ON THE NATURAL $q^2$-ANALOGUE OF THE GENERALIZED GEGENBAUER FORM

I. BEN SALAH AND L. KHÉRIJI

Abstract. The aim of this paper is to highlight a $q^2$-analogue of the generalized Gegenbauer polynomials orthogonal with respect to the form $G(\alpha, \beta, q^2)$. Integral representation and discrete measure of $G(\alpha, \beta, q^2)$ are given for some values of parameters.

1. Introduction

The generalized Gegenbauer orthogonal polynomials is the one of monic orthogonal polynomials sequences which appear in many applications like the weighted $\ell^p$ mean convergence of Hermite-Fejér interpolation, the Clifford analysis and the Lie algebra $A_2$ [5] [13] [15]. The generalized Gegenbauer orthogonal polynomials is in connection with the Dunkl-classical character [3].

Denoting by $\{S_n\}_{n \geq 0}$ the (MOPS) of the generalized Gegenbauer polynomials and let $G(\alpha, \beta)$ be its corresponding regular form. The (MOPS) $\{S_n\}_{n \geq 0}$ satisfies the three-term recurrence relation (see (8) below) [4]

$$
\begin{align*}
\beta_n &= 0, \\
\gamma_{2n+1} &= \frac{(n + \alpha + 1)(n + \alpha + \beta + 1)}{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)} (n + 1)(n + \alpha + 1), \\
\gamma_{2n+2} &= \frac{(2n + \alpha + \beta + 2)(2n + \alpha + \beta + 3)}{(2n + \alpha + \beta + 2)(2n + \alpha + \beta + 3)},
\end{align*}
$$

with the positive-definite case occurring for $\alpha > -1$, $\beta > -1$.

In [1], the authors proved that the generalized Gegenbauer form $G(\alpha, \beta)$ is $D$-semiclassical of class one satisfying the functional equation

$$
D\left(x(x^2 - 1)G(\alpha, \beta)\right) + \left\{ -2(\alpha + \beta + 2)x^2 + 2(\beta + 1) \right\}G(\alpha, \beta) = 0
$$

for $\alpha \neq -n - 1$, $\beta \neq -n - 1$, $\beta \neq -\frac{1}{2}$, $\alpha + \beta \neq -n - 1$, $n \geq 0$ from which they recovered an integral representation and derived the moments of $G(\alpha, \beta)$ for all $f \in P$, $\Re \alpha > -1$, $\Re \beta > -1$

$$
\langle G(\alpha, \beta), f \rangle = \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)} \int_{-1}^{+1} (1 - x^2)^{\alpha} |x|^{2\beta + 1} f(x) dx,
$$

$$
G(\alpha, \beta)_{2n} = \frac{\Gamma(\alpha + \beta + 2)\Gamma(n + \alpha + \beta + 2)}{\Gamma(\beta + 1)\Gamma(n + \alpha + \beta + 2)}, \quad (G(\alpha, \beta))_{2n+1} = 0, \quad n \geq 0.
$$

For other characterizations of the generalized Gegenbauer polynomials as a consequence of its $D$-semiclassical character see [2]. To enrich the quantum calculus it is interesting to build some $q$-analogous of the generalized Gegenbauer polynomials. In fact, the problem

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of defining $q$-analogous of symmetrical (MOPS) has been the interest of some authors from different point of views [4, 6, 7, 13].

In [7], the classification of the symmetric $H_q$-semiclassical orthogonal $q$-polynomials of class one is given where $H_q$ is the $q$-difference operator. Among the obtained canonical situations we get the natural $H$ class one is given where

\[
\begin{cases}
\beta_n = 0, \\
\gamma_{2n+1} = q^{2n} \frac{(\alpha + \beta + 2 + [n-1]_q)\beta + 1 + [n]_q}{(\alpha + \beta + 2 + [2n-1]_q)(\alpha + \beta + 2 + [2n]_q)}, \\
\gamma_{2n+2} = q^{2n}[n+1]_q \frac{(\alpha + \beta + 2 + [2n]_q)(\alpha + \beta + 2 + [2n+1]_q)}{\alpha + \beta + 2 - (\beta + 1)q^{2n} + [n]_q^2},
\end{cases}
\]

where

\[|n|_q := \frac{q^n - 1}{q - 1}, \quad q \neq 1, \quad n \geq 0.\]  

Also in that work it is showed that the form $\mathcal{G}(\alpha, \beta, q^2)$ is $H_q$-semiclassical of class one for $\alpha + \beta \neq \frac{2-q^{(n+1)}}{q-1}$. The corresponding $(MOPS)$ is $H_q$-semiclassical orthogonal with respect to the form $\mathcal{G}(\alpha, \beta, q^2)$ for $\alpha + \beta \neq \frac{2-q^{(n+1)}}{q-1}$, $\gamma = 1$, $\alpha + \beta + 2 - (\beta + 1)q^{2n} + [n]_q^2 \neq 0$, $n \geq 0$ and having the recurrence coefficients

\[
\begin{cases}
\beta_n = 0, \\
\gamma_{2n+1} = q^{2n} \frac{(\alpha + \beta + 2 + [n-1]_q)(\beta + 1 + [n]_q)}{(\alpha + \beta + 2 + [2n-1]_q)(\alpha + \beta + 2 + [2n]_q)}, \\
\gamma_{2n+2} = q^{2n}[n+1]_q \frac{(\alpha + \beta + 2 + [2n]_q)(\alpha + \beta + 2 + [2n+1]_q)}{\alpha + \beta + 2 - (\beta + 1)q^{2n} + [n]_q^2},
\end{cases}
\]

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\[|n|_q := \frac{q^n - 1}{q - 1}, \quad q \neq 1, \quad n \geq 0.\]  

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\[
H_q \left( x(x-1)\mathcal{G}(\alpha, \beta, q^2) \right) - (q+1) \left( (\alpha + \beta + 2)x^2 - (\beta + 1) \right) \mathcal{G}(\alpha, \beta, q^2) = 0.
\]  

When $q \to 1$ in (5) and (7) we recover (1)-(2) since $|n|_q$ tends to $n$ and $H_q$ tends to $D$. So the aim of our contribution is to highlight the moments, integral representation and discrete measure for $\mathcal{G}(\alpha, \beta, q^2)$ when it is possible.

2. Preliminaries

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $(u, f)$ the effect of a form $u \in \mathcal{P}'$ (linear functional) on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := (u, x^n)$, $n \geq 0$ the moments of $u$. Let $\{P_n\}_{n \geq 0}$ be a sequence of monic polynomials with $\deg P_n = n$, $n \geq 0$. The sequence $\{P_n\}_{n \geq 0}$ is called orthogonal (MOPS) if we can associate with it a form $u$ ($(u)_0 = 1$) and a sequence of numbers $\{r_n\}_{n \geq 0}$ ($r_n \neq 0$, $n \geq 0$) such that

\[
(u, P_n P_m) = r_n \delta_{n,m}, \quad n, m \geq 0
\]

and the form $u$ is then said regular. The (MOPS) $\{P_n\}_{n \geq 0}$ fulfills the three-term recurrence relation

\[
\begin{cases}
P_0(x) = 1, \quad P_1(x) = x - \beta_0, \\
P_{n+2}(x) = (x - \beta_{n+1})P_{n+1}(x) - \gamma_{n+1}P_n(x), \quad n \geq 0,
\end{cases}
\]

where

\[
\beta_n = \frac{(u, x P_{n}^2)}{r_n}; \quad \gamma_{n+1} = \frac{r_{n+1}}{r_n} \neq 0, \quad n \geq 0.
\]

The regular form $u$ is positive definite if and only if $\forall n \geq 0$, $\beta_n \in \mathbb{R}$, $\gamma_{n+1} > 0$. Also, its corresponding (MOPS) $\{P_n\}_{n \geq 0}$ is symmetric if and only if $\beta_n = 0$, $n \geq 0$ or equivalently $(u)_{2n+1} = 0$, $n \geq 0$. 

\[60 \quad \text{I. BEN SALAH AND L. KHÉRIJI}\]
Let us introduce some useful operations in $\mathcal{P}$. For any form $u$, any $a \in \mathbb{C} - \{0\}$, any $c \in \mathbb{C}$ and any $q \neq 1$, we let $Du = u'$, $h_a u$, $(x - c)^{-1} u$ and $H_q u$, be the forms defined by duality [11, 12]

$$\langle u', f \rangle := -\langle u, f' \rangle, \langle h_a u, f \rangle := \langle u, h_a f \rangle,$$

and

$$\langle H_q u, f \rangle := -\langle u, H_q f \rangle,$$

for all $f \in \mathcal{P}$ where

$$\langle h_a f \rangle(x) = f(ax), \quad \langle \theta_c f \rangle(x) = \frac{f(x) - f(c)}{x - c}, \quad \langle H_q f \rangle(x) = \frac{f(qx) - f(x)}{(q - 1)x} \ [8].$$

We will usually suppose that $q \in \tilde{C} := \mathbb{C} - \left(\{0\} \cup \bigcup_{n \geq 0} \{z \in \mathbb{C}, z^n = 1\}\right)$. When $q \to 1$, we again meet the derivative $D$.

A form $u$ is called $H_q$-semiclassical when it is regular and there exist two polynomials $\Phi$ and $\Psi$, $\Phi$ monic, $\deg \Phi = t \geq 0$, $\deg \Psi = p \geq 1$ such that

$$H_q(\Phi u) + \Psi u = 0 \ (9)$$

the corresponding orthogonal sequence $\{P_n\}_{n \geq 0}$ is called $H_q$-semiclassical [9]. The $H_q$-semiclassical form $u$ is said to be of class $s = \max(p - 1, t - 2) \geq 0$ if and only if [10]

$$\prod_{c \in \mathcal{Z}_\Phi} \{|q(h_q \Psi)(c) + (H_q \Phi)(c)| + |(u, q(\theta_c \Psi) + (\theta_c \circ \theta_c \Phi))|\} > 0, \quad (10)$$

where $\mathcal{Z}_\Phi$ is the set of zeros of $\Phi$.

**Remark 1.** When $q \to 1$ in [9]-[10] we meet the $D$-semiclassical character [11, 12].

Regarding integral representations through weight-functions for a $H_q$-semiclassical form $u$ satisfying [9], we look for a function $U$ such that

$$\langle u, f \rangle = \int_{-\infty}^{+\infty} U(x) f(x) dx, \quad f \in \mathcal{P}, \quad (11)$$

where we suppose that $U$ is regular as far as necessary. On account of [9], we get [9]

$$\int_{-\infty}^{+\infty} \{q^{-1}(H_{q^{-1}}(\Phi U)) (x) + \Psi(x)U(x)\} f(x) dx = 0, \quad f \in \mathcal{P},$$

with the additional condition [9]

$$\lim_{\varepsilon \to +0} \int_{\varepsilon}^{1} \frac{U(x) - U(-x)}{x} dx \quad (12)$$

exists or is continuous at the origin. Therefore

$$q^{-1}(H_{q^{-1}}(\Phi U)) (x) + \Psi(x)U(x) = \lambda g(x), \quad (13)$$

where $\lambda \in \mathbb{C}$ and $g$ is a locally integrable function with rapid decay representing the null form. For instance

$$g(x) = \begin{cases} 0, & x \leq 0, \\ e^{-x^2} \sin x^2, & x > 0, \end{cases}$$

was given by Stieltjes [16]. When $\lambda = 0$, the equation (13) becomes

$$\Phi(q^{-1}x)U(q^{-1}x) = \{\Phi(x) + (q - 1)x\Psi(x)\} U(x),$$
so that, if \( q > 1 \), we have
\[
U(q^{-1}x) = \frac{\Phi(x) + (q - 1)q\Psi(x)}{\Phi(q^{-1}x)} U(x), \ x \in \mathbb{R}, \quad (14)
\]
and if \( 0 < q < 1 \), with \( x \to qx \), we have
\[
U(qx) = \frac{\Phi(x)}{\Phi(qx) + (q - 1)qx\Psi(qx)} U(x), \ x \in \mathbb{R}. \quad (15)
\]
Lastly, let us recall the following standard expressions needed to the sequel \[6, 9\]
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\]
\[
(a; q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}), \ n \geq 1, \quad (16)
\]
\[
(a; q)_\infty := \prod_{k=0}^{+\infty} (1 - aq^k), \ |q| < 1, \quad (17)
\]
\[
(a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}, \ 0 < q < 1. \quad (18)
\]
\[
(a; q)_n = (-1)^n a^n(a^{-1}; q^{-1})_n q^{\frac{1}{2}n(n-1)}, \ n \geq 0, \quad (19)
\]
the \( q \)-binomial theorem
\[
\sum_{k=0}^{+\infty} (a; q)_k (q; q)_k z^k = \frac{(a z; q)_\infty}{(z; q)_\infty}, \ |z| < 1, \ |q| < 1, \quad (20)
\]
the \( q \)-analogue of the exponential function
\[
\sum_{k=0}^{+\infty} \frac{q^{\frac{1}{2}k(k-1)}}{(q; q)_k} z^k = (1 - z; q)_\infty, \ |q| < 1. \quad (21)
\]

3. MOMENTS, DISCRETE MEASURE AND INTEGRAL REPRESENTATION OF \( G(\alpha, \beta, q^2) \)

Firstly, let us state this technical lemma needed to the sequel and is easy to establish:

**Lemma 1.** Let
\[
\xi_{\mu}(q) = 1 + (\mu + 1)(1 - q^2), \quad q > 0, \ \mu > -1, \quad (22)
\]
and
\[
q(\mu, \omega) = \sqrt{1 + \frac{\omega}{\mu + 1}}, \ \mu > -1, \ \omega > -\mu - 1. \quad (23)
\]
We have
\[
\xi_{\mu}(q) = 1 \iff q = 1, \ \xi_{\mu}(q) = 0 \iff q = q(\mu, 1), \ \xi_{\mu}(q) = -1 \iff q = q(\mu, 2), \quad (24)
\]
\[
0 < \xi_{\mu}(q) < 1 \iff q \in [q(\mu, 1), q(\mu, 2)[, \ \xi_{\mu}(q) > 1 \iff q \in ]0, 1[. \quad (24)
\]
Secondly, from \[5\] and according to the lemma \[1\], the natural \( q \)-analogue of the generalized Gegenbauer orthogonal polynomials is positive definite for \( 0 < q < 1, \ \alpha > -1, \ \beta > -1 \) or \( 1 < q < q(\beta, 1), \ \alpha > -1, \ \beta > -1 \).

Thirdly, from the \( H_q \)-semiclassical of class one conditions \( \alpha + \beta \neq \frac{3 - 2q^2}{q^2 - 1}, \ \alpha + \beta \neq -[n]q^2 - 2, \ \beta \neq -[n]q^2 - 1, \ \alpha + \beta + 2 - (\beta + 1)q^{2n} + [n]q^2 \neq 0, \ n \geq 0, \ \beta \neq \frac{1}{q(q+1)} - 1 \) concerning the form \( G(\alpha, \beta, q^2) \) and by virtue of the lemma \[1\], another time we get
\[
\left\{ \begin{array}{l}
\xi_{\alpha + \beta + 1}(q) \neq 0, \ \xi_{\beta}(q) \neq q^{-1}, \\
\xi_{\alpha + \beta + 1}(q) \neq q^{2n}, \ \xi_{\beta}(q) \neq q^{2n}, \ \xi_{\alpha + \beta + 1}(q) \neq q^{2n}\xi_{\beta}(q), \ n \geq 0.
\end{array} \right. \quad (25)
\]
Now, we are able to highlight discrete measure and integral representations of $G(\alpha, \beta, q^2)$ in the positive definite case and for some values of parameters.

**Proposition 1.** The form $G(\alpha, \beta, q^2)$ has the following properties.

1. The moments of $G(\alpha, \beta, q^2)$ are

$$G(\alpha, \beta, q^2)_{2n+1} = 0,$$

$$G(\alpha, \beta, q^2)_0 = 1, G(\alpha, \beta, q^2)_{2n} = \prod_{k=1}^{n} \left(q^{2k-2} - \xi_\beta(q)\right), \quad n \geq 0. \quad (26)$$

2. For all $\alpha > -1, \beta > -1$ and $0 < q < 1$, the form $G(\alpha, \beta, q^2)$ has the discrete measure

$$G(\alpha, \beta, q^2) = \frac{(\xi_\beta(q)^{-1}; q^2)}{(\xi_{\alpha+\beta+1}(q)^{-1}; q^2)} \sum_{k=0}^{+\infty} \Delta_k \left(q^k \sqrt{\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}} - q^k \sqrt{\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}}\right) \quad (27)$$

where

$$\Delta_k = \frac{\xi_\beta(q)^{-1}}{2} \sum_{l=0}^{k} \frac{q^{2l} \xi_\beta(q)^{l}}{(q^2; q^2)^{l}(q^{-2}; q^{-2})_{k-l}} \left(-\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}\right)^l, \quad k \geq 0. \quad (28)$$

3. For all $\alpha > -1, \beta > -1$ and $1 < q < q(\beta, 1)$, the form $G(\alpha, \beta, q^2)$ has the discrete measure

$$G(\alpha, \beta, q^2) = \frac{(\xi_\beta(q); q^{-2})}{(\xi_{\alpha+\beta+1}(q); q^{-2})} \sum_{k=0}^{+\infty} \Lambda_k \left(q^{-k} - \delta q^{-k}\right) \quad (29)$$

where

$$\Lambda_k = \frac{(\xi_\beta(q)^{-1})^k}{2} \sum_{l=0}^{k} \frac{q^{-2l} \xi_\beta(q)^{l}}{(q^{-2}; q^{-2})^{l}(q^2; q^2)_{k-l}} \left(-\frac{q\xi_{\alpha+\beta+1}(q)}{\xi_\beta(q)}\right)^l, \quad k \geq 0. \quad (30)$$

**Proof.** For [1], equivalently with [7], we have

$$(H_q \left(x^2 - 1\right) G(\alpha, \beta, q^2)) - (q + 1) \left((\alpha + \beta + 2)x^2 - (\beta + 1)\right) G(\alpha, \beta, q^2), x^n = 0, \quad n \geq 0. \quad (22)$$

Consequently, according to the symmetric character of this form and the definition in [22], this yields the recurrence relation

$$\left\{ \begin{array}{l} (G(\alpha, \beta, q^2))_0 = 1; \ G(\alpha, \beta, q^2))_1 = 0, \\
(q^n - \xi_{\alpha+\beta+1}(q)) (G(\alpha, \beta, q^2))_{n+2} = (q^n - \xi_\beta(q)) \left(G(\alpha, \beta, q^2)\right)_n, \quad n \geq 0. \end{array} \right. \quad (27)$$

Thus the desired result [26] since the properties in [25].

To establish [27] and [29], by virtue of [24]-[25] and [16]-[19] we may write the moment of index even in [26] as follows: for all $n \geq 0$

$$(G(\alpha, \beta, q^2))_{2n} = \left\{ \begin{array}{ll} \left(\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}\right)^n (\xi_\beta(q)^{-1}; q^2)_{\infty}, & 0 < q < 1, \\
\left(\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)}\right)^n (\xi_\beta(q)^{-1}; q^{-2})_{\infty}, & q > 1. \end{array} \right. \quad (31)$$

But, by the $q$-binomial theorem [20], the $q$-analogue of the exponential function [21], the two latest properties in [24] and since

$$\forall n \geq 0, \quad \forall q \in ]0, 1[, \quad 0 < q^{2n} (\xi_\beta(q)^{-1})^{-1} < 1; \quad \forall n \geq 0, \quad \forall q \in ]1, q(\beta, 1)[, \quad 0 < q^{-2n} \xi_\beta(q) < 1,$$
the equality in (31) yields to
\[
c^{C}(G(\alpha, q, \beta), x^{2n}) = \left( \frac{\xi_{\beta}(q)}{\xi_{\alpha+\beta+1}(q)} \right)^{n} \frac{((\xi_{\beta}(q))^{-1}; q^{2})_{\infty}}{((\xi_{\alpha+\beta+1}(q))^{-1}; q^{2})_{\infty}} \times \sum_{k=0}^{+\infty} \frac{(-1)^{k}q^{k(1-\beta)}(\xi_{\alpha+\beta+1}(q))^{-k}}{(q^{2}; q^{2})_{k}} \sum_{k=0}^{+\infty} (-1)^{k}q^{k(1-\beta)}(\xi_{\alpha+\beta+1}(q))^{-k} q^{2nk}, \quad 0 < q < 1, \quad n \geq 0,
\]
and
\[
\langle G(\alpha, \beta, q^{2}), x^{2n} \rangle = \frac{((\xi_{\beta}(q))^{-1}; q^{2})_{\infty}}{((\xi_{\alpha+\beta+1}(q)); q^{2})_{\infty}} \times \sum_{k=0}^{+\infty} \frac{(-1)^{k}q^{k(1-\beta)}(\xi_{\alpha+\beta+1}(q))^{-k}}{(q^{2}; q^{2})_{k}} \sum_{k=0}^{+\infty} (-1)^{k}q^{k(1-\beta)}(\xi_{\alpha+\beta+1}(q))^{-k} q^{2nk}, \quad 1 < q < q(\beta, 1), \quad n \geq 0.
\]

Using the Cauchy product between the two power series in (32) since and those in (33), according to the definitions in (28) and (30) we get successively for all \( n \geq 0 \)
\[
\langle G(\alpha, \beta, q^{2}), x^{2n} \rangle = 2 \frac{((\xi_{\beta}(q))^{-1}; q^{2})_{\infty}}{((\xi_{\alpha+\beta+1}(q)); q^{2})_{\infty}} \sum_{k=0}^{+\infty} \Lambda_{k}(q^{2}n), \quad 0 < q < 1,
\]
\[
\langle G(\alpha, \beta, q^{2}), x^{2n} \rangle = 2 \frac{((\xi_{\beta}(q))^{-1}; q^{2})_{\infty}}{((\xi_{\alpha+\beta+1}(q)); q^{2})_{\infty}} \sum_{k=0}^{+\infty} \Lambda_{k}(q^{2}n), \quad 1 < q < q(\beta, 1).
\]

By the fact that the form \( G(\alpha, \beta, q^{2}) \) is symmetric we obtain the desired results and (29). Thus, the points (2)–(3) are proved.

**Proposition 2.** The form \( G(\alpha, \beta, q^{2}) \) has the following integral representations.

1. For \(-1 < \alpha < 0, \beta > -1, \alpha \geq q(\beta, 1) < q < 1\) and for all \( f \in \mathcal{P} \)
\[
\langle G(\alpha, \beta, q^{2}), f \rangle = K_{1} \int_{-1}^{+1} \left| x \right| \frac{\ln \left( \left| x \right| \right)}{\ln q} \frac{\left( q^{2} \xi_{\alpha+\beta+1}(q) \right) x^{2}; q^{2}}{(x^{2}; q^{2})_{\infty}} \sin \left( 2\pi \frac{\ln \left( \left| x \right| \right)}{\ln q} \right) f(x) dx,
\]
where
\[
K_{1}^{-1} = 2 \int_{0}^{+1} x \frac{\ln \left( \left| x \right| \right)}{\ln q} \frac{\left( q^{2} \xi_{\alpha+\beta+1}(q) \right) x^{2}; q^{2}}{(x^{2}; q^{2})_{\infty}} \sin \left( 2\pi \frac{\ln \left( \left| x \right| \right)}{\ln q} \right) dx.
\]
2. For \( \alpha \geq 0, \beta > -1, q(\alpha+\beta+1, -\alpha) < q < 1\) and for all \( f \in \mathcal{P} \)
\[
\langle G(\alpha, \beta, q^{2}), f \rangle = K_{2} \int_{-1}^{+1} \left| x \right| \frac{\ln \left( \left| x \right| \right)}{\ln q} \frac{\left( q^{2} \xi_{\alpha+\beta+1}(q) \right) x^{2}; q^{2}}{(x^{2}; q^{2})_{\infty}} \sin \left( 2\pi \frac{\ln \left( \left| x \right| \right)}{\ln q} \right) f(x) dx,
\]
where
\[
K_{2}^{-1} = 2 \int_{0}^{+1} x \frac{\ln \left( \left| x \right| \right)}{\ln q} \frac{\left( q^{2} \xi_{\alpha+\beta+1}(q) \right) x^{2}; q^{2}}{(x^{2}; q^{2})_{\infty}} \sin \left( 2\pi \frac{\ln \left( \left| x \right| \right)}{\ln q} \right) dx.
\]
(3) For $\alpha \geq 0$, $\beta > -1$, $1 < q < q_{(\alpha + \beta + 1, -1)}$ and for all $f \in \mathcal{P}$

$$\langle G(\alpha, \beta, q^2), f \rangle = K_3 \int_{-q}^{q} |x|^{-\frac{\ln \xi_{\beta}(q)}{\ln q} - 1} \left( \frac{q^{-2}x^2; q^{-2}}{\xi_{\alpha+\beta+1}(q) x^2; q^{-2}} \right) f(x) dx,$$

where

$$K_3^{-1} = 2 \int_{0}^{q} x^{-\frac{\ln \xi_{\beta}(q)}{\ln q} - 1} \left( \frac{q^{-2}x^2; q^{-2}}{\xi_{\alpha+\beta+1}(q) x^2; q^{-2}} \right) dx.$$  

(4) For $-1 < \alpha < 0$, $\beta > -1$, $1 < q < q_{(\alpha + \beta + 1, 1)}$ and for all $f \in \mathcal{P}$

$$\langle G(\alpha, \beta, q^2), f \rangle = K_4 \int_{-q}^{q} \left| x \right|^{-\frac{\ln \xi_{\beta}(q)}{\ln q} - 1} \left( \frac{q^{-2}x^2; q^{-2}}{\xi_{\alpha+\beta+1}(q) x^2; q^{-2}} \right) \sin\left( 2\pi \frac{\ln \left| \frac{\xi_{\alpha+\beta+1}(q)}{\xi_{\beta}(q)} x \right|}{\ln q^{-1}} \right) f(x) dx,$$

where

$$K_4^{-1} = 2 \int_{0}^{q} x^{-\frac{\ln \xi_{\beta}(q)}{\ln q} - 1} \left( \frac{q^{-2}x^2; q^{-2}}{\xi_{\alpha+\beta+1}(q) x^2; q^{-2}} \right) \sin\left( 2\pi \frac{\ln \left| \frac{\xi_{\alpha+\beta+1}(q)}{\xi_{\beta}(q)} x \right|}{\ln q^{-1}} \right) dx.$$  

Proof. To establish the integral representations in (3)-(4) and by virtue of (11), we look for a function $U$ representing $G(\alpha, \beta, q^2)$. It is seen from the $q$-distributional equation (7) that

$$\Phi(x) = x(x^2 - 1); \quad \Psi(x) = -(q + 1) ((\alpha + \beta + 2)x^2 - (\beta + 1)).$$  

For (1)-(2), according to (11), (42) and (22), the $q$-difference equation (15) becomes

$$U(qx) = (q^2\xi_{\beta}(q))^{-1} \left( 1 - \frac{q^2}{1 - q^2\xi_{\alpha+\beta+1}(q) x^2} \right) U(x).$$  

(43) But, taking $\alpha > -1$, $\beta > -1$, $0 < q < 1$, and using (23)-(24) it is quite straightforward to get the following equivalences

$$0 < \frac{\xi_{\beta}(q)}{q^2\xi_{\alpha+\beta+1}(q)} < 1 \iff q > q_{(\alpha + \beta + 1, -\alpha)},$$

(44)

$$0 < q_{(\alpha + \beta + 1, -\alpha)} < 1 \iff \alpha \geq 0,$$

(45)

and

$$q_{(\alpha + \beta + 1, -\alpha)} > 1 \iff \alpha < 0.$$  

(46) Consequently, if $-1 < \alpha < 0$, $\beta > -1$, $0 < q < 1$ we seek $U$ as

$$U(x) = \begin{cases} V(x) \left( \frac{q^2\xi_{\alpha+\beta+1}(q)}{\xi_{\beta}(q)} x^2; q^2 \right) \infty, & |x| < 1, \\ 0, & |x| \geq 1. \end{cases}$$  

(47) Replacing in (43) this leads to $V(qx) = (q^2\xi_{\beta}(q))^{-1} V(x)$, therefore

$$V(x) = |x|^{-\frac{\ln \xi_{\beta}(q)}{\ln q} - 1} W(x)$$ 
with \( W(qx) = W(x) \). Taking into account (47) we choose
\[
W(x) = K_1 \left| \sin \left( 2\pi \frac{\ln |x|}{\ln q} \right) \right|.
\]

Thus, for \( 0 < |x| < \frac{1}{2} \) we have
\[
0 \leq U(x) \leq K_1 |x|^{-\frac{\ln \xi_\beta(q)}{\ln q} - 1} \frac{\left( q^2 \xi_{\alpha+\beta+1}(q) x^2; q^2 \right)_\infty}{(x^2; q^2)_\infty} \sim_{x \to 0} \frac{K_1}{|x|^{-\frac{\ln \xi_\beta(q)}{\ln q} - 1}} \frac{\ln \xi_\beta(q)}{\ln q} + 1 < 1,
\]
and
\[
U(x) \sim_{|x| \to 1} 2\pi K_1 \left( q^2 \xi_{\alpha+\beta+1}(q); q^2 \right)_\infty \ln |x| |\prod_{k=0}^{\infty} (1 - q^{2k})|^{-1} x^2 |x|^{-1} \frac{\ln q}{|\prod_{k=0}^{\infty} (1 - q^{2k})|^{-1}}.
\]

It follows the result in (34) with (35) since the first condition in (12) is valid.

Also, if \( \alpha \geq 0, \beta > -1, q(\alpha+\beta+1,-\alpha) < q < 1 \) we seek \( U \) as
\[
U(x) = \begin{cases} V(x) \frac{\left( q^2 \xi_{\alpha+\beta+1}(q) x^2; q^2 \right)_\infty}{(x^2; q^2)_\infty}, & |x| \leq \sqrt{\frac{\xi_\beta(q)}{q^2 \xi_{\alpha+\beta+1}(q)}}, \\ 0, & |x| > \sqrt{\frac{\xi_\beta(q)}{q^2 \xi_{\alpha+\beta+1}(q)}}. \end{cases}
\]

Replacing in (43) this leads to \( V(qx) = (q \xi_\beta(q))^{-1} V(x) \), therefore
\[
V(x) = K_2 |x|^{-\frac{\ln \xi_\beta(q)}{\ln q} - 1}.
\]

It follows the result in (36) with (37) since the first condition in (12) is valid.

From the hypothesis of (3)-(4), we have \( \alpha > -1, \beta > -1, 1 < q < q(\alpha+\beta+1,1) \). By virtue of (11), (42) and (22), the \( q \)-difference equation (14) becomes
\[
U(q^{-1}x) = q \xi_\beta(q) \frac{1 - \xi_{\alpha+\beta+1}(q) x^2}{1 - q^{-2} x^2} U(x).
\]

According to (24) and (45)-(46) we have
\[
0 < \xi_{\alpha+\beta+1}(q) < \xi_\beta(q) < 1, 1 < q < \min(q(\alpha+\beta+1,1), q(\beta,1)) = q(\alpha+\beta+1,1),
\]
\[
\frac{\xi_\beta(q)}{\xi_{\alpha+\beta+1}(q)} > q^2 \iff q > q(\alpha+\beta+1,-\alpha).
\]

Consequently, if \( \alpha \geq 0, \beta > -1, 1 < q < q(\alpha+\beta+1,1) \) we seek \( U \) as
\[
U(x) = \begin{cases} V(x) \frac{\left( q^{-2} x^2; q^{-2} \right)_\infty}{\xi_\beta(q)}, & |x| \leq q, \\ 0, & |x| > q. \end{cases}
\]

Replacing in (48) this leads to \( V(qx) = q \xi_\beta(q) V(x) \), therefore
\[
V(x) = K_3 |x|^{-\frac{\ln \xi_\beta(q)}{\ln q} - 1}.
\]

It follows the result in (38) with (39) since the first condition in (12) is valid.
Moreover, if \(-1 < \alpha < 0, \beta > -1, 1 < q < \min(q(\alpha + \beta + 1, -\alpha), q(\alpha + \beta + 1, 1)) = q(\alpha + \beta + 1, 1)\) we seek \(U\) as

\[
U(x) = \begin{cases} 
V(x) \left( \frac{q^{-2} x^2; q^{-2}}{\xi(q)} \right), & |x| < \sqrt{\frac{\xi(q)}{\xi(\alpha + \beta + 1)}} \\
0 & |x| \geq \sqrt{\frac{\xi(q)}{\xi(\alpha + \beta + 1)}}.
\end{cases}
\]  

(49)

Replacing in (48) this leads to \(V(qx) = q\xi(q) V(x)\), therefore

\[
V(x) = |x|^{-\ln \xi(q)} W(x),
\]

with \(W(q^{-1}x) = W(x)\). According to (49), one may choose

\[
W(x) = K_4 \left| \frac{\ln \xi(q)}{\ln q^{-1}} \right| \sin \left( 2\pi \frac{\xi(q)}{\xi(q)} x \right).
\]

It follows the result in (40) with (41) since the first condition in (12) is valid and by a similar reasoning likewise in (1).

\[\square\]

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