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ON THE NATURAL q^2-ANALOGUE OF THE GENERALIZED GEGENBAUER FORM

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Abstract. The aim of this paper is to highlight a q^2-analogue of the generalized Gegenbauer polynomials orthogonal with respect to the form $\mathcal{G}(\alpha,\beta,q^2)$. Integral representation and discrete measure of $\mathcal{G}(\alpha,\beta,q^2)$ are given for some values of parameters.

1. Introduction

The generalized Gegenbauer orthogonal polynomials is the one of monic orthogonal polynomials sequences which appear in many applications like the weighted L^p mean convergence of Hermite-Fejér interpolation, the Clifford analysis and the Lie algebra A_2 [5, 14, 15]. The generalized Gegenbauer orthogonal polynomials is in connection with the Dunkl-classical character [3].

Denoting by $\{S_n\}_{n\geq 0}$ the (MOPS) of the generalized Gegenbauer polynomials and let $\mathcal{G}(\alpha,\beta)$ be its corresponding regular form. The (MOPS) $\{S_n\}_{n\geq 0}$ satisfies the three-term recurrence relation (see (8) below) [4]

$$\begin{cases} \beta_n = 0, \\ \gamma_{2n+1} = \dfrac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \quad n\geq 0, \\ \gamma_{2n+2} = \dfrac{(n+1)(n+\alpha+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)}, \end{cases} \tag{1}$$

with the positive-definite case occurring for $\alpha > -1$, $\beta > -1$.
In [1], the authors proved that the generalized Gegenbauer form $\mathcal{G}(\alpha,\beta)$ is D-semiclassical of class one satisfying the functional equation

$$D\Big(x(x^2-1)\mathcal{G}(\alpha,\beta)\Big) + \Big\{-2(\alpha+\beta+2)x^2 + 2(\beta+1)\Big\}\mathcal{G}(\alpha,\beta) = 0 \tag{2}$$

for $\alpha \neq -n-1$, $\beta \neq -n-1$, $\beta \neq -\frac{1}{2}$, $\alpha+\beta \neq -n-1$, $n\geq 0$ from which they recovered an integral representation and derived the moments of $\mathcal{G}(\alpha,\beta)$ for all $f\in\mathcal{P}$, $\Re\alpha > -1$, $\Re\beta > -1$

$$\langle \mathcal{G}(\alpha,\beta), f\rangle = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\int_{-1}^{+1}(1-x^2)^\alpha |x|^{2\beta+1} f(x)dx, \tag{3}$$

$$(\mathcal{G}(\alpha,\beta))_{2n} = \frac{\Gamma(\alpha+\beta+2)\Gamma(n+\beta+1)}{\Gamma(\beta+1)\Gamma(n+\alpha+\beta+2)},\ (\mathcal{G}(\alpha,\beta))_{2n+1} = 0,\ n\geq 0. \tag{4}$$

For other characterizations of the generalized Gegenbauer polynomials as a consequence of its D-semiclassical character see [2]. To enrich the quantum calculus it is interesting to build some q-analogous of the generalized Gegenbauer polynomials. In fact, the problem

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of defining q-analogous of symmetrical (MOPS) has been the interest of some authors from different point of views [4, 6, 7, 13].

In [7], the classification of the symmetric H_q-semiclassical orthogonal q-polynomials of class one is given where H_q is the q-difference operator. Among the obtained canonical situations we get the natural q^2-analogue of the generalized Gegenbauer polynomials $\{S_n(.,q^2)\}_{n\geq 0}$ orthogonal with respect to the form $\mathcal{G}(\alpha,\beta,q^2)$ for $\alpha+\beta\neq\frac{3-2q^2}{q^2-1}$, $\alpha+\beta\neq -[n]_{q^2}-2$, $\beta\neq -[n]_{q^2}-1$, $\alpha+\beta+2-(\beta+1)q^{2n}+[n]_{q^2}\neq 0$, $n\geq 0$ and having the recurrence coefficients

$$\begin{cases} \beta_n = 0, \\ \gamma_{2n+1} = q^{2n}\dfrac{(\alpha+\beta+2+[n-1]_{q^2})(\beta+1+[n]_{q^2})}{(\alpha+\beta+2+[2n-1]_{q^2})(\alpha+\beta+2+[2n]_{q^2})}, \\ \gamma_{2n+2} = q^{2n}[n+1]_{q^2}\dfrac{\alpha+\beta+2-(\beta+1)q^{2n}+[n]_{q^2}}{(\alpha+\beta+2+[2n]_{q^2})(\alpha+\beta+2+[2n+1]_{q^2})}, \end{cases} \qquad n\geq 0, \tag{5}$$

where

$$[n]_q := \frac{q^n-1}{q-1}, \quad q\neq 1, \quad n\geq 0. \tag{6}$$

Also in that work it is showed that the form $\mathcal{G}(\alpha,\beta,q^2)$ is H_q-semiclassical of class one for $\alpha+\beta\neq\frac{3-2q^2}{q^2-1}$, $\alpha+\beta\neq -[n]_{q^2}-2$, $\beta\neq -[n]_{q^2}-1$, $\alpha+\beta+2-(\beta+1)q^{2n}+[n]_{q^2}\neq 0$, $n\geq 0$, $\beta\neq\frac{1}{q(q+1)}-1$ satisfying the q-distributional equation

$$H_q\Big(x(x^2-1)\mathcal{G}(\alpha,\beta,q^2)\Big)-(q+1)\Big((\alpha+\beta+2)x^2-(\beta+1)\Big)\mathcal{G}(\alpha,\beta,q^2)=0. \tag{7}$$

When $q\to 1$ in (5) and (7) we recover (1)-(2) since $[n]_q$ tends to n and H_q tends to D. So the aim of our contribution is to highlight the moments, integral representation and discrete measure for $\mathcal{G}(\alpha,\beta,q^2)$ when it is possible.

2. Preliminaries

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u,f\rangle$ the effect of a form $u\in\mathcal{P}'$ (linear functional) on $f\in\mathcal{P}$. In particular, we denote by $(u)_n := \langle u,x^n\rangle$, $n\geq 0$ the moments of u. Let $\{P_n\}_{n\geq 0}$ be a sequence of monic polynomials with $\deg P_n = n$, $n\geq 0$. The sequence $\{P_n\}_{n\geq 0}$ is called orthogonal (MOPS) if we can associate with it a form u ($(u)_0=1$) and a sequence of numbers $\{r_n\}_{n\geq 0}$ ($r_n\neq 0$, $n\geq 0$) such that

$$\langle u, P_mP_n\rangle = r_n\delta_{n,m}, \quad n,m\geq 0$$

and the form u is then said regular. The (MOPS) $\{P_n\}_{n\geq 0}$ fulfils the three-term recurrence relation

$$\begin{cases} P_0(x)=1,\ P_1(x)=x-\beta_0, \\ P_{n+2}(x)=(x-\beta_{n+1})P_{n+1}(x)-\gamma_{n+1}P_n(x),\ n\geq 0, \end{cases} \tag{8}$$

where

$$\beta_n=\frac{\langle u,xP_n^2\rangle}{r_n};\quad \gamma_{n+1}=\frac{r_{n+1}}{r_n}\neq 0,\quad n\geq 0.$$

The regular form u is positive definite if and only if $\forall n\geq 0$, $\beta_n\in\mathbb{R}$, $\gamma_{n+1}>0$. Also, its corresponding (MOPS) $\{P_n\}_{n\geq 0}$ is symmetric if and only if $\beta_n=0$, $n\geq 0$ or equivalently $(u)_{2n+1}=0$, $n\geq 0$.

Let us introduce some useful operations in $\mathcal{P}'$. For any form u, any $a \in \mathbb{C} - \{0\}$, any $c \in \mathbb{C}$ and any $q \neq 1$, we let $Du = u'$, $h_a u$, $(x-c)^{-1}u$ and $H_q u$, be the forms defined by duality [11, 12]

$$\langle u', f\rangle := -\langle u, f'\rangle,\ \langle h_a u, f\rangle := \langle u, h_a f\rangle,\ \langle (x-c)^{-1}u, f\rangle := \langle u, \theta_c f\rangle,$$

and

$$\langle H_q u, f\rangle := -\langle u, H_q f\rangle,$$

for all $f \in \mathcal{P}$ where

$$(h_a f)(x) = f(ax), \quad (\theta_c f)(x) = \frac{f(x) - f(c)}{x - c}, \quad (H_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}\ [8].$$

We will usually suppose that $q \in \widetilde{\mathbb{C}} := \mathbb{C} - \left(\{0\} \bigcup \left(\bigcup_{n \geq 0} \{z \in \mathbb{C},\ z^n = 1\}\right)\right)$. When $q \to 1$, we again meet the derivative D.

A form u is called H_q-semiclassical when it is regular and there exist two polynomials Φ and Ψ, Φ monic, $\deg \Phi = t \geq 0$, $\deg \Psi = p \geq 1$ such that

$$H_q(\Phi u) + \Psi u = 0 \tag{9}$$

the corresponding orthogonal sequence $\{P_n\}_{n\geq 0}$ is called H_q-semiclassical [9]. The H_q-semiclassical form u is said to be of class $s = \max(p-1, t-2) \geq 0$ if and only if [10]

$$\prod_{c \in \mathcal{Z}_\Phi} \{|q(h_q \Psi)(c) + (H_q \Phi)(c)| + |\langle u, q(\theta_{cq}\Psi) + (\theta_{cq} \circ \theta_c \Phi)\rangle|\} > 0, \tag{10}$$

where $\mathcal{Z}_\Phi$ is the set of zeros of Φ.

Remark 1. *When $q \to 1$ in* (9)-(10) *we meet the D-semiclassical character* [11, 12].

Regarding integral representations through weight-functions for a H_q-semiclassical form u satisfying (9), we look for a function U such that

$$\langle u, f\rangle = \int_{-\infty}^{+\infty} U(x) f(x) dx,\ f \in \mathcal{P}, \tag{11}$$

where we suppose that U is regular as far as necessary. On account of (9), we get [9]

$$\int_{-\infty}^{+\infty} \left\{q^{-1}\left(H_{q^{-1}}(\Phi U)\right)(x) + \Psi(x) U(x)\right\} f(x) dx = 0,\ f \in \mathcal{P},$$

with the additional condition [9]

$$\lim_{\epsilon \to +0} \int_\epsilon^1 \frac{U(x) - U(-x)}{x} dx \tag{12}$$

exists or is continuous at the origin. Therefore

$$q^{-1}\left(H_{q^{-1}}(\Phi U)\right)(x) + \Psi(x) U(x) = \lambda g(x), \tag{13}$$

where $\lambda \in \mathbb{C}$ and g is a locally integrable function with rapid decay representing the null form. For instance

$$g(x) = \begin{cases} 0\,, & x \leq 0, \\ \mathrm{e}^{-x^{\frac{1}{4}}} \sin x^{\frac{1}{4}}, & x > 0, \end{cases}$$

was given by Stieltjes [16]. When $\lambda = 0$, the equation (13) becomes

$$\Phi(q^{-1}x) U(q^{-1}x) = \{\Phi(x) + (q-1)x\Psi(x)\} U(x),$$

so that, if $q > 1$, we have

$$U(q^{-1}x) = \frac{\Phi(x) + (q-1)x\Psi(x)}{\Phi(q^{-1}x)} U(x),\; x \in \mathbb{R}, \tag{14}$$

and if $0 < q < 1$, with $x \to qx$, we have

$$U(qx) = \frac{\Phi(x)}{\Phi(qx) + (q-1)qx\Psi(qx)} U(x),\; x \in \mathbb{R}. \tag{15}$$

Lastly, let us recall the following standard expressions needed to the q-calculus in the sequel [6, 9]

$$(a;q)_0 := 1;\; (a;q)_n := \prod_{k=1}^{n}(1 - aq^{k-1}),\; n \geq 1, \tag{16}$$

$$(a;q)_\infty := \prod_{k=0}^{+\infty}(1 - aq^k),\; |q| < 1, \tag{17}$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty},\; 0 < q < 1. \tag{18}$$

$$(a;q)_n = (-1)^n a^n (a^{-1};q^{-1})_n\, q^{\frac{1}{2}n(n-1)},\; n \geq 0, \tag{19}$$

the q-binomial theorem

$$\sum_{k=0}^{+\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty},\; |z| < 1,\; |q| < 1, \tag{20}$$

the q-analogue of the exponential function

$$\sum_{k=0}^{+\infty} \frac{q^{\frac{1}{2}k(k-1)}}{(q;q)_k} z^k = (-z;q)_\infty,\; |q| < 1. \tag{21}$$

3. Moments, discrete measure and integral representation of $\mathcal{G}(\alpha, \beta, q^2)$

Firstly, let us state this technical lemma needed to the sequel and is easy to establish:

Lemma 1. *Let*

$$\xi_\mu(q) = 1 + (\mu+1)(1-q^2), \quad q > 0,\; \mu > -1, \tag{22}$$

and

$$q_{(\mu,\omega)} = \sqrt{1 + \frac{\omega}{\mu+1}},\; \mu > -1,\; \omega > -\mu - 1. \tag{23}$$

We have

$$\begin{gathered} \xi_\mu(q) = 1 \Longleftrightarrow q = 1,\; \xi_\mu(q) = 0 \Longleftrightarrow q = q_{(\mu,1)},\; \xi_\mu(q) = -1 \Longleftrightarrow q = q_{(\mu,2)}, \\ \xi_\mu(q) < -1 \Longleftrightarrow q \in]q_{(\mu,2)}, +\infty[,\; -1 < \xi_\mu(q) < 0 \Longleftrightarrow q \in]q_{(\mu,1)}, q_{(\mu,2)}[, \\ 0 < \xi_\mu(q) < 1 \Longleftrightarrow q \in]1, q_{(\mu,1)}[,\; \xi_\mu(q) > 1 \Longleftrightarrow q \in]0, 1[. \end{gathered} \tag{24}$$

Secondly, from (5) and according to the lemma 1, the natural q^2-analogue of the generalized Gegenbauer orthogonal polynomials is positive definite for $0 < q < 1$, $\alpha > -1$, $\beta > -1$ or $1 < q < q_{(\beta,1)}$, $\alpha > -1$, $\beta > -1$.

Thirdly, from the H_q-semiclassical of class one conditions $\alpha + \beta \neq \frac{3-2q^2}{q^2-1}$, $\alpha + \beta \neq -[n]_{q^2} - 2$, $\beta \neq -[n]_{q^2} - 1$, $\alpha + \beta + 2 - (\beta+1)q^{2n} + [n]_{q^2} \neq 0$, $n \geq 0$, $\beta \neq \frac{1}{q(q+1)} - 1$ concerning the form $\mathcal{G}(\alpha, \beta, q^2)$ and by virtue of the lemma 1 another time we get

$$\begin{cases} \xi_{\alpha+\beta+1}(q) \neq 0,\; \xi_\beta(q) \neq q^{-1}, \\ \xi_{\alpha+\beta+1}(q) \neq q^{2n},\; \xi_\beta(q) \neq q^{2n},\; \xi_{\alpha+\beta+1}(q) \neq q^{2n}\xi_\beta(q),\; n \geq 0. \end{cases} \tag{25}$$

Now, we are able to highlight discrete measure and integral representations of $\mathcal{G}(\alpha,\beta,q^2)$ in the positive definite case and for some values of parameters.

Proposition 1. *The form $\mathcal{G}(\alpha,\beta,q^2)$ has the following properties.*

(1) The moments of $\mathcal{G}(\alpha,\beta,q^2)$ are

$$\begin{aligned} &(\mathcal{G}(\alpha,\beta,q^2))_{2n+1}=0, && n\geq 0,\\ &(\mathcal{G}(\alpha,\beta,q^2))_0=1, (\mathcal{G}(\alpha,\beta,q^2))_{2n}=\frac{\prod_{k=1}^{n}\left(q^{2k-2}-\xi_\beta(q)\right)}{\prod_{k=1}^{n}\left(q^{2k-2}-\xi_{\alpha+\beta+1}(q)\right)}, && n\geq 1.\end{aligned} \tag{26}$$

(2) For all $\alpha>-1$, $\beta>-1$ and $0<q<1$, the form $\mathcal{G}(\alpha,\beta,q^2)$ has the discrete measure

$$\mathcal{G}(\alpha,\beta,q^2)=\frac{\left((\xi_\beta(q))^{-1};q^2\right)_\infty}{\left((\xi_{\alpha+\beta+1}(q))^{-1};q^2\right)_\infty}\sum_{k=0}^{+\infty}\Delta_k\left(\delta_{q^k\sqrt{\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}}}+\delta_{-q^k\sqrt{\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}}}\right) \tag{27}$$

where

$$\Delta_k=\frac{(\xi_\beta(q))^{-k}}{2}\sum_{l=0}^{k}\frac{q^{l^2}}{(q^2;q^2)_l(q^2;q^2)_{k-l}}\left(-\frac{\xi_\beta(q)}{q\xi_{\alpha+\beta+1}(q)}\right)^l,\ k\geq 0. \tag{28}$$

(3) For all $\alpha>-1$, $\beta>-1$ and $1<q<q_{(\beta,1)}$, the form $\mathcal{G}(\alpha,\beta,q^2)$ has the discrete measure

$$\mathcal{G}(\alpha,\beta,q^2)=\frac{\left(\xi_\beta(q);q^{-2}\right)_\infty}{\left(\xi_{\alpha+\beta+1}(q);q^{-2}\right)_\infty}\sum_{k=0}^{+\infty}\Lambda_k\left(\delta_{-q^{-k}}+\delta_{q^{-k}}\right) \tag{29}$$

where

$$\Lambda_k=\frac{(\xi_\beta(q))^{k}}{2}\sum_{l=0}^{k}\frac{q^{-l^2}}{(q^{-2};q^{-2})_l(q^{-2};q^{-2})_{k-l}}\left(-\frac{q\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}\right)^l,\ k\geq 0. \tag{30}$$

Proof. For (1), equivalently with (7), we have

$$\langle H_q\left(x(x^2-1)\mathcal{G}(\alpha,\beta,q^2)\right)-(q+1)\left((\alpha+\beta+2)x^2-(\beta+1)\right)\mathcal{G}(\alpha,\beta,q^2),x^n\rangle=0,\ n\geq 0.$$

Consequently, according to the symmetric character of this form and the definition in (22), this yields the recurrence relation

$$\begin{cases}\left(\mathcal{G}(\alpha,\beta,q^2)\right)_0=1;\ \left(\mathcal{G}(\alpha,\beta,q^2)\right)_1=0,\\ \left(q^n-\xi_{\alpha+\beta+1}(q)\right)\left(\mathcal{G}(\alpha,\beta,q^2)\right)_{n+2}=\left(q^n-\xi_\beta(q)\right)\left(\mathcal{G}(\alpha,\beta,q^2)\right)_n,\ n\geq 0.\end{cases}$$

Thus the desired result (26) since the properties in (25).

To establish (27) and (29), by virtue of (24)-(25) and (16)-(19) we may write the moment of index even in (26) as follows: for all $n\geq 0$

$$\left(\mathcal{G}(\alpha,\beta,q^2)\right)_{2n}=\begin{cases}\left(\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}\right)^n\frac{\left((\xi_\beta(q))^{-1};q^2\right)_\infty}{\left((\xi_{\alpha+\beta+1}(q))^{-1};q^2\right)_\infty}\frac{\left((\xi_{\alpha+\beta+1}(q))^{-1}q^{2n};q^2\right)_\infty}{\left((\xi_\beta(q))^{-1}q^{2n};q^2\right)_\infty}, & 0<q<1,\\ \frac{\left(\xi_\beta(q);q^{-2}\right)_\infty}{\left(\xi_{\alpha+\beta+1}(q);q^{-2}\right)_\infty}\frac{\left(\xi_{\alpha+\beta+1}(q)q^{-2n};q^{-2}\right)_\infty}{\left(\xi_\beta(q)q^{-2n};q^{-2}\right)_\infty}, & q>1.\end{cases} \tag{31}$$

But, by the q-binomial theorem (20), the q-analogue of the exponential function (21), the two latest properties in (24) and since

$$\forall n\geq 0,\ \forall q\in]0,1[,\ 0<q^{2n}(\xi_\beta(q))^{-1}<1;\ \forall n\geq 0,\ \forall q\in]1,q_{(\beta,1)}[,\ 0<q^{-2n}\xi_\beta(q)<1,$$

the equality in (31) yields to

$$c\langle \mathcal{G}(\alpha,\beta,q^2), x^{2n}\rangle = \left(\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}\right)^n \frac{\left((\xi_\beta(q))^{-1};q^2\right)_\infty}{\left((\xi_{\alpha+\beta+1}(q))^{-1};q^2\right)_\infty}$$
$$\times \sum_{k=0}^{+\infty} \frac{(\xi_\beta(q))^{-k}\, q^{2nk}}{(q^2;q^2)_k} \sum_{k=0}^{+\infty} \frac{(-1)^k q^{k(k-1)}(\xi_{\alpha+\beta+1}(q))^{-k}\, q^{2nk}}{(q^2;q^2)_k},\ 0<q<1,\ n\geq 0, \tag{32}$$

and

$$\langle \mathcal{G}(\alpha,\beta,q^2), x^{2n}\rangle = \frac{\left(\xi_\beta(q);q^{-2}\right)_\infty}{\left(\xi_{\alpha+\beta+1}(q);q^{-2}\right)_\infty}$$
$$\times \sum_{k=0}^{+\infty} \frac{(-1)^k q^{-k(k-1)}(\xi_{\alpha+\beta+1}(q))^k q^{-2nk}}{(q^{-2};q^{-2})_k} \sum_{k=0}^{+\infty} \frac{(\xi_\beta(q))^k q^{-2nk}}{(q^{-2};q^{-2})_k},\ 1<q<q_{(\beta,1)},\ n\geq 0. \tag{33}$$

Using the Cauchy product between the two power series in (32) since and those in (33), according to the definitions in (28) and (30) we get successively for all $n\geq 0$

$$\langle \mathcal{G}(\alpha,\beta,q^2), x^{2n}\rangle = 2\frac{\left((\xi_\beta(q))^{-1};q^2\right)_\infty}{\left((\xi_{\alpha+\beta+1}(q))^{-1};q^2\right)_\infty}\sum_{k=0}^{+\infty}\Delta_k\left(q^k\sqrt{\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}}\right)^{2n},\ 0<q<1,$$

$$\langle \mathcal{G}(\alpha,\beta,q^2), x^{2n}\rangle = 2\frac{\left(\xi_\beta(q);q^{-2}\right)_\infty}{\left(\xi_{\alpha+\beta+1}(q);q^{-2}\right)_\infty}\sum_{k=0}^{+\infty}\Lambda_k(q^{-k})^{2n},\ 1<q<q_{(\beta,1)}.$$

By the fact that the form $\mathcal{G}(\alpha,\beta,q^2)$ is symmetric we obtain the desired results (27) and (29). Thus, the points (2)-(3) are proved. □

Proposition 2. *The form $\mathcal{G}(\alpha,\beta,q^2)$ has the following integral representations.*

(1) For $-1<\alpha<0$, $\beta>-1$, $0<q<1$ and for all $f\in\mathcal{P}$

$$\langle \mathcal{G}(\alpha,\beta,q^2), f\rangle = K_1\int_{-1}^{+1}|x|^{-\frac{\ln\xi_\beta(q)}{\ln q}-1}\frac{\left(\frac{q^2\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x^2;q^2\right)_\infty}{(x^2;q^2)_\infty}\left|\sin\left(2\pi\frac{\ln|x|}{\ln q}\right)\right| f(x)dx, \tag{34}$$

where

$$K_1^{-1} = 2\int_0^{+1}x^{-\frac{\ln\xi_\beta(q)}{\ln q}-1}\frac{\left(\frac{q^2\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x^2;q^2\right)_\infty}{(x^2;q^2)_\infty}\left|\sin\left(2\pi\frac{\ln|x|}{\ln q}\right)\right| dx. \tag{35}$$

(2) For $\alpha\geq 0$, $\beta>-1$, $q_{(\alpha+\beta+1,-\alpha)}<q<1$ and for all $f\in\mathcal{P}$

$$\langle \mathcal{G}(\alpha,\beta,q^2), f\rangle = K_2\int_{-\sqrt{\frac{\xi_\beta(q)}{q^2\xi_{\alpha+\beta+1}(q)}}}^{+\sqrt{\frac{\xi_\beta(q)}{q^2\xi_{\alpha+\beta+1}(q)}}}|x|^{-\frac{\ln\xi_\beta(q)}{\ln q}-1}\frac{\left(\frac{q^2\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x^2;q^2\right)_\infty}{(x^2;q^2)_\infty}f(x)dx, \tag{36}$$

where

$$K_2^{-1} = 2\int_0^{+\sqrt{\frac{\xi_\beta(q)}{q^2\xi_{\alpha+\beta+1}(q)}}}x^{-\frac{\ln\xi_\beta(q)}{\ln q}-1}\frac{\left(\frac{q^2\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x^2;q^2\right)_\infty}{(x^2;q^2)_\infty}dx. \tag{37}$$

(3) For $\alpha \geq 0$, $\beta > -1$, $1 < q < q_{(\alpha+\beta+1,1)}$ and for all $f \in \mathcal{P}$

$$\langle \mathcal{G}(\alpha,\beta,q^2), f\rangle = K_3 \int_{-q}^{q} |x|^{-\frac{\ln \xi_\beta(q)}{\ln q}-1} \frac{(q^{-2}x^2;q^{-2})_\infty}{\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x^2;q^{-2}\right)_\infty} f(x)dx, \tag{38}$$

where

$$K_3^{-1} = 2\int_0^q x^{-\frac{\ln \xi_\beta(q)}{\ln q}-1} \frac{(q^{-2}x^2;q^{-2})_\infty}{\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x^2;q^{-2}\right)_\infty} dx. \tag{39}$$

(4) For $-1 < \alpha < 0$, $\beta > -1$, $1 < q < q_{(\alpha+\beta+1,1)}$ and for all $f \in \mathcal{P}$

$$\langle \mathcal{G}(\alpha,\beta,q^2), f\rangle = K_4$$

$$\times \int_{-\sqrt{\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}}}^{\sqrt{\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}}} |x|^{-\frac{\ln \xi_\beta(q)}{\ln q}-1} \frac{(q^{-2}x^2;q^{-2})_\infty}{\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x^2;q^{-2}\right)_\infty} \left|\sin\left(2\pi \frac{\ln\left|\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x\right|}{\ln q^{-1}}\right)\right| f(x)dx, \tag{40}$$

where

$$K_4^{-1} = 2 \int_0^{\sqrt{\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}}} x^{-\frac{\ln \xi_\beta(q)}{\ln q}-1} \frac{(q^{-2}x^2;q^{-2})_\infty}{\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x^2;q^{-2}\right)_\infty} \left|\sin\left(2\pi \frac{\ln\left|\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x\right|}{\ln q^{-1}}\right)\right| dx. \tag{41}$$

Proof. To establish the integral representations in (1)-(4) and by virtue of (11), we look for a function U representing $\mathcal{G}(\alpha,\beta,q^2)$. It is seen from the q-distributional equation (7) that

$$\Phi(x) = x(x^2-1);\ \Psi(x) = -(q+1)\left((\alpha+\beta+2)x^2 - (\beta+1)\right). \tag{42}$$

For (1)-(2), according to (11), (42) and (22), the q-difference equation (15) becomes

$$U(qx) = (q\xi_\beta(q))^{-1} \frac{1-x^2}{1-\frac{q^2\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x^2} U(x). \tag{43}$$

But, taking $\alpha > -1$, $\beta > -1$, $0 < q < 1$, and using (23)-(24) it is quite straightforward to get the following equivalences

$$0 < \frac{\xi_\beta(q)}{q^2\xi_{\alpha+\beta+1}(q)} < 1 \Longleftrightarrow q > q_{(\alpha+\beta+1,-\alpha)}, \tag{44}$$

$$0 < q_{(\alpha+\beta+1,-\alpha)} < 1 \Longleftrightarrow \alpha \geq 0, \tag{45}$$

and

$$q_{(\alpha+\beta+1,-\alpha)} > 1 \Longleftrightarrow \alpha < 0. \tag{46}$$

Consequently, if $-1 < \alpha < 0$, $\beta > -1$, $0 < q < 1$ we seek U as

$$U(x) = \begin{cases} V(x)\dfrac{\left(\frac{q^2\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x^2;q^2\right)_\infty}{(x^2;q^2)_\infty} & , \quad |x| < 1, \\ 0 & , \quad |x| \geq 1. \end{cases} \tag{47}$$

Replacing in (43) this leads to $V(qx) = (q\xi_\beta(q))^{-1}\,V(x)$, therefore

$$V(x) = |x|^{-\frac{\ln \xi_\beta(q)}{\ln q}-1} W(x)$$

with $W(qx) = W(x)$. Taking into account (47) we choose

$$W(x) = K_1 \left| \sin\left(2\pi \frac{\ln|x|}{\ln q}\right) \right|.$$

Thus, for $0 < |x| < \frac{1}{2}$ we have

$$0 \le U(x) \le K_1 |x|^{-\frac{\ln \xi_\beta(q)}{\ln q} - 1} \frac{\left(\frac{q^2 \xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)} x^2; q^2\right)_\infty}{(x^2; q^2)_\infty} \underset{x\to 0}{\sim} \frac{K_1}{|x|^{\frac{\ln \xi_\beta(q)}{\ln q}+1}}, \ \frac{\ln \xi_\beta(q)}{\ln q} + 1 < 1,$$

and

$$U(x) \underset{|x|\to 1}{\sim} \frac{2\pi K_1 \left(\frac{q^2 \xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}; q^2\right)_\infty}{|\ln q| \prod_{k=1}^{+\infty}(1-q^{2k})} \frac{|\ln|x||}{1-x^2} \underset{|x|\to 1}{\longrightarrow} \frac{\pi K_1 \left(\frac{q^2 \xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}; q^2\right)_\infty}{|\ln q| \prod_{k=1}^{+\infty}(1-q^{2k})}.$$

It follows the result in (34) with (35) since the first condition in (12) is valid.

Also, if $\alpha \ge 0$, $\beta > -1$, $q_{(\alpha+\beta+1,-\alpha)} < q < 1$ we seek U as

$$U(x) = \begin{cases} V(x) \dfrac{\left(\frac{q^2 \xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)} x^2; q^2\right)_\infty}{(x^2; q^2)_\infty} & , \quad |x| \le \sqrt{\dfrac{\xi_\beta(q)}{q^2 \xi_{\alpha+\beta+1}(q)}}, \\ 0 & , \quad |x| > \sqrt{\dfrac{\xi_\beta(q)}{q^2 \xi_{\alpha+\beta+1}(q)}}. \end{cases}$$

Replacing in (43) this leads to $V(qx) = (q\xi_\beta(q))^{-1} V(x)$, therefore

$$V(x) = K_2 |x|^{-\frac{\ln \xi_\beta(q)}{\ln q} - 1}.$$

It follows the result in (36) with (37) since the first condition in (12) is valid.

From the hypothesis of (3)-(4), we have $\alpha > -1$, $\beta > -1$, $1 < q < q_{(\alpha+\beta+1,1)}$. By virtue of (11), (42) and (22), the q-difference equation (14) becomes

$$U(q^{-1}x) = q\xi_\beta(q) \frac{1 - \frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)} x^2}{1 - q^{-2}x^2} U(x). \tag{48}$$

According to (24) and (45)-(46) we have

$$0 < \xi_{\alpha+\beta+1}(q) < \xi_\beta(q) < 1, \ 1 < q < \min\left(q_{(\alpha+\beta+1,1)}, q_{(\beta,1)}\right) = q_{(\alpha+\beta+1,1)},$$

$$\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)} > q^2 \Longleftrightarrow q > q_{(\alpha+\beta+1,-\alpha)}.$$

Consequently, if $\alpha \ge 0$, $\beta > -1$, $1 < q < q_{(\alpha+\beta+1,1)}$ we seek U as

$$U(x) = \begin{cases} V(x) \dfrac{\left(q^{-2}x^2; q^{-2}\right)_\infty}{\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)} x^2; q^{-2}\right)_\infty} & , \quad |x| \le q, \\ 0 & , \quad |x| > q. \end{cases}$$

Replacing in (48) this leads to $V(qx) = q\xi_\beta(q) V(x)$, therefore

$$V(x) = K_3 |x|^{-\frac{\ln \xi_\beta(q)}{\ln q} - 1}.$$

It follows the result in (38) with (39) since the first condition in (12) is valid.

Moreover, if $-1 < \alpha < 0$, $\beta > -1$, $1 < q < \min(q_{(\alpha+\beta+1,-\alpha)}, q_{(\alpha+\beta+1,1)}) = q_{(\alpha+\beta+1,1)}$ we seek U as

$$U(x) = \begin{cases} V(x)\dfrac{\left(q^{-2}x^2;q^{-2}\right)_\infty}{\left(\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x^2;q^{-2}\right)_\infty} & , \quad |x| < \sqrt{\dfrac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}}, \\ 0 & , \quad |x| \geq \sqrt{\dfrac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}}. \end{cases} \tag{49}$$

Replacing in (48) this leads to $V(qx) = q\xi_\beta(q)V(x)$, therefore

$$V(x) = |x|^{-\frac{\ln \xi_\beta(q)}{\ln q}-1}W(x),$$

with $W(q^{-1}x) = W(x)$. According to (49), one may choose

$$W(x) = K_4\left|\sin\left(2\pi\frac{\ln\left|\frac{\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}x\right|}{\ln q^{-1}}\right)\right|.$$

It follows the result in (40) with (41) since the first condition in (12) is valid and by a similar reasoning likewise in (1). □

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