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A NOTE TO GEOMETRY OF COSSERAT MEDIA AND DEFORMATION BUNDLES

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ABSTRACT. We study Cosserat media from the geometric point of view; in particular, we present a construction of Cosserat deformation bundles and demonstrate the role of the velocities bundles.

1. INTRODUCTION

In the paper, we present some ideas on two views on Cosserat media: both are geometric, but the second one uses more modern formalism of higher order frame bundles. Surely, the development of appropriate mathematical tools for a characterization of new materials is lagging behind the technological advances. Materials, in which changes occur in the molecular or crystalline texture at various microscopic scales (substructure) and influence the macroscopic behaviour through peculiar interactions are commonly known, used and created. Materials such as liquid crystals, ferroelectrics, quasicrystals, polymeric fluids are paradigmatic examples. The attribute complex is assigned to bodies made of these materials in order to underline that significant substructural effects must be accounted for.

So, our mathematical tools are mainly differential geometry methods. In modern categorical approach to differential geometry, if we interpret geometric objects as bundle functors, then natural transformations represent a number of geometric constructions. In this context, the result finding the bijection between natural transformations between two Weil functors (generalizing well-known functors of higher order velocities and, of course, the tangent functor as first of them) and corresponding morphisms of Weil algebras has the fundamental importance. Roughly speaking, Weil bundles generalize higher order velocities bundles including also higher order semiholonomic velocities bundles.Semiholonomic jets and velocities play an important role in the modern physics. However, the description of Weil algebras associated to functors of higher order semiholonomic velocities is not known in general because of their complicated structure mainly in view of some technical problems of a combinatorial character. The case of the order 1 is trivial, the second order case is known in the community of specialists (and described e.g. in [6], where one reference is also to unpublished notes of Ivan Kolář) and we have some new results about third order nonholonomic and semiholonomic velocities bundles and corresponding Weil algebras, see e.g. [6]. In this paper, we present applications of such methods to microstructures as was studied by Marcelo Epstein and his collaborators, see [1], [2], [3].

The brief introduction to the theory of Cosserat continua is in the Section 2, based mainly on [4], but using also [5] and [7].

In the Section 3, we present a construction of Cosserat deformation bundles and demonstrate the role of the velocities bundles in this geometric approach. When studied the

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prolongations of a differential system of higher order connections, Claude Ehresmann was led to use the terminology of mechanics: so, he called jets in question nonholonomic and semiholonomic jets. The construction of their generalized involution, the classification of all symmetrised nonholonomic jets and the geometrical interpretation are studied in detail only for the second order up to now; however, third order semiholonomic velocities are newly described (as mentioned) in [6] by the stucture of their Weil algebra. Hence the her order gradient theory for Cosserat bodies is in a relation with such results and one may find in future very interesting application of what now appears as a pure algebra.

2. Some basics from the Cosserat brothers' theory

Traditional mechanics of continua endowes particles of a material body with translational degrees of freedom, the Cosserat brothers' approach endowes them with both translational and rotational degrees of freedom. In elementary approach a body B of dimension 1 (rods, beams) or 2 (plates, shells) or 3 in $\mathbb{R}^3$ is considered. For each particle of such a body we denote by $\vec{X}$ an initial position and by $\vec{d}^{10}$, $\vec{d}^{20}$ and $\vec{d}^{30}$ initial settings of orthornormal directors (they are 3 for $\dim B = 4$). In time t we have

$\vec{x}(\vec{X},t)$ the actual position of the particle having the initial position $\vec{X}$

$\vec{d}^i(\vec{X},t)$ the actual orientation of the i-th director of the particle having the initial position $\vec{X}$ ($i = 1,2,3$);

so, $\vec{x}(\vec{X},t_0) = \vec{X}$, $\vec{d}^i(\vec{X},t_0) = \vec{d}^{i0}$ and

$$\begin{aligned} \vec{x}(\vec{X},t) &= \vec{X} + \vec{u}(\vec{X},t) \\ \vec{d}^i(\vec{X},t) &= \mathbf{R}(\vec{X},t)\vec{d}^{i0}, \end{aligned}$$

where $\vec{u}(\vec{X},t)$ is the *displacement field*, $\vec{u}(\vec{X},t_0) = \vec{o}$, and $\mathbf{R}(\vec{X},t)$ is the *rotation field*, $\mathbf{R}(\vec{X},t_0) = \mathbf{I}$ (the identity matrix), $\mathbf{R}(\vec{X},t)\mathbf{R}^\top(\vec{X},t) = \mathbf{I}$, $\det \mathbf{R}(\vec{X},t) = 1$; naturally, these fields are assumed smooth. Hence $\mathbf{R}(\vec{X},t) \in \mathrm{SO}(3,\mathbb{R})$. Associating to the Lie group $G = \mathrm{SO}(3,\mathbb{R})$ its Lie algebra $\mathfrak{g} = T_eG = \mathfrak{so}(3,\mathbb{R})$, we recall the well known construction: for $\mathbf{W}(\vec{X},t) \in \mathfrak{so}(3,\mathbb{R})$ we consider the one-parameter subgroup $\gamma_{\mathbf{W}} \colon \mathbb{R} \to \mathfrak{so}(3,\mathbb{R})$ corresponding to $\mathbf{W}$ (i.e. $\dot\gamma_{\mathbf{W}}(0) = \mathbf{W}$); then $\exp \mathbf{W} = \mathbf{I} + \mathbf{W} + \frac{\mathbf{W}^2}{2!} + \frac{\mathbf{W}^3}{3!} + \cdots = \mathbf{R}$ and $\mathbf{W}$ is a skew-symmetric matrix field called the *infinitesimal rotation*. We associate a vector field (which is called the *microrotation field*) $\vec{\Phi}(\vec{X},t)$ to $\mathbf{W}(\vec{X},t)$ by

$$\mathbf{W} = -\epsilon\dot{\vec{\Phi}} \quad \text{(where } \epsilon \text{ is the Levi-Civita tensor)},$$

in coordinates [1], $W_{ij} = -\epsilon_{ijk}\dot\Phi^k$, i.e.

$$\begin{pmatrix} 0 & W_{12} & -W_{31} \\ -W_{12} & 0 & W_{23} \\ W_{31} & -W_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\dot\Phi^3 & \dot\Phi^2 \\ \dot\Phi^3 & 0 & -\dot\Phi^1 \\ -\dot\Phi^2 & \dot\Phi^1 & 0 \end{pmatrix}.$$

Then we obtain *virtual velocities* of a Cosserat continuum as

$$\mathcal{V} = \left(\vec{u}(\vec{X},t), \dot{\vec{\Phi}}(\vec{X},t)\right) = \left((\dot u^1, \dot u^2, \dot u^3), (\dot\Phi^1, \dot\Phi^2, \dot\Phi^3)\right)$$

[1] in three dimensions, ϵ_{ijk} is 1 if (i,j,k) is an even permutation of $(1,2,3)$, -1 if it is an odd permutation, 0 if any index is repeated

where $\vec{u}$ and $\vec{\dot{\Phi}}$ are independent. In the *first gradient theory*, gradients of fields $\vec{u}$, $\vec{\dot{\Phi}}$ are considered and put into the set

$$\mathcal{V}^{\nabla} = \left(\nabla\vec{\dot{u}}, \nabla\vec{\dot{\Phi}}\right) = \left(\begin{pmatrix} \frac{\partial \dot{u}^1}{\partial x^1} & \frac{\partial \dot{u}^1}{\partial x^2} & \frac{\partial \dot{u}^1}{\partial x^3} \\ \frac{\partial \dot{u}^2}{\partial x^1} & \frac{\partial \dot{u}^2}{\partial x^2} & \frac{\partial \dot{u}^2}{\partial x^3} \\ \frac{\partial \dot{u}^3}{\partial x^1} & \frac{\partial \dot{u}^3}{\partial x^2} & \frac{\partial \dot{u}^3}{\partial x^3} \end{pmatrix}, \begin{pmatrix} \frac{\partial \dot{\Phi}^1}{\partial x^1} & \frac{\partial \dot{\Phi}^1}{\partial x^2} & \frac{\partial \dot{\Phi}^1}{\partial x^3} \\ \frac{\partial \dot{\Phi}^2}{\partial x^1} & \frac{\partial \dot{\Phi}^2}{\partial x^2} & \frac{\partial \dot{\Phi}^2}{\partial x^3} \\ \frac{\partial \dot{\Phi}^3}{\partial x^1} & \frac{\partial \dot{\Phi}^3}{\partial x^2} & \frac{\partial \dot{\Phi}^3}{\partial x^3} \end{pmatrix}\right).$$

However, second and higher order gradient theory is also studied.

Further, the deformation measures *distortion* β and *contortion* κ are defined (see [5]) by

$$\begin{aligned} \beta_{ij} &= \nabla_i u_j - W_{ij} \\ \kappa_{ijk} &= \nabla_i W_{jk}; \end{aligned}$$

in classical elasticity, the only deformation measure is the *strain* $\varepsilon_{ij} = \frac{1}{2}(\beta_{ij} + \beta_{ji})$.

We can incorporate also the *macrorotation field* $\vec{\omega}$ defined by $\omega_i = \frac{1}{4}\epsilon_{ijk}(\nabla_k u_j - \nabla_j u_k)$ into the picture. As the usual *macrostrain* is $\bar{\varepsilon}_{ij} = \frac{1}{2}(\nabla_i u_j + \nabla_j u_i)$, the *relative strain* (cf. e. g. [7])

$$\varepsilon^{\mathrm{rel}}_{ij} = \bar{\varepsilon}_{ij} + \epsilon_{ijk}\left(\omega_k + \dot{\Phi}_k\right)$$

and we can directly verify $\varepsilon_{ij} = \varepsilon^{\mathrm{rel}}_{ij} - \epsilon_{ijk}\omega_k$.

3. The Cosserat configuration

In the modern differential geometry language, the *material body* B is a manifold without boundary of dimension 1, 2 or 3 covered by a single coordinate chart. The *configuration* of a material body B is an embedding

$$\kappa\colon B \to \mathbb{R}^3$$

($\mathbb{R}^3$ is understood as an affine space with the euclidean inner product). Some fixed configuration

$$\kappa_0\colon B \to \mathbb{R}^3$$

is usually called the *reference configuration* and for an other configuration κ a composition $\chi = \kappa \circ \kappa_0$ is called the *deformation* (of the macrostructure) relative to the chosen reference configuration κ_0. Denoting the coordinates in the reference configuration by X^i ($i = 1, 2, 3$) and in a new configuration by x^i the deformation χ has a coordinate expression

$$x^i = \chi^i(\vec{X})$$

where smooth functions χ^i have smooth inverses.

Further, by the *Cosserat body* B we mean the frame bundle FB where microparticles are represented by fibers F_xB. The group $\mathrm{GL}(3,\mathbb{R})$ acts on the bundle FB. Here we distinguish between the micromorphic continuum (action of $\mathrm{GL}(3,\mathbb{R})$) and micropolar continuum (action of $\mathrm{SO}(3,\mathbb{R})$), more generally, we can consider even arbitrary G-structures.

Then the *Cosserat configuration* of a material body B is a principal bundle morphism

$$\begin{array}{ccc} FB & \xrightarrow{\;K\;} & F\mathbb{R}^3 \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\;\kappa\;} & \mathbb{R}^3 \end{array}$$

in coordinates

$$\begin{aligned} x^i &= \kappa^i(X) = x^i(X) \\ K^i_j &= K^i_j(X); \end{aligned}$$

then $\frac{\partial x^i}{\partial X^j}$ represents the *deformation gradient*. So, the Cosserat deformation means the change of vectors with respect to the deformation gradient of macromedia or independently, i.e. in general $\frac{\partial x^i}{\partial X^j} \neq K^i_j$. However, they can occur kinematic restrictions or constitutive restrictions (depending on the material).

4. Nonholonomic, semiholonomic and holonomic jets of deformations

Let us denote

$$x^i_j = \frac{\partial x^i}{\partial X^j},\ K^i_{jk} = \frac{\partial K^i_j}{\partial X^k}.$$

Then the deformation gradient is represented by the nonholonomic second order frame bundle with local coordinates x^i, K^i_j, x^i_j, K^i_{jk}. Nevertheless, if we require in a neighborhood of $X \in B$

$$K^i_j = x^i_j$$

we easily derive

$$K^i_{jk} = \frac{\partial K^i_j}{\partial X^k} = \frac{\partial x^i_j}{\partial X^k} = \frac{\partial^2 x^i}{\partial X^j \partial X^k} = \frac{\partial x^i_k}{\partial X^j} = K^i_{kj}$$

(symmetry in lower indexes) and obtain the holonomic second order frame bundle. It is more difficult to realize the semiholonomic Cosserat media, but Marcelo Epstein in [1] proposes a suitable constitutive rule for the material. Then we have obtained just the semiholonomic second order frame bundle, in which $K^i_j = x^i_j$ but there is no symmetry in lower indexes of K^i_{jk}.

For a clearer geometric point of view, let us deal with a more general situation now. Now, more general geometric point of view. Let us consider the *nonholonomic bundle of k–dimensional velocities of r–th order* $\tilde{T}^r_k M = \tilde{J}^r_0(\mathbb{R}^k, M)$. The functor $\tilde{T}^r_k$ is naturally equivalent to the r-times iterated functor T^1_k. Given some local coordinates x^i on M $i = 1, \dots, m = \dim M$ and t^j on $\mathbb{R}^k$, $j = 1, \dots, k$, the iterated differentiation of $x^i(t^1, \dots, t^k)$ determines the induced coordinates on $\tilde{T}^r_k M$ as $y^i_{j_1 \dots j_r}$, $j_1, \dots, j_r = 0, 1, \dots, k$, which are not symmetric in the subscripts.

For every s, $0 \leq s \leq r$, we denote by $\pi^s \colon \tilde{T}^s_k M \to M$ the canonical projection to the base. Further, we denote $\pi^s_b = \pi^s_{\tilde{T}^b_k M} \colon \tilde{T}^s_k(\tilde{T}^b_k M) \to \tilde{T}^b_k M$ projection with $\tilde{T}^b_k M$ as the base space, ${}_a\pi^s = \tilde{T}^a_k \pi^s \colon \tilde{T}^a_k(\tilde{T}^s_k M) \to \tilde{T}^a_k M$ induced projection originating by the posterior application of the functor $\tilde{T}^a_k$ and ${}_a\pi^s_b = \tilde{T}^a \pi^s_{\tilde{T}^b M}$ the general case containing both previous cases. If a or b equal zero, we do not write them. In local coordinates, just the coordinates with s adjoining zero subscripts (the $(b+1)$–th subscript is at the beginning) are remained after the application of ${}_a\pi^s_b$. Projections $\tilde{T}^p M \to \tilde{T}^q M$ are of a type ${}_a\pi^s_b$ or they are a composition of projections of such types.

In general, we can obtain subbundles of $\tilde{T}^r_k M$ by equalizations of some projections. An element $Z \in \tilde{T}^r_k M$ is called the *semiholonomic* k–dimensional velocity of r–th order, if for all $q = 1, \dots, r$

$$\pi^1_{r-1}(Z) = {}_{q-1}\pi^r_{r-q}(Z)$$

is satisfied. We denote by $\bar{T}^r_k M$ the bundle of semiholonomic k–dimensional velocities of r–th order. In local coordinates, we identify the coordinates subscripts of which become equal if we delete zeros in each of them.

We say that $Z \in \tilde{J}^r_a(M,N)_b$ is *invertible* if there exists a $Z^{-1} \in \tilde{J}^r_b(N,M)_a$ such that $Z^{-1} \circ Z = j^r_a \operatorname{id}_M$ and $Z \circ Z^{-1} = j^r_b \operatorname{id}_N$. We denote by $\operatorname{inv} \tilde{J}^r_a(M,N)_b$ the open submanifold of all invertible nonholonomic r–jets and we define the *r–th order nonholonomic frame bundle* as $\tilde{F}^r M = \operatorname{inv} \tilde{J}^r_0(\mathbb{R}^m, M)$. The group $\tilde{G}^r_m = \operatorname{inv} \tilde{J}^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ acts smoothly on $\tilde{F}^r M$ on the right by the jet composition. Similarly for the *semiholonomic* and *holonomic* cases.

So, we can apply this approach for a material body B on the place of a manifold M, then $m = \dim B = 3$; we take invertible jets from $\mathbb{R}^3$ to B and obtain

$$F^2 B = \operatorname{inv} J^2(\mathbb{R}^3, B), \bar{F}^2 B = \operatorname{inv} \bar{J}^2(\mathbb{R}^3, B) \text{ and } \tilde{F}^2 B = \operatorname{inv} \tilde{J}^2(\mathbb{R}^3, B)$$

as *holonomic*, *semiholonomic* and *nonholonomic Cosserat deformation bundles*, respectively.

We have thus shown what is the role of the velocities bundles in the geometric approach to Cosserat media.

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