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STATE SPACE APPROACH FOR SOLVING TRANSIENT QUEUING SYSTEMS

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Abstract. In the present paper, we consider a transient queuing system and find its solution using the state space approach. For this, the given interval of study $[0, T]$ is sub divided into small sub intervals, where it is reasonable to assume that the instantaneous arrival and service rates are constants. The system of equations together with the initial conditions is solved for an analytical expression for P_ns' in each interval. The solution obtained at the right end point of the previous sub interval is taken as the initial condition in the sub sequent interval. The developed method is illustrated for an (M/M/1):(FCFS/m/∞) system.

1. Introduction

Queuing theory, owing to its applications in various real time situations has attracted many researchers from diverse fields like computer science, natural sciences like biology etc. This theory was first introduced by Erlang [1] in 1909, where it was used to describe telephone exchange operations. Later, it found applications in varied fields as can be seen in the references [2, 3, 4].

Queuing systems can be described to follow Markov process as they are characterized by the feature that if a transition occurs, then it leads to a neighboring state. An important subclass of Markov chains, called the Birth death process (BDP) is best suited to model queuing systems.

2. BDP model

Consider a Queueing system with the state of system as n. Let λ_n represents the instantaneous mean arrival rate and μ_n denote the instantaneous mean service rate, and let P_n denote the probability that there are n units in the system at time t. Then, we have

$$P_n' = -\left(\lambda_n + \mu_n\right) P_n(t) + \mu_{n+1} P_{n+1}(t) + \lambda_{n-1} P_{n-1}(t) \quad \forall n = 1, 2, 3.. \tag{1}$$

Assume that $\mu_0 = 0$ and $P_{-1} = 0$.

If the system has i customers initially at time zero, then the initial conditions are

$$P_0 = 1, P_n = 0 \quad \text{for} \quad n \neq i. \tag{2}$$

The conditions for existence and uniqueness of the solutions for above system can be found in the works of Feller [5].

In the analysis of the BDP, emphasis was laid only on its steady state solutions, as the closed form of the transient solutions are, in general, extremely difficult to obtain. In this regard, one may refer to the works done by L. Kleinrock in 1975, who found an explicit solution to this system in terms of infinite sum of modified Bessel functions [6, 7] and Karlin and McGregor, who developed a formal theory of general BDPs that

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expresses the transition probabilities in terms of a sequence of orthogonal polynomials and a spectral measure [8, 9]. But, due to the difficulty of finding computationally useful formulas for transition probabilities in general BDPs, many applied researchers resort to analyses using moments, equilibrium probabilities, continued fraction etc [10, 11]. As can be seen in Literature, closed form solutions are available only for a very few cases, the present paper attempts in obtaining closed form of solutions to the system described in [1], in each sub interval of the given interval $[0, T]$, by applying the state space.

3. State Space Solution to The Problem

Let us write the equation (1) in the form:

$$\begin{aligned} P_0'(t) &= -\lambda_0 P_0(t) + \mu_1 P_1(t) \\ P_1'(t) &= -\mu_0 P_0(t) - (\lambda_{n-1} + \mu_1) + \mu_2 P_2(t) \\ &. \\ &. \\ P_n'(t) &= -\mu_{n-1} P_{n-1}(t) - (\lambda_n + \mu_n)\, P_n(t) \end{aligned} \tag{3}$$

The above system can be written in the matrix form as,

$$\begin{pmatrix} P_0'(t) \\ P_1'(t) \\ P_2'(t) \\ . \\ . \\ P_n'(t) \end{pmatrix} = \begin{pmatrix} -\lambda_0 & \mu_1 & 0 & . & . & . & 0 & 0 & 0 \\ \lambda_0 & -\lambda_0 - \mu_1 & \mu_2 & . & . & . & 0 & 0 & 0 \\ & & & . & . & . & & & \\ & & & . & . & . & & & \\ 0 & 0 & 0 & . & . & . & \lambda_{n-1} & -\lambda_n - \mu_n & \end{pmatrix} \begin{pmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \\ . \\ . \\ P_n(t) \end{pmatrix} \tag{4}$$

If $k_1, \ldots k_n$ are the eigen values of the above system, then the solution to the system (4) is

$$\begin{pmatrix} P_0'(t) \\ P_1'(t) \\ P_2'(t) \\ . \\ . \\ P_n'(t) \end{pmatrix} = e^{At} \begin{pmatrix} P_0(0) \\ P_1(0) \\ P_2(0) \\ . \\ . \\ P_n(0) \end{pmatrix} \tag{5}$$

where

$$e^{At} = a_0 I + a_1 At + \ldots \tag{6}$$

and

$$p_0(0) = 1, p_1(0) = 0, p_2(0) = 0 \ldots \tag{7}$$

Though e^{At} is an infinite series in powers of A, using Cayley Hamilton theorem, it reduces into a finite expression involving n constants $a_0, a_1 ... a_{n-1}$ that are to be determined using:

$$\begin{aligned} a_0 + a_1 K_1 + ... + a_{n-1}(k_1)^{n-1}) &= e^{k_1 t} \\ a_0 + a_1 K_2 + ... + a_{n-1}(k_2)^{n-1}) &= e^{k_2 t} \\ &. \\ &. \\ a_0 + a_1 K_n + ... + a_{n-1}(k_n)^{n-1}) &= e^{k_n t} \end{aligned} \tag{8}$$

Thus, an analytical solution to (1) is obtained in a sub interval $[0, t_0]$. This solution, at t_0, is taken as the initial condition to the subsequent interval $[t_1, t_1]$ and so on.

The above method is illustrated by an example as shown below:

Consider a queuing system with single server where it is assumed that the arrival rate $\lambda_n = 2/(n+1)$ (arrivals tend to get discouraged when more customers are present in the system) $\mu_n = 3$ for $n > 0$. In fact, these arrival and departure rates can be different for different sub intervals, so as to take care of the busy and the non busy hours of a queuing system. Also, the model described above, can be used for multi server systems. For convenient sake, they are taken to be the same constants in all the intervals for this example.

Further, assume that at $T = 0$, there are no customers in the system. For $n = 2$, we have, from (5), that

$$A = \begin{pmatrix} -\lambda_0 & \mu_1 & 0 \\ \lambda_0 & (\lambda_1 + \mu_1) & \mu_2 \\ 0 & \lambda_1 & -(\lambda_2 + \mu_2) \end{pmatrix} = \begin{pmatrix} -2 & 3 & 0 \\ 2 & -4 & 3 \\ 0 & 1 & -\frac{11}{3} \end{pmatrix} \tag{9}$$

The characteristic equation of A is

$$\begin{vmatrix} -2-\lambda & 3 & 0 \\ 2 & -4-\lambda & 3 \\ 0 & 1 & -\frac{11}{3}-\lambda \end{vmatrix} = 0 \tag{10}$$

which on solving gives

$$\begin{aligned} k_1 &= -6.436688664346754 \\ k_2 &= -3.167588925842283 \\ k_3 &= -0.0657224098109674 \end{aligned} \tag{11}$$

Note that, for higher order matrices, diagonalization of a tri diagonal matrix can be found using Francis algorithm. In this paper, evaluations are done using MATHEMATICA.

Substituting the values of k_1, k_2 and k_3 in equation (7) and solving the system of algebraic equations, we have,

$$\begin{aligned} a_0 &= 0.009995598874351606e^{-6.436688664346754t} \\ &\quad -0.041718117870172114e^{-3.167588925842283t} \\ &\quad +1.0317225189958206e^{-0.0657224098109647t} \\ a_1 &= 0.15524372298297265e^{-6.436688664346754t} \\ &\quad -0.6412437511649962e^{-3.167588925842283t} \\ &\quad +0.48600002818202365e^{-0.0657224098109647t} \\ a_2 &= 0.0480138492297735e^{-6.436688664346754t} \\ &\quad -0.09861630460637386e^{-3.675889258422283t} \\ &\quad +0.050602455376600367e^{-0.0657224098109647t} \end{aligned} \tag{12}$$

For $t = 0.1$, the solutions in the interval $[0, 0.1]$

$$
\begin{aligned}
P_0(t) = &+0.009995598874351606e^{-6.436688664346754t}\\
&-0.041718117870172114e^{-3.167588925842283t}\\
&+1.0317225189958206e^{-0.0657224098109647t}\\
&+1.031722189958206e^{-0.06657224098109647t}\\
&+10.(0.0480138492297735e^{-6.436688664346754t}\\
&-0.09861630460637386e^{-3.167588925842283t}\\
&+0.050602455376600367e^{-0.06657224098109647t})\\
&+2.(0.15524372298297265e^{-6.436688664346754t}\\
&-0.6412437511649962e^{-3.167588925842283t}\\
&+0.48600002818202365e^{-0.06657224098109647t})\\
P_1(t) = &-12.(0.0480138492297735e^{-6.436688664346754t}\\
&-0.09861630460637386e^{-3.167588925842283t}\\
&+0.050602455376600367e^{-0.0657224098109647t})\\
&+2.(0.155243722982972265e^{-6.436688664346754t}\\
&-0.6412437511649961e^{-3.167588925842283t}\\
&+0.48600002818202365e^{-0.0657224098109647t})\\
P_2(t) = &+2.(0.0480138492297735e^{-6.436688664346754t}\\
&-0.09861630460637386e^{-3.67588925842283t}\\
&+0.050602455376600367e^{-0.0657224098109647t})
\end{aligned} \tag{13}
$$

And at $t = 0.1$, $P_0(0.1) = 0.842$, $P_1(0.1) = 0.151$, $P_2(0.1) = 0.007$ which is the initial condition for the system (1) in the second interval $[0.1, 0.2]$. Proceeding as above, we have $P_0(0.2) = 0.743$, $P_1(0.2) = 0.234$, $P_2(0.2) = 0.022$. Hence, the queuing system is completely described, using which the queue characteristics can be studied.

S.No	t	P_0	P_1	P_2	L	W
1	0.1	0.842	0.151	0.007	0.165	0.08969
2	0.2	0.743	0.234	0.022	0.278	0.160261
3	0.3	0.679	0.281	0.037	0.355	0.213384
4	0.4	0.636	0.307	0.054	0.415	0.256966
5	0.5	0.607	0.322	0.064	0.45	0.285051
6	0.6	0.586	0.331	0.072	0.475	0.306254
7	0.7	0.571	0.336	0.078	0.492	0.321569
8	0.8	0.559	0.338	0.082	0.502	0.332304
9	0.9	0.550	0.339	0.085	0.509	0.340316
10	1.0	0.542	0.338	0.087	0.512	0.345946
11	1.1	0.536	0.337	0.088	0.513	0.349534
12	1.2	0.530	0.336	0.089	0.514	0.353184
13	1.3	0.525	0.334	0.090	0.514	0.355956
14	1.4	0.520	0.332	0.090	0.512	0.357542
15	1.5	0.516	0.330	0.090	0.51	0.35865
16	1.6	0.515	0.330	0.090	0.51	0.359155

Here L denotes the expected number of customers in the system and W is the average waiting time of a customer in the system. Steady state solution is: $P_0 = 0.513$, $P_1 = 0.342$, $P_2 = 0.114$, $L = 0.67$ and $W = 0.108$.

4. Conclusion

This paper provides analytical solution to the transient queuing system using state space approach. It is observed that, the solutions using the present method approach the steady state solutions. The advantage of the present method of solution is that, the queue characteristics can be found at any desired point of time, which helps in better decision analysis.

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