla t \\ \int_{\rho(0)}^{\sigma(1)} h(t)\int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)v(s)\nabla s\nabla t \\ + \int_{\rho(0)}^{\sigma(1)} h(t)\sum_{k=1}^{m} G(t,t_k)p(t_k)\theta_k\nabla t \end{pmatrix},$$

and so

$$\begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} = k_0^{-1}\begin{pmatrix} k_4 & k_2 \\ k_3 & k_1 \end{pmatrix}\begin{pmatrix} \int_{\rho(0)}^{\sigma(1)} g(t)\int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)v(s)\nabla s\nabla t \\ + \int_{\rho(0)}^{\sigma(1)} g(t)\sum_{k=1}^{m} G(t,t_k)p(t_k)\theta_k\nabla t \\ \int_{\rho(0)}^{\sigma(1)} h(t)\int_{\rho(0)}^{\sigma(1)} G(t,s)p(s)v(s)\nabla s\nabla t \\ + \int_{\rho(0)}^{\sigma(1)} h(t)\sum_{k=1}^{m} G(t,t_k)p(t_k)\theta_k\nabla t \end{pmatrix}, \tag{7}$$

Hence, (4) follows from (6) and (7).

Second, if (4) holds, then for $t \in [t_i, t_{i+1}]_{\mathbb{T}}$, we have

$$\begin{aligned} u(t) =& \frac{1}{\phi_1^{\triangle}(\rho(0))}\left(\int_{\rho(0)}^{t} \phi_1(s)\phi_2(t)p(s)v(s)\nabla s + \int_{t}^{\sigma(1)} \phi_1(t)\phi_2(s)p(s)v(s)\nabla s\right) \\ &+ \frac{\phi_1(t)k_3+\phi_2(t)k_4}{k_0}\int_{\rho(0)}^{\sigma(1)} v(s)\left(\int_{\rho(0)}^{\sigma(1)} G(\tau,s)p(s)g(\tau)\nabla\tau\right)\nabla s \\ &+ \frac{\phi_1(t)k_1+\phi_2(t)k_2}{k_0}\int_{\rho(0)}^{\sigma(1)} v(s)\left(\int_{\rho(0)}^{\sigma(1)} G(\tau,s)p(s)h(\tau)\nabla\tau\right)\nabla s \\ &+ \frac{1}{\phi_1^{\triangle}(\rho(0))}\left(\sum_{k=1}^{i}\phi_1(t_k)\phi_2(t)p(t_k)\theta_k + \sum_{k=i+1}^{m}\phi_1(t)\phi_2(t_k)p(t_k)\theta_k\right) \\ &+ \frac{\phi_1(t)k_3+\phi_2(t)k_4}{k_0}\sum_{k=1}^{m}\left(\int_{\rho(0)}^{\sigma(1)} G(\tau,t_k)p(t_k)g(\tau)\nabla\tau\right)\theta_k \\ &+ \frac{\phi_1(t)k_1+\phi_2(t)k_2}{k_0}\sum_{k=1}^{m}\left(\int_{\rho(0)}^{\sigma(1)} G(\tau,t_k)p(t_k)h(\tau)\nabla\tau\right)\theta_k. \end{aligned} \tag{8}$$

Direct differentiation of (8) implies

$$\begin{aligned} u^{\triangle}(t) =& \frac{1}{\phi_1^{\triangle}(\rho(0))}\left(\phi_2^{\triangle}(t)\int_{\rho(0)}^{t} \phi_1(s)p(s)v(s)\nabla s + \phi_1^{\triangle}(t)\int_{t}^{\sigma(1)} \phi_2(s)p(s)v(s)\nabla s\right) \\ &+ \frac{\phi_1^{\triangle}(t)k_3+\phi_2^{\triangle}(t)k_4}{k_0}\int_{\rho(0)}^{\sigma(1)} v(s)\left(\int_{\rho(0)}^{\sigma(1)} G(\tau,s)p(s)g(\tau)\nabla\tau\right)\nabla s \\ &+ \frac{\phi_1^{\triangle}(t)k_1+\phi_2^{\triangle}(t)k_2}{k_0}\int_{\rho(0)}^{\sigma(1)} v(s)\left(\int_{\rho(0)}^{\sigma(1)} G(\tau,s)p(s)h(\tau)\nabla\tau\right)\nabla s \\ &+ \frac{1}{\phi_1^{\triangle}(\rho(0))}\left(\sum_{k=1}^{i}\phi_1(t_k)\phi_2^{\triangle}(t)p(t_k)\theta_k + \sum_{k=i+1}^{m}\phi_1^{\triangle}(t)\phi_2(t_k)p(t_k)\theta_k\right) \\ &+ \frac{\phi_1^{\triangle}(t)k_3+\phi_2^{\triangle}(t)k_4}{k_0}\sum_{k=1}^{m}\left(\int_{\rho(0)}^{\sigma(1)} G(\tau,t_k)p(t_k)g(\tau)\nabla\tau\right)\theta_k \\ &+ \frac{\phi_1^{\triangle}(t)k_1+\phi_2^{\triangle}(t)k_2}{k_0}\sum_{k=1}^{m}\left(\int_{\rho(0)}^{\sigma(1)} G(\tau,t_k)p(t_k)h(\tau)\nabla\tau\right)\theta_k, \end{aligned}$$

and

$$\begin{aligned} u^{\triangle\nabla}(t) =& \frac{1}{\phi_1^{\triangle}(\rho(0))}\left(\phi_2^{\triangle\nabla}(t)\int_{\rho(0)}^{\rho(t)} \phi_1(s)p(s)v(s)\nabla s + \phi_2^{\triangle}(t)\phi_1(t)p(t)v(t)\right. \\ &\left. +\phi_1^{\triangle\nabla}(t)\int_{\rho(t)}^{\sigma(1)} \phi_2(s)p(s)v(s)\nabla s - \phi_1^{\triangle}(t)\phi_2(t)p(t)v(t)\right) \\ &+ \frac{\phi_1^{\triangle\nabla}(t)k_3+\phi_2^{\triangle\nabla}(t)k_4}{k_0}\int_{\rho(0)}^{\sigma(1)} v(s)\left(\int_{\rho(0)}^{\sigma(1)} G(\tau,s)p(s)g(\tau)\nabla\tau\right)\nabla s \\ &+ \frac{\phi_1^{\triangle\nabla}(t)k_1+\phi_2^{\triangle\nabla}(t)k_2}{k_0}\int_{\rho(0)}^{\sigma(1)} v(s)\left(\int_{\rho(0)}^{\sigma(1)} G(\tau,s)p(s)h(\tau)\nabla\tau\right)\nabla s \end{aligned}$$

$$+\frac{1}{\phi_1^{\triangle}(\rho(0))}\left(\sum_{k=1}^{i}\phi_1(t_k)\phi_2^{\triangle\nabla}(t)p(t_k)\theta_k+\sum_{k=i+1}^{m}\phi_1^{\triangle\nabla}(t)\phi_2(t_k)p(t_k)\theta_k\right)$$
$$+\frac{\phi_1^{\triangle\nabla}(t)k_3+\phi_2^{\triangle\nabla}(t)k_4}{k_0}\sum_{k=1}^{m}\left(\int_{\rho(0)}^{\sigma(1)}G(\tau,t_k)p(t_k)g(\tau)\nabla\tau\right)\theta_k$$
$$+\frac{\phi_1^{\triangle\nabla}(t)k_1+\phi_2^{\triangle\nabla}(t)k_2}{k_0}\sum_{k=1}^{m}\left(\int_{\rho(0)}^{\sigma(1)}G(\tau,t_k)p(t_k)h(\tau)\nabla\tau\right)\theta_k.$$

Replacing the derivatives in (3), by Liouville's formula, we deduce that

$$\begin{aligned}\Delta u^{\triangle}|_{t=t_k}=&u^{\triangle}(t_k^+)-u^{\triangle}(t_k^-)\\=&\frac{1}{\phi_1^{\triangle}(\rho(0))}p(t_k)\theta_k\left(\phi_2^{\triangle}(t_k)\phi_1(t_k)-\phi_1^{\triangle}(t_k)\phi_2(t_k)\right)\\=&\frac{1}{\phi_1^{\triangle}(\rho(0))}\begin{vmatrix}\phi_1(t_k)&\phi_2(t_k)\\\phi_1^{\triangle}(t_k)&\phi_2^{\triangle}(t_k)\end{vmatrix}p(t_k)\theta_k\\=&\frac{1}{\phi_1^{\triangle}(\rho(0))}p(t_k)\theta_k e_{-a}(\rho(t_k),\rho(0))(-\phi_1^{\triangle}(\rho(0)))=-\theta_k,\end{aligned}$$

$$\begin{aligned}&u^{\triangle\nabla}(t)+a(t)u^{\triangle}(t)+b(t)u(t)\\=&\frac{1}{\phi_1^{\triangle}(\rho(0))}\left(\phi_2^{\triangle\nabla}(t)\int_t^{\rho(t)}\phi_1(s)p(s)v(s)\nabla s+\phi_2^{\triangle}(t)\phi_1(t)p(t)v(t)\right.\\&\left.+\phi_1^{\triangle\nabla}(t)\int_{\rho(t)}^{t}\phi_2(s)p(s)v(s)\nabla s-\phi_1^{\triangle}(t)\phi_2(t)p(t)v(t)\right)\\=&\frac{1}{\phi_1^{\triangle}(\rho(0))}\left(\phi_2^{\triangle\nabla}(t)(\rho(t)-t)\phi_1(t)p(t)v(t)+\phi_2^{\triangle}(t)\phi_1(t)p(t)v(t)\right.\\&\left.-\phi_1^{\triangle\nabla}(t)(\rho(t)-t)\phi_2(t)p(t)v(t)-\phi_1^{\triangle}(t)\phi_2(t)p(t)v(t)\right)\\=&\frac{1}{\phi_1^{\triangle}(\rho(0))}p(t)v(t)\left((\phi_2^{\triangle}(t)\phi_1(t)-\phi_1^{\triangle}(t)\phi_2(t))\right.\\&\left.+(\rho(t)-t)(\phi_2^{\triangle}(t)\phi_1(t)-\phi_1^{\triangle}(t)\phi_2(t))^{\nabla}\right)\\=&\frac{1}{\phi_1^{\triangle}(\rho(0))}p(t)v(t)\left(\phi_2^{\triangle}(\rho(t))\phi_1(\rho(t))-\phi_1^{\triangle}(\rho(t))\phi_2(\rho(t))\right)\\=&\frac{1}{\phi_1^{\triangle}(\rho(0))}p(t)v(t)e_{-a}(\rho(t),\rho(0))(-\phi_1^{\triangle}(\rho(0)))=-v(t).\end{aligned}$$

on the other hand, it is not difficult to verify that $u(\rho(0))=\int_{\rho(0)}^{\sigma(1)}g(t)u(t)\nabla t$ and $u(\sigma(1))=\int_{\rho(0)}^{\sigma(1)}h(t)u(t)\nabla t$. This complete the proof. □

Lemma 4. *Assume that* (H_1) *and* (H_2) *hold. For any* $t,s\in[0,1]_{\mathbb{T}}$*, we have*

$$G(t,s)\le G(t,t),\quad 0\le H(t,s)\le H(s),\quad H(t,s)\ge\gamma H(s),$$

where $\gamma = \min_{t\in[0,1]_{\mathbb{T}}}\{\phi_1(t), \phi_2(t)\}$,

$$\begin{aligned}H(s) =&G(s,s)p(s) + \frac{k_3+k_4}{k_0}\int_{\rho(0)}^{\sigma(1)} G(\tau,s)p(s)g(\tau)\nabla\tau\\&+\frac{k_1+k_2}{k_0}\int_{\rho(0)}^{\sigma(1)} G(\tau,s)p(s)h(\tau)\nabla\tau.\end{aligned}$$

Proof. The proofs of the Lemmas 4 can be obtained easily by Lemmas 2 and 3, so we omit it. □

Let

$$K = \left\{u \in PC([0,1]_{\mathbb{T}}, [0,+\infty)) : u(t) \geq 0, t \in [0,1]_{\mathbb{T}}, \min_{t\in[\delta,1-\delta]_{\mathbb{T}}} u(t) \geq \gamma\|u\|_{PC}\right\},$$

where $0 < \delta < \min\{1/2, t_1, t_m\}$. Then K is a cone of $PC([0,1]_{\mathbb{T}}, [0,+\infty))$. For $r > 0$, let $K_r = \{u \in K, \|u\|_{PC} < r\}$, $\partial K = \{u \in K, \|u\|_{PC} = r\}$. Defining operator $T_\lambda : K \to PC([0,1]_{\mathbb{T}}, [0,+\infty))$ as follow:

$$T_\lambda u(t) = \lambda\int_{\rho(0)}^{\sigma(1)} H(t,s)c(s)f(s,u(s))\nabla s + \lambda\sum_{k=1}^{m} H(t,t_k)I_k(u(t_k)),\ \ t\in[0,1]_{\mathbb{T}}. \tag{9}$$

It is well known that $u \in K$ is a positive fixed point of operator T_λ if and only if u be a solution of boundary value problem (2).

Lemma 5. *Assume that* (H_1) *and* (H_3) *hold. then* $T_\lambda : K \to K$ *is completely continuous.*

Proof. For any $u \in K$, $t \in [0,1]$, by (9) and Lemma 4, we have

$$\begin{aligned}T_\lambda u(t) =&\lambda\int_{\rho(0)}^{\sigma(1)} H(t,s)c(s)f(s,u(s))\nabla s + \lambda\sum_{k=1}^{m} H(t,t_k)I_k(u(t_k))\\ \leq&\lambda\int_{\rho(0)}^{\sigma(1)} H(s)c(s)f(s,u(s))\nabla s + \lambda\sum_{k=1}^{m} H(t_k)I_k(u(t_k)),\ \ t\in[0,1]_{\mathbb{T}}.\end{aligned}$$

Then

$$\|T_\lambda u(t)\|_{PC} \leq \lambda\int_{\rho(0)}^{\sigma(1)} H(s)c(s)f(s,u(s))\nabla s + \lambda\sum_{k=1}^{m} H(t_k)I_k(u(t_k)),\ \ t\in[0,1]_{\mathbb{T}}.$$

For $t \in [\delta, 1-\delta]_{\mathbb{T}}$,

$$\begin{aligned}T_\lambda u(t) =&\lambda\int_{\rho(0)}^{\sigma(1)} H(t,s)c(s)f(s,u(s))\nabla s + \lambda\sum_{k=1}^{m} H(t,t_k)I_k(u(t_k))\\ \geq&\lambda\int_{\rho(0)}^{\sigma(1)} \gamma H(s)c(s)f(s,u(s))\nabla s + \lambda\sum_{k=1}^{m} \gamma H(t_k)I_k(u(t_k))\\ \geq&\gamma\|T_\lambda u(t)\|_{PC},\ \ t\in[0,1]_{\mathbb{T}}.\end{aligned}$$

Next by standard methods and the Ascoli-Arzela theorem one can prove that $T_\lambda : K \to K$ is completely continuous. So it is omitted. □

3. Main results

In this section, we state main results, including existence, multiplicity and nonexistence results of positive solutions.

For convenience and simplicity in the following discussion, we use the following notations:

$$f^\beta = \limsup_{x\to\beta} \sup_{[0,1]_{\mathbb{T}}} \frac{f(t,x)}{x}, \quad f_\beta = \liminf_{x\to\beta} \inf_{[0,1]_{\mathbb{T}}} \frac{f(t,x)}{x},$$
$$I_k^\beta = \limsup_{x\to\beta} \sup_{[0,1]_{\mathbb{T}}} \frac{I_k(x)}{x}, \quad I_{k\beta} = \liminf_{x\to\beta} \inf_{[0,1]_{\mathbb{T}}} \frac{I_k(x)}{x},$$

where $\beta \to 0^+$ or $+\infty$.

Theorem 1. *Assume that (H_1) and (H_3) hold. Suppose that*

$$\max\left\{\gamma^2 \int_\delta^{1-\delta} H(s)c(s)\nabla s f_\infty, I_{1\infty}, I_{2\infty}, \ldots, I_{m\infty}\right\} > \int_{\rho(0)}^{\sigma(1)} H(s)c(s)\nabla s f^0 + \sum_{k=1}^m H(t_k) I_k^0,$$

where

$$f^0 + \sum_{k=1}^m I_k^0 < +\infty \quad \text{and} \quad \max\left\{\gamma^2 \int_\delta^{1-\delta} H(s)c(s)\nabla s f_\infty, I_{1\infty}, I_{2\infty}, \ldots, I_{m\infty}\right\} > 0,$$

then Problem (2) has at least one positive solution for

$$\frac{1}{\max\left\{\gamma^2 \int_\delta^{1-\delta} H(s)c(s)\nabla s f_\infty, I_{1\infty}, I_{2\infty}, \ldots, I_{m\infty}\right\}} < \lambda < \frac{1}{\int_{\rho(0)}^{\sigma(1)} H(s)c(s)\nabla s f^0 + \sum_{k=1}^m H(t_k) I_k^0}. \tag{10}$$

Proof. In view of (10), there exists $\varepsilon > 0$ such that

$$\frac{1}{\max\left\{\gamma^2 \int_\delta^{1-\delta} H(s)c(s)\nabla s (f_\infty - \varepsilon), I_{1\infty} - \varepsilon, I_{2\infty} - \varepsilon, \ldots, I_{m\infty} - \varepsilon\right\}} < \lambda < \frac{1}{\int_{\rho(0)}^{\sigma(1)} H(s)c(s)\nabla s (f^0 + \varepsilon) + \sum_{k=1}^m H(t_k)(I_k^0 + \varepsilon)}. \tag{11}$$

Let ε be fixed. Since $f^0 + \sum_{k=1}^m I_k^0 < +\infty$, there exists $r_1 > 0$ such that

$$f(t,x) < (f^0 + \varepsilon)x, \quad I_k(x) < (I_k^0 + \varepsilon)x, \quad 0 \le x \le r_1, \ t \in [0,1]_{\mathbb{T}}. \tag{12}$$

Then for $u \in \partial K_{r_1}$, from (11) and (12), we have

$$\begin{aligned}
T_\lambda u(t) \le& \lambda \int_{\rho(0)}^{\sigma(1)} H(s)c(s) f(s,u(s))\nabla s + \lambda \sum_{k=1}^m H(t_k) I_k(u(t_k)) \\
\le& \lambda \int_{\rho(0)}^{\sigma(1)} H(s)c(s)(f^0+\varepsilon)u(s)\nabla s + \lambda \sum_{k=1}^m H(t_k)(I_k^0 + \varepsilon) u_k(t_k) \\
\le& \lambda \left(\int_{\rho(0)}^{\sigma(1)} H(s)c(s)(f^0+\varepsilon)\nabla s + \sum_{k=1}^m H(t_k)(I_k^0+\varepsilon)\right) \|u\|_{PC} \le \|u\|_{PC}.
\end{aligned}$$

On the other hand, since $\max\left\{\gamma^2\int_\delta^{1-\delta}H(s)c(s)\nabla s f_\infty, I_{1\infty}, I_{2\infty},\dots,I_{m\infty}\right\}>0$. Without loss of generality, we assume that

$$\max\left\{\gamma^2\int_\delta^{1-\delta}H(s)c(s)\nabla s f_\infty, I_{1\infty}, I_{2\infty},\dots,I_{m\infty}\right\}=\gamma^2\int_\delta^{1-\delta}H(s)c(s)\nabla s f_\infty,$$

which means there exists $\tilde{r}_2>0$ such that $f(t,x)>(f_\infty-\varepsilon)x$, $x\geq\tilde{r}_2$, $t\in[\delta,1-\delta]_{\mathbb{T}}$. Let $r_2\geq\max\{2r_1,\tilde{r}_2/\gamma\}$, then for $u\in\partial K_{r_2}$, $t\in[\delta,1-\delta]_{\mathbb{T}}$, we have

$$\begin{aligned}T_\lambda u(t)\geq&\lambda\int_\delta^{1-\delta}\gamma H(s)c(s)f(s,u(s))\nabla s\geq\lambda\int_\delta^{1-\delta}\gamma H(s)c(s)(f_\infty-\varepsilon)u(s)\nabla s\\ \geq&\lambda\gamma^2(f_\infty-\varepsilon)\int_\delta^{1-\delta}H(s)c(s)\nabla s\|u\|_{PC}\geq\|u\|_{PC},\end{aligned}$$

then $\|T_\lambda u\|_{PC}\geq\|u\|_{PC}$, $u\in\partial K_{r_2}$. It follows from Lemma 1 that T_λ has a fixed point u^* with $r_1\leq\|u^*\|_{PC}\leq r_2$, which is a positive solution of Problem (2). Thus, Theorem 1 is completed. □

Corollary 1. *Assume that* (H_1) *and* (H_3) *hold. Suppose that*

$$f^0+\sum_{k=1}^m I_k^0=0 \text{ and } \max\left\{\gamma^2\int_\delta^{1-\delta}H(s)c(s)\nabla s f_\infty, I_{1\infty}, I_{2\infty},\dots,I_{m\infty}\right\}=\infty,$$

then Problem (2) has at least one positive solution for $\lambda>0$.

Theorem 2. *Assume that* (H_1) *and* (H_3) *hold. Suppose that*

$$\max\left\{\gamma^2\int_\delta^{1-\delta}H(s)c(s)\nabla s f_0, I_{10}, I_{20},\dots,I_{m0}\right\}>\int_{\rho(0)}^{\sigma(1)}H(s)c(s)\nabla s f^\infty+\sum_{k=1}^m H(t_k)I_k^\infty,$$

where

$$f^\infty+\sum_{k=1}^m I_k^\infty<+\infty \text{ and } \max\left\{\gamma^2\int_\delta^{1-\delta}H(s)c(s)\nabla s f_0, I_{10}, I_{20},\dots,I_{m0}\right\}>0,$$

then Problem (2) has at least one positive solution for

$$\frac{1}{\max\left\{\gamma^2\int_\delta^{1-\delta}H(s)c(s)\nabla s f_0, I_{10}, I_{20},\dots,I_{m0}\right\}}<\lambda<\frac{1}{\int_{\rho(0)}^{\sigma(1)}H(s)c(s)\nabla s f^\infty+\sum_{k=1}^m H(t_k)I_k^\infty}. \tag{13}$$

Proof. In view of (13), there exists $\varepsilon>0$ such that

$$\frac{1}{\max\left\{\gamma^2\int_\delta^{1-\delta}H(s)c(s)\nabla s(f_0-\varepsilon), I_{10}-\varepsilon, I_{20}-\varepsilon,\dots,I_{m0}-\varepsilon\right\}}<\lambda<\frac{1}{\int_{\rho(0)}^{\sigma(1)}H(s)c(s)\nabla s(f^\infty+\varepsilon)+\sum_{k=1}^m H(t_k)(I_k^\infty+\varepsilon)}. \tag{14}$$

Let ε be fixed. Since $\max\left\{\gamma^2\int_\delta^{1-\delta}H(s)c(s)\nabla sf_0, I_{10}, I_{20}, \ldots, I_{m0}\right\} > 0$. Without loss of generality, we assume that

$$\max\left\{\gamma^2\int_\delta^{1-\delta}H(s)c(s)\nabla sf_0, I_{10}, I_{20}, \ldots, I_{m0}\right\} = \gamma^2\int_\delta^{1-\delta}H(s)c(s)\nabla sf_0,$$

which means there exists $r_1 > 0$ such that $f(t,x) > (f_0-\varepsilon)x$, $x \geq r_1$, $t \in [\delta, 1-\delta]_{\mathbb{T}}$. Then for $u \in \partial K_{r_1}$, $t \in [\delta, 1-\delta]_{\mathbb{T}}$, we have

$$\begin{aligned}T_\lambda u(t) \geq& \lambda\int_\delta^{1-\delta}\gamma H(s)c(s)f(s,u(s))\nabla s \geq \lambda\int_\delta^{1-\delta}\gamma H(s)c(s)(f_0-\varepsilon)u(s)\nabla s\\ \geq& \lambda\gamma^2(f_0-\varepsilon)\int_\delta^{1-\delta}H(s)c(s)\nabla s\|u\|_{PC} \geq \|u\|_{PC},\end{aligned}$$

then $\|T_\lambda u\|_{PC} \geq \|u\|_{PC}$, $u \in \partial K_{r_1}$. On the other hand, since $f^\infty + \sum_{k=1}^m I_k^\infty < +\infty$, there exists $\tilde{r}_2 > 0$ such that

$$f(t,x) < (f^\infty+\varepsilon)x,\ \ I_k(x) < (I_k^\infty+\varepsilon)x,\ \ 0 \leq x \leq \tilde{r}_2,\ \ t \in [0,1]_{\mathbb{T}}.$$

Set

$$\begin{aligned}M = &\max\left\{\max_{t\in[0,1]_{\mathbb{T}}, x\in[0,\tilde{r}_2]}f(t,x), \max_{1\leq k\leq m}\left\{\max_{x\in[0,\tilde{r}_2]}I_k(x)\right\}\right\},\\ r_2 \geq &\max\left\{r_1, \tilde{r}_2, \frac{M}{f^\infty+\varepsilon}, \max\left\{\frac{M}{I_m^\infty+\varepsilon}\right\}\right\},\end{aligned}$$

then

$$f(t,x) < (f^\infty+\varepsilon)r_2,\ \ I_k(x) < (I_k^\infty+\varepsilon)r_2,\ \ 0 \leq x \leq r_2,\ \ t \in [0,1]_{\mathbb{T}}. \tag{15}$$

Hence, for $u \in \partial K_{r_2}$, from (14) and (15), we have

$$\begin{aligned}T_\lambda u(t) \leq& \lambda\int_{\rho(0)}^{\sigma(1)}H(s)c(s)f(s,u(s))\nabla s + \lambda\sum_{k=1}^m H(t_k)I_k(u(t_k))\\ \leq& \lambda\int_{\rho(0)}^{\sigma(1)}H(s)c(s)(f^\infty+\varepsilon)r_2\nabla s + \lambda\sum_{k=1}^m H(t_k)(I_k^\infty+\varepsilon)r_2\\ \leq& \lambda\left(\int_{\rho(0)}^{\sigma(1)}H(s)c(s)(f^\infty+\varepsilon)\nabla s + \sum_{k=1}^m H(t_k)(I_k^\infty+\varepsilon)\right)r_2\\ \leq& r_2 = \|u\|_{PC}.\end{aligned}$$

It follows from Lemma 1 that T_λ has at least one fixed point u^* with $r_1 \leq \|u^*\|_{PC} \leq r_2$, which is a positive solution of Problem (2), which is a positive solution of problem. Thus, Theorem 2 is completed. □

Corollary 2. *Assume that* (H_1) *and* (H_3) *hold. suppose that*

$$f^\infty + \sum_{k=1}^m I_k^\infty = 0 \ \text{and}\ \max\left\{\gamma^2\int_\delta^{1-\delta}H(s)c(s)\nabla sf_0, I_{10}, I_{20}, \ldots, I_{m0}\right\} = \infty,$$

then problem (2) has at least one positive solution for $\lambda > 0$.

Theorem 3. *Assume that* (H_1) *and* (H_4) *hold, if* $f^0+\sum_{k=1}^m I_k^0 = 0$ *or* $f^\infty+\sum_{k=1}^m I_k^\infty = 0$, *then there exists* $\lambda_0 > 0$ *such that Problem (2) has at least one positive solution for* $\lambda > \lambda_0$.

Proof. Choose a number $r_1 > 0$ and

$$\lambda_0 = \frac{r_1}{\gamma \int_\delta^{1-\delta} H(s)c(s)\nabla s \min_{t\in[\delta,1-\delta]_{\mathbb{T}},x\in[\gamma r_1,r_1]} f(t,x)}.$$

For $\lambda > \lambda_0$, $u \in \partial K_{r_1}$, $[\delta, 1-\delta]_{\mathbb{T}}$, one has

$$\begin{aligned} T_\lambda u(t) \geq& \lambda \int_\delta^{1-\delta} \gamma H(s)c(s)f(s,u(s))\nabla s \\ \geq& \lambda_0 \int_\delta^{1-\delta} \gamma H(s)c(s)u(s)\nabla s \min_{t\in[\delta,1-\delta]_{\mathbb{T}},x\in[\gamma r_1,r_1]} f(t,x) = r_1 = \|u\|_{PC}. \end{aligned} \tag{16}$$

Thus, $\|T_\lambda u\|_{PC} \geq \|u\|_{PC}$, $u \in \partial K_{r_1}$. If $f^0 + \sum_{k=1}^m I_k^0 = 0$, we can choose $r_2 \in (0, \gamma r_1)$ such that $f(t,x) < \varepsilon x$ and $I_k(x) < \varepsilon x$ for all $t \in [0,1]_{\mathbb{T}}$ and $x \in [0, r_2]$, where $\varepsilon > 0$ satisfying

$$\lambda\varepsilon \left(\int_{\rho(0)}^{\sigma(1)} H(s)c(s)\nabla s + \sum_{k=1}^m H(t_k) \right) < 1. \tag{17}$$

Hence, for $u \in \partial K_{r_2}$, we get

$$\begin{aligned} T_\lambda u(t) \leq& \lambda \int_{\rho(0)}^{\sigma(1)} H(s)c(s)f(s,u(s))\nabla s + \lambda \sum_{k=1}^m H(t_k)I_k(u(t_k)) \\ \leq& \lambda \int_{\rho(0)}^{\sigma(1)} H(s)c(s)\varepsilon u(s)\nabla s + \lambda \sum_{k=1}^m H(t_k)\varepsilon u(t_k) \\ \leq& \lambda\varepsilon \left(\int_{\rho(0)}^{\sigma(1)} H(s)c(s)\nabla s + \sum_{k=1}^m H(t_k) \right) \|u\|_{PC} \leq \|u\|_{PC}. \end{aligned}$$

If $f^\infty + \sum_{k=1}^m I_k^\infty = 0$, there exists $\tilde{r} > 0$ such that $f(t,x) < \varepsilon x$ and $I_k(x) < \varepsilon x$ for all $t \in [0,1]_{\mathbb{T}}$ and $x \geq \tilde{r}_3$, where $\varepsilon > 0$ satisfying (17). Let

$$M = \max\left\{ \max_{t\in[0,1]_{\mathbb{T}},x\in[0,\tilde{r}_3]} f(t,x), \max_{1\leq k\leq m} \left\{ \max_{x\in[0,\tilde{r}_3]} I_k(x) \right\} \right\},$$

$r_3 > \max\{r_1, \tilde{r}_3, M/\varepsilon\}$, then $f(t,x) < \varepsilon r_3$ and $I_k(x) < \varepsilon r_3$ for all $t \in [0,1]_{\mathbb{T}}$, $0 \leq x \leq r_3$. So for $u \in \partial K_{r_3}$, we have

$$\begin{aligned} T_\lambda u(t) \leq& \lambda \int_{\rho(0)}^{\sigma(1)} H(s)c(s)f(s,u(s))\nabla s + \lambda \sum_{k=1}^m H(t_k)I_k(u(t_k)) \\ \leq& \lambda \int_{\rho(0)}^{\sigma(1)} H(s)c(s)\varepsilon r_3\nabla s + \lambda \sum_{k=1}^m H(t_k)\varepsilon r_3 \\ \leq& \lambda\varepsilon \left(\int_{\rho(0)}^{\sigma(1)} H(s)c(s)\nabla s + \sum_{k=1}^m H(t_k) \right) r_3 \leq r_3 = \|u\|_{PC}. \end{aligned}$$

Then from Guo-Krasnosel'skii fixed point theorem, $T_\lambda u$ has a fixed point in $\overline{K}_{r_2} \setminus K_{r_1}$ or $\overline{K}_{r_3} \setminus K_{r_1}$, according to whether $f^0 + \sum_{k=1}^m I_k^0 = 0$ or $f^\infty + \sum_{k=1}^m I_k^\infty = 0$, respectively. Consequently, Problem (2) has at least one positive solution for $\lambda > \lambda_0$. □

Theorem 4. *Assume that* (H_1) *and* (H_4) *hold, if* $\max\{f_0, I_{10}, I_{20}, \dots, I_{m0}\} = \infty$ *or* $\max\{f_\infty, I_{1\infty}, I_{2\infty}, \dots, I_{m\infty}\} = \infty$, *then there exists* $\lambda_0 > 0$ *such that problem (2) has at least one positive solution for* $0 < \lambda < \lambda_0$.

Proof. Choose a number $r_1 > 0$ and

$$\lambda_0 = \frac{r_1}{\int_{\rho(0)}^{\sigma(1)} H(s)c(s)\nabla s \max_{t\in[0,1]_{\mathbb{T}},x\in[0,r_1]} f(t,x) + \sum_{k=1}^{m} \max_{x\in[0,r_1]} I_k(t_k)}.$$

For $u \in \partial K_{r_1}$, $0 < \lambda < \lambda_0$,

$$\begin{aligned}\|T_\lambda u\|_{PC} \leq& \lambda \int_{\rho(0)}^{\sigma(1)} H(s)c(s)f(s,u(s))\nabla s + \sum_{k=1}^{m} H(t_k))I_k(u(t_k)) \\ <& \lambda_0 \int_{\rho(0)}^{\sigma(1)} H(s)c(s)\nabla s \max_{t\in[0,1]_{\mathbb{T}},x\in[0,r_1]} f(t,x) + \lambda_0 \sum_{k=1}^{m} H(t_k)) \max_{x\in[0,r_1]} I_k(x) \\ =& r_1 = \|u\|_{PC}.\end{aligned} \tag{18}$$

If $\max\{f_0, I_{10}, I_{20}, \ldots, I_{m0}\} = \infty$. Without loss of generality, we assume $f_0 = \infty$. Then there exists $r_2 \in (0, r_1)$ such that $f(t,x) \geq \eta x$ for all $t \in [\delta, 1-\delta]_{\mathbb{T}}$, $x \in [0, r_2]$, where $\eta > 0$ satisfying

$$\lambda\gamma^2\eta \int_{\delta}^{1-\delta} H(s)c(s)\nabla s > 1. \tag{19}$$

Then for $u \in \partial K_{r_2}$, $t \in [\delta, 1-\delta]_{\mathbb{T}}$,

$$\begin{aligned}T_\lambda u(t) \geq& \lambda\gamma \int_{\delta}^{1-\delta} H(s)c(s)f(s,u(s))\nabla s \geq \lambda\gamma\eta \int_{\delta}^{1-\delta} H(s)c(s)u(s)\nabla s \\ \geq& \lambda\gamma^2\eta \int_{\delta}^{1-\delta} H(s)c(s)\nabla s \|u\|_{PC} \geq \|u\|_{PC}.\end{aligned} \tag{20}$$

Thus, $\|T_\lambda u\|_{PC} \geq \|u\|_{PC}$, for $u \in \partial K_{r_2}$.

If $\max\{f_\infty, I_{1\infty}, I_{2\infty}, \ldots, I_{m\infty}\} = \infty$. Without loss of generality, we assume $f^\infty = \infty$. So there exists $\tilde{r} > 0$ such that $f(t,x) > \eta x$ for $t \in [\delta, 1-\delta]_{\mathbb{T}}$, $x \in [\tilde{r}_3, +\infty)$, where $\eta > 0$ satisfies (19). Let $r_3 > \max\{r_1, \tilde{r}_3/\gamma\}$, then for $u \in \partial K_{r_3}$, $\min_{t\in[\delta,1-\delta]_{\mathbb{T}}} u(t) \geq \gamma\|u\|_{PC} > \tilde{r}_3$. As a result, we have (20). Thus, $\|T_\lambda u\|_{PC} \geq \|u\|_{PC}$, for $u \in \partial K_{r_3}$.

Then Guo-Krasnosel'skii fixed point theorem implies that T_λ has a fixed point in $\overline{K}_{r_1} \setminus K_{r_2}$ or $\overline{K}_{r_3} \setminus K_{r_1}$ or according to whether $\max\{f_0, I_{10}, I_{20}, \ldots, I_{m0}\} = \infty$ or $\max\{f_\infty, I_{1\infty}, I_{2\infty}, \ldots, I_{m\infty}\} = \infty$, respectively. Consequently, Problem (2) has a positive solution for $0 < \lambda < \lambda_0$. □

Theorem 5. *Assume that* (H_1) *and* (H_4) *hold, if* $f^0 + \sum_{k=1}^{m} I_k^0 = f^\infty + \sum_{k=1}^{m} I_k^\infty = 0$, *then there exists* $\lambda_0 > 0$ *such that Problem (2) has two positive solutions for* $\lambda > \lambda_0$.

Proof. Choose two numbers $0 < r_3 < \gamma r_4$ and

$$\lambda_0 = \frac{r_1}{\gamma \int_{\delta}^{1-\delta} H(s)c(s)\nabla s \min_{t\in[\delta,1-\delta]_{\mathbb{T}},x\in[\gamma r_3,r_3]\cup[\gamma r_4,r_4]} f(t,x)}.$$

Similar to (16), we have $\|T_\lambda u\|_{PC} \geq \|u\|_{PC}$, for $\lambda > \lambda_0$, $u \in \partial K_{r_3}$, and $\|T_\lambda u\|_{PC} \geq \|u\|_{PC}$, for $\lambda > \lambda_0$, $u \in \partial K_{r_4}$.

Since $f^0 + \sum_{k=1}^{m} I_k^0 = 0$, from the proof of Theorem 3, we choose $r_1 \in (0, \gamma r_3)$ such that $\|T_\lambda u\|_{PC} \leq \|u\|_{PC}$, for $u \in \partial K_{r_1}$.

Since $f^\infty + \sum_{k=1}^{m} I_k^\infty = 0$, from the proof of Theorem 3, we choose $r_2 > r_4/\gamma$ such that $\|T_\lambda u\|_{PC} \leq \|u\|_{PC}$, for $u \in \partial K_{r_2}$.

From Guo-Krasnosel'skii fixed point theorem, T_λ has two fixed points u^* and u^{**} such that $u^* \in \overline{K}_{r_3} \setminus K_{r_1}$ and $u^{**} \in \overline{K}_{r_2} \setminus K_{r_4}$. These are the desired distinct positive solutions of Problem (2) for $\lambda > \lambda_0$ satisfying $r \leq \|u^*\|_{PC} \leq r_3 < r_4 \leq \|u^{**}\|_{PC} \leq r_2$. □

Theorem 6. *Assume that* (H_1) *and* (H_4) *hold, suppose* $\max\{f_0, I_{10}, I_{20}, \ldots, I_{m0}\} = \max\{f_\infty, I_{1\infty}, I_{2\infty}, \ldots, I_{m\infty}\} = \infty$*, then there exists* $\lambda_0 > 0$ *such that Problem (2) has at least two positive solutions for* $0 < \lambda < \lambda_0$.

Proof. Choose two numbers $0 < r_3 < r_4$ and

$$\lambda_0 = \frac{r_1}{\int_{\rho(0)}^{\sigma(1)} H(s)c(s)\nabla s \max_{t\in[0,1]_{\mathbb{T}}, x\in[0,r_4]} f(t,x) + \sum_{k=1}^m H(t_k)\max_{x\in[0,r_4]} I_k(x)}.$$

Similar to (18), we have $\|T_\lambda u\|_{PC} \leq \|u\|_{PC}$, for $0 < \lambda < \lambda_0$, $u \in \partial K_{r_3}$, and $\|T_\lambda u\|_{PC} \leq \|u\|_{PC}$, for $0 < \lambda < \lambda_0$, $u \in \partial K_{r_4}$.

Since $\max\{f_0, I_{10}, I_{20}, \ldots, I_{m0}\} = \infty$, from the proof of Theorem 4, we choose $r_1 \in (0, \gamma r_3)$ such that $\|T_\lambda u\|_{PC} \geq \|u\|_{PC}$, for $u \in \partial K_{r_1}$.

Since $\max\{f_\infty, I_{1\infty}, I_{2\infty}, \ldots, I_{m\infty}\} = \infty$, from the proof of Theorem 4, we choose $r_2 > r_4/\gamma$ such that $\|T_\lambda u\|_{PC} \geq \|u\|_{PC}$, for $u \in \partial K_{r_2}$.

From Guo-Krasnosel'skii fixed point theorem, T_λ has two fixed points u^* and u^{**} such that $u^* \in \overline{K}_{r_3} \setminus K_{r_1}$ and $u^{**} \in \overline{K}_{r_2} \setminus K_{r_4}$. These are the desired distinct positive solutions of Problem (2) for $\lambda > \lambda_0$ satisfying $r \leq \|u^*\|_{PC} \leq r_3 < r_4 \leq \|u^{**}\|_{PC} \leq r_2$. □

Theorem 7. *Assume that* (H_1) *and* (H_4) *hold, if* $f^0 + \sum_{k=1}^m I_k^0 < \infty$ *and* $f^\infty + \sum_{k=1}^m I_k^\infty < \infty$*, then there exists* $\lambda_0 > 0$ *such that Problem (2) has no positive solution for* $0 < \lambda < \lambda_0$.

Proof. Since $f^0 + \sum_{k=1}^m I_k^0 < \infty$ and $f^\infty + \sum_{k=1}^m I_k^\infty < \infty$, then there exist positive numbers ε_1, ε_2, r_1 and r_2 such that $r_1 < r_2$ and

$$f(t,x) \geq \varepsilon_1 x,\ \ I_1(x) \leq \varepsilon_1 x,\ \ x \in [0, r_1],\ \ t \in [0,1]_{\mathbb{T}},$$
$$f(t,x) \geq \varepsilon_2 x,\ \ I_1(x) \leq \varepsilon_2 x,\ \ x \in [r_2, +\infty),\ \ t \in [0,1]_{\mathbb{T}}.$$

Let the positive number ε_3 be defined by

$$\varepsilon_3 = \max\left\{\varepsilon_1, \varepsilon_2, \max_{t\in[0,1]_{\mathbb{T}}, x\in[r_1,r_2]} \frac{f(t,x)}{x}, \max_{1\leq k\leq m}\max_{x\in[r_1,r_2]} \frac{I_k(x)}{x}\right\}.$$

Then

$$f(t,x) \geq \varepsilon_3 x,\ \ I_1(x) \leq \varepsilon_3 x,\ \ x \in [0, +\infty),\ \ t \in [0,1]_{\mathbb{T}}.$$

Assume $v(t)$ is a positive solution of Problem (2). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0 = 1/\left(\varepsilon_3(\int_{\rho(0)}^{\sigma(1)} H(s)c(s)\nabla s + \sum_{k=1}^m H(t_k))\right)$. Since $T_\lambda v(t) = v(t)$ for $t \in [0,1]_{\mathbb{T}}$, then for $0 < \lambda < \lambda_0$ and $\|v\|_{PC} \in [0, r_2]$,

$$\begin{aligned}\|v\|_{PC} =& \|T_\lambda v\|_{PC} \leq \lambda \int_{\rho(0)}^{\sigma(1)} H(s)c(s)f(s,v(s))\nabla s + \sum_{k=1}^m H(t_k))I_k(v(t_k))\\ <& \lambda_0\varepsilon_3\left(\int_{\rho(0)}^{\sigma(1)} H(s)c(s)\nabla s + \sum_{k=1}^m H(t_k))\right)\|v\|_{PC} \geq \|v\|_{PC}\end{aligned}$$

which is a contradiction. Therefore, Problem (2) has no positive solution. □

Theorem 8. *Assume that* (H_1) *and* (H_4) *hold, if* $\max\{f_0, I_{10}, I_{20}, \ldots, I_{m0}\} > 0$ *and* $\max\{f_\infty, I_{1\infty}, I_{2\infty}, \ldots, I_{m\infty}\} > 0$*, then there exists* $\lambda_0 > 0$ *such that Problem (2) has no positive solution for* $\lambda > \lambda_0$.

Proof. Since $\max\{f_0, I_{10}, I_{20}, \ldots, I_{m0}\} > 0$ and $\max\{f_\infty, I_{1\infty}, I_{2\infty}, \ldots, I_{m\infty}\} > 0$. Without loss of generality, we assume that $\max\{f_0, I_{10}, I_{20}, \ldots, I_{m0}\} = I_{10}$ and $\max\{f_\infty, I_{1\infty}, I_{2\infty}, \ldots, I_{m\infty}\} = f_\infty$. Then there exist positive numbers η_1, η_2, r_1 and r_2 such that $r_1 < \gamma r_2$ and

$$I_1(x) \geq \eta_1 x,\ \ x \in [0, r_1],\ \ f(t,x) \geq \eta_2 x,\ \ x \in [r_2, +\infty],\ \ t \in [\delta, 1-\delta]_{\mathbb{T}}.$$

Let the positive number η_3 be defined by

$$\eta_3 = \min\left\{\eta_1, \eta_2, \min_{t\in[\delta,1-\delta]_{\mathbb{T}}, x\in[r_1,r_2]} \frac{f(t,x)}{x}, \min_{x\in[r_1,r_2]} \frac{I_1(x)}{x}\right\}.$$

Then

$$I_1(x) \geq \eta_3 x,\ \ x \in [0, r_2],\ \ f(t,x) \geq \eta_3 x,\ \ x \in [r_1, +\infty],\ \ t \in [\delta, 1-\delta]_{\mathbb{T}}.$$

Assume $v(t)$ is a positive solution of Problem (2). We will show that this leads to a contradiction for $\lambda > \lambda_0 = \max\{1/(\gamma^2\eta_3 h(t_1)), 1/(\gamma^2\eta_3 \int_\delta^{1-\delta} h(s)c(s)\nabla s)\}$. Since $T_\lambda v(t) = v(t)$ for $t \in [0,1]_{\mathbb{T}}$, then for $\lambda > \lambda_0$ and $\|v\|_{PC} \in [0, r_2]$,

$$\|v\|_{PC} = \|T_\lambda v\|_{PC} \geq T_\lambda v(\frac{1}{2}) > \lambda_0 H(\frac{1}{2}, t_1) I_1(v(t_1)) > \lambda_0 \gamma^2 H(t_1)\eta_3 \|v\|_{PC} \geq \|v\|_{PC}$$

which is a contradiction. For $\lambda > \lambda_0$ and $\|v\|_{PC} \in [r_1/\gamma, +\infty)$,

$$\begin{aligned}\|v\|_{PC} =& \|T_\lambda v\|_{PC} \geq T_\lambda v(\frac{1}{2}) > \lambda_0 \int_\delta^{1-\delta} H(\frac{1}{2}, s)c(s)f(s, v(s))\nabla s\\ >& \lambda_0\gamma^2\eta_3 \int_\delta^{1-\delta} H(s)c(s)\nabla s \|v\|_{PC} \geq \|v\|_{PC}\end{aligned}$$

which is a contradiction. Therefore, Problem (2) has no positive solution. □

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