TJMM
6 (2014), No. 1, 29-38

ON THE MAXIMUM TERM AND LOWER ORDER OF ENTIRE MONOGENIC FUNCTIONS

SUSHEEL KUMAR AND G.S. SRIVASTAVA

Abstract. In the present paper, we study the growth properties of entire monogenic functions. The characterizations of lower order of entire monogenic functions have been obtained in terms of their Taylor's series coefficients. Also we have obtained some inequalities between order, type, maximum term and central index of entire monogenic functions.

1. Introduction

Clifford analysis offers possibility of generalizing complex function theory to higher dimensions. It considers Clifford algebra valued functions that are defined in open subsets of $\mathbb{R}^n$ for arbitrary finite $n \in \mathbb{N}$ and that are solutions of higher dimensional Cauchy-Riemann systems. These are often called Clifford holomorphic or monogenic functions. In order to make calculations more concise we use following notations, where $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ be $n-$dimensional multi-index and $\mathbf{x} \in \mathbb{R}^n$:

$$\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \ldots x_n^{m_n}. \quad \mathbf{m!} = m_1! \ldots m_n!, \quad |\mathbf{m}| = m_1 + \ldots + m_n.$$

Following Constales, Almeida and Krausshar ([2], [3]), we give some definitions and associated properties.

By $\{e_1, e_2, \ldots, e_n\}$ we denote the canonical basis of the Euclidean vector space $\mathbb{R}^n$. The associated real Clifford algebra Cl_{0n} is the free algebra generated by $\mathbb{R}^n$ modulo $\mathbf{x}^2 = -||\mathbf{x}||^2 e_0$. where e_0 is the neutral element with respect to multiplication of the Clifford algebra Cl_{0n}. In the Clifford algebra Cl_{0n} following multiplication rule holds:

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0, \quad i, j = 1, 2, \ldots, n,$$

where δ_{ij} is Kronecker symbol. A basis for Clifford algebra Cl_{0n} is given by the set $\{e_A : A \subseteq \{1, 2, \ldots, n\}\}$ with $e_A = e_{l_1} e_{l_2} \ldots e_{l_r}$, where $1 \leq l_1 < l_2 < \ldots < l_r \leq n$, $e_\phi = e_0 = 1$. Each $a \in Cl_{0n}$ can be written in the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$. The conjugation in Clifford algebra Cl_{0n} is defined by $\bar{a} = \sum_A a_A \bar{e}_A$, where $\bar{e}_A = \bar{e}_{l_r} \bar{e}_{l_{r-1}} \ldots \bar{e}_{l_1}$ and $\bar{e}_j = -e_j$ for $j = 1, 2, \ldots, n$, $\bar{e}_0 = e_0 = 1$. The linear subspace $\text{span}_R\{1, e_1, \ldots, e_n\} = \mathbb{R} \oplus \mathbb{R}^n \subset Cl_{0n}$ is the so called space of para vectors $z = x_0 + x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$ which we simply identify with $\mathbb{R}^{n+1}$. Here $x_0 = \text{Sc}(z)$ is scalar part and $\mathbf{x} = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n = \text{Vec}(z)$ is vector part of para vector z. The Clifford norm of an arbitrary $a = \sum_A a_A e_A$ is given by

$$||a|| = \left(\sum_A |a_A|^2 \right)^{1/2}.$$

Each para vector $z \in \mathbb{R}^{n+1} \backslash \{0\}$ has an inverse element in $\mathbb{R}^{n+1}$ which can be represented in the form $z^{-1} = \overline{z}/||z||^2$.

2010 *Mathematics Subject Classification.* 30G35.
Key words and phrases. entire monogenic function, order, type, maximum term, central index.

The generalized Cauchy-Riemann operator in $\mathbb{R}^{n+1}$ is given by

$$D \equiv \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}.$$

If $U \subseteq \mathbb{R}^{n+1}$ is an open set, then a function $g : U \to Cl_{0n}$ is called left (right) monogenic at a point $z \in U$ if $Dg(z) = 0$ ($gD(z) = 0$). The functions which are left (right) monogenic in the whole space are called left (right) entire monogenic functions.

Let A_{n+1} be n-dimensional surface area of $(n+1)$-dimensional unit ball and $q_{\mathbf{0}}(z) = \frac{\bar{z}}{||z||^{n+1}}$ be Cauchy kernel function. Then every function g which is monogenic in a neighborhood of closure $\overline{G}$ of domain G satisfies the following equation ([3, p. 766])

$$g(z) = \frac{1}{A_{n+1}} \int_{\partial G} q_{\mathbf{0}}(z-\zeta)\, d\tau(\zeta) g(\zeta), \text{ for all } z \in G,$$

where

$$d\tau(\zeta) = \sum_{j=0}^{n} (-1)^j e_j \widehat{d\zeta_j}$$

with

$$\widehat{d\zeta_j} = d\zeta_0 \wedge \ldots \wedge d\zeta_{j-1} \wedge d\zeta_{j+1} \wedge \ldots \wedge d\zeta_n$$

is the oriented outer normal surface measure. If g is a left monogenic function in a ball $||z|| < R$, then for all $||z|| < r$ with $0 < r < R$,

$$g(z) = \sum_{|\mathbf{m}|=0}^{\infty} V_{\mathbf{m}}(z) a_{\mathbf{m}}. \tag{1}$$

In (1) $V_{\mathbf{m}}(z)$ are called Fueter polynomials and are given as

$$V_{\mathbf{m}}(z) = \frac{\mathbf{m}!}{|\mathbf{m}|!} \sum_{\pi \in perm(\mathrm{m})} z_{\pi(m_1)} \ldots z_{\pi(m_n)},$$

where $perm(\mathbf{m})$ is the set of all permutations of the sequence $(m_1, m_2, \ldots, m_n)$ and $z_i = x_i - x_0 e_i$ for $i = 1, \ldots, n$ and $V_{\mathbf{0}}(z) = 1$. Also in (1), $\{a_{\mathbf{m}}\}$ are Clifford numbers which are defined by

$$a_{\mathbf{m}} = \frac{1}{\mathbf{m}! A_{n+1}} \int_{||\zeta||<r} q_{\mathbf{m}}(\zeta)\, d\tau(\zeta) g(\zeta)$$

and satisfy the inequality

$$||a_{\mathbf{m}}|| \leq c(n, \mathbf{m}) \frac{M(r)}{r^{|\mathbf{m}|}}.$$

Here $M(r) = M(r, g) = \max_{||z||=r} \{||g(z)||\}$ denotes the maximum modulus of the function g in the closed ball of radius r and

$$q_{\mathbf{m}}(z) = \frac{\partial^{m_0+m_1+\ldots+m_n}}{\partial x_0^{m_0} \partial x_1^{m_1} \ldots \partial x_n^{m_n}} q_{\mathbf{0}}(z),$$

$$c(n, \mathbf{m}) = \frac{n(n+1)\ldots(n+|\mathbf{m}|-1)}{\mathbf{m}!}.$$

Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be an entire monogenic function. Constales, Almeida and Krausshar [3], defined the order ρ and lower order λ of $g(z)$ as

$$\begin{matrix} \rho \\ \lambda \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log\log M(r)}{\log r}. \tag{2}$$

Also the lower order λ of $g(z)$ satisfies following inequality [3, Thm. 2]

$$\lambda \geq \lim_{|\mathbf{m}|\to\infty} \inf \frac{|\mathbf{m}| \log |\mathbf{m}|}{\log ||a_{\mathbf{m}}/c(n, \mathbf{m})||^{-1}}.$$

In the present paper, we have obtained a sharp estimate of lower order of entire monogenic functions in terms of their Taylor's series expansion which generalized the result of Constales, Almeida and Kraußhar [3, Thm. 2].

Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be an entire monogenic function having order $\rho(0 < \rho < \infty)$, then the type σ of $g(z)$ is given by [2]

$$\sigma = \lim_{r\to\infty} \sup \frac{\log M(r)}{r^{\rho}}.$$

We define the lower type of entire monogenic function $g(z)$ as

$$\varpi = \lim_{r\to\infty} \inf \frac{\log M(r)}{r^{\rho}}.$$

Almeida and Kraußhar [1], introduced the concept of the maximum term and central index of entire monogenic functions. Hence, let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be a left entire monogenic function whose Taylor's series representation is given by (1). Then for $r > 0$ the maximum term of this entire monogenic function is given by

$$\mu(r) = \mu(r, g) = \max_{|\mathbf{m}|\geq 0} \{||a_{\mathbf{m}}||r^{|\mathbf{m}|}\}.$$

Also the index $\mathbf{m}$ with maximal length $|\mathbf{m}|$ for which maximum term is achieved is called the central index and is denoted by

$$\nu(r) = \nu(r, g) = \mathbf{m}.$$

Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be a left entire monogenic function with the property that in its Taylor's series expansion, first coefficient $a_0 \neq 0$. Then Almeida and Kraußhar proved that [1, p. 803]

$$\log \{\mu(r)\} - \log ||a_0|| = \int_0^r \frac{|v(t)|}{t} dt.$$

Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be a left entire monogenic function of order ρ and lower order λ and put

$$\begin{matrix}\rho_1 \\ \lambda_1\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log \mu(r)}{\log r}$$

and

$$\begin{matrix}\rho_2 \\ \lambda_2\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log |\nu(r)|}{\log r}. \tag{3}$$

Then Almeida and Kraußhar proved that [1, Prop. 5.3]

$$\rho \leq \rho_1 = \rho_2$$

and

$$\lambda \leq \lambda_1 = \lambda_2.$$

Also we define

$$\begin{matrix}\omega_1 \\ \omega_2\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log \mu(r)}{r^{\rho}}$$

and

$$\begin{matrix}\tau_1 \\ \tau_2\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{|\nu(r)|}{r^{\rho}}.$$

2. Main results

We now prove

Theorem 1. *Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be an entire monogenic function whose Taylor's series representation is given by $g(z) = \sum_{|\boldsymbol{m}|=0}^{\infty} a_{\boldsymbol{m}} V_{\boldsymbol{m}}(z)$. Then the lower order λ of this entire monogenic function $g(z)$ satisfies*

$$\lambda \geq \lim_{|\boldsymbol{m}|\to\infty} \inf \frac{|\boldsymbol{m}| \log |\boldsymbol{m}|}{-\log ||a_{\boldsymbol{m}}/c(n, \boldsymbol{m})||}. \tag{4}$$

Also if

$$\psi(k) = \max_{|\boldsymbol{m}|=k} \left\{ \frac{||a_{\boldsymbol{m}}||}{||a_{\boldsymbol{m}'}||}, ||\boldsymbol{m}'|| = ||\boldsymbol{m}|| + 1 \right\}$$

is a non-decreasing function of k, then equality holds in (4).

Proof. Write

$$\Phi = \frac{|\mathbf{m}| \log |\mathbf{m}|}{-\log ||a_{\mathbf{m}}/c(n, \mathbf{m})||}.$$

First we prove that $\lambda \geq \Phi$. The coefficients of an entire monogenic Taylor's series satisfy Cauchy's inequality, that is

$$||a_{\mathbf{m}}|| \leq M(r)c(n, \mathbf{m})r^{-|\mathbf{m}|}. \tag{5}$$

From (2), for arbitrary $\varepsilon > 0$ and a sequence $r = r_k \to \infty$ as $k \to \infty$, we have

$$M(r) \leq \exp\left(r^{\overline{\lambda}}\right), \quad \overline{\lambda} = \lambda + \varepsilon.$$

Now from (5), we get

$$||a_{\mathbf{m}}|| \leq c(n, \mathbf{m})r^{-|\mathbf{m}|} \exp\left(r^{\overline{\lambda}}\right).$$

Putting $r = \left(|\mathbf{m}|/\overline{\lambda}\right)^{1/\overline{\lambda}}$ in the above inequality we get

$$||a_{\mathbf{m}}|| \leq \left(|\mathbf{m}|/\overline{\lambda}\right)^{-|\mathbf{m}|/\overline{\lambda}} \exp\left(|\mathbf{m}|/\overline{\lambda}\right)$$

or

$$-\log ||a_{\mathbf{m}}|| \geq \frac{|\mathbf{m}| \log |\mathbf{m}|}{\overline{\lambda}} \left[1 - \frac{\log \overline{\lambda}}{\log |\mathbf{m}|} - \frac{1}{\log |\mathbf{m}|}\right]$$

or

$$\lim_{|\mathbf{m}|\to\infty} \inf \frac{|\mathbf{m}| \log |\mathbf{m}|}{-\log ||a_{\mathbf{m}}||} \leq \overline{\lambda}$$

or

$$\Phi \leq \overline{\lambda}.$$

Since $\varepsilon > 0$ is arbitrarily small so finally we get

$$\Phi \leq \lambda.$$

Now we prove that $\lambda \leq \Phi$. From the assumption on ψ, $\psi(k) \to \infty$ as $k \to \infty$. By the definition given in section 1, if $||a_{\mathbf{m}}||r^{|\mathbf{m}|}$ is the maximum term for r, then for $|\mathbf{m}_1| \leq |\mathbf{m}| < |\mathbf{m}_2|$,

$$||a_{\mathbf{m}_1}||r^{|\mathbf{m}_1|} \leq ||a_{\mathbf{m}}||r^{|\mathbf{m}|} > ||a_{\mathbf{m}_2}||r^{|\mathbf{m}_2|}$$

and for $|\mathbf{m}| = k$

$$\psi(k-1) \leq r < \psi(k).$$

Now suppose that $||a_{\mathbf{m}^1}||r^{|\mathbf{m}^1|}$ and $||a_{\mathbf{m}^2}||r^{|\mathbf{m}^2|}$ are two consecutive maximum terms. Then

$$|\mathbf{m}^1| \leq |\mathbf{m}^2| - 1.$$

Let

$$|\mathbf{m}^1| \leq k \leq |\mathbf{m}^2|.$$

Then

$$|\nu(r)| = |\mathbf{m}^1|$$

for

$$\psi(|\mathbf{m}^{1^*}|) \leq r < \psi(|\mathbf{m}^1|),$$

where $|\mathbf{m}^{1^*}| = |\mathbf{m}^1| - 1$. Hence from (3) and [1, prop. 5.3], for arbitrary $\varepsilon > 0$ and all $r > r_0(\varepsilon)$, we have

$$|\mathbf{m}^1| = |\nu(r)| > r^{\lambda'}, \quad \lambda' = \lambda - \varepsilon$$

or

$$|\mathbf{m}^1| = |\nu(r)| \geq \left\{\psi(|\mathbf{m}^1|) - q\right\}^{\lambda'},$$

where q is a constant such that $0 < q < \min\left\{1, [\psi(|\mathbf{m}^1|) - \psi(|\mathbf{m}^{1^*}|)]/2\right\}$
or

$$\log \psi(|\mathbf{m}^1|) \leq O(1) + \frac{\log |\mathbf{m}^1|}{\lambda'}.$$

Further we have

$$\psi(|\mathbf{m}^1|) = \psi(|\mathbf{m}^1| + 1) = \ldots = \psi(|\mathbf{m}| - 1).$$

Now we can write

$$\psi(|\mathbf{m}^0|) \ldots \psi(|\mathbf{m}^*|) = \frac{||a_{\mathbf{m}^0}||}{||a_{\mathbf{m}}||} \leq [\psi(|\mathbf{m}^*|)]^{|\mathbf{m}| - |\mathbf{m}^0|},$$

where $|\mathbf{m}^*| = |\mathbf{m}| - 1$ and $|\mathbf{m}| \gg |\mathbf{m}^0|$.
Hence

$$||c(n, \mathbf{m})|| \frac{||a_{\mathbf{m}^0}||}{||a_{\mathbf{m}}||} \leq ||c(n, \mathbf{m})|| [\psi(|\mathbf{m}^*|)]^{|\mathbf{m}| - |\mathbf{m}^0|}$$

or

$$\begin{aligned} \log ||a_{\mathbf{m}}/c(n, \mathbf{m})||^{-1} &\leq |\mathbf{m}| \log \psi(|\mathbf{m}^1|) + O(1) \\ &\leq |\mathbf{m}| \frac{\log |\mathbf{m}^1|}{\lambda'} [1 + o(1)] \end{aligned}$$

or

$$\frac{1}{|\mathbf{m}|} \log ||a_{\mathbf{m}}/c(n, \mathbf{m})||^{-1} \leq \frac{\log |\mathbf{m}^1|}{\lambda'} [1 + o(1)]$$

or

$$\frac{1}{|\mathbf{m}|} \log ||a_{\mathbf{m}}/c(n, \mathbf{m})||^{-1} \leq \frac{\log |\mathbf{m}|}{\lambda'} [1 + o(1)]$$

or

$$\lambda' \leq \frac{|\mathbf{m}| \log |\mathbf{m}|}{-\log ||a_{\mathbf{m}}/c(n, \mathbf{m})||} [1 + o(1)].$$

Now taking limits as $|\mathbf{m}| \to \infty$, we get $\lambda \leq \Phi$. Hence the Theorem 1 is proved. □

Next we prove

Theorem 2. *Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be an entire monogenic function whose Taylor's series representation is given by $g(z) = \sum_{|\boldsymbol{m}|=0}^{\infty} a_{\boldsymbol{m}} V_{\boldsymbol{m}}(z)$. Also let $\rho_2 < \infty$. Then*

$$\lim_{r \to \infty} \sup \frac{\log M(r)}{\log \mu(r)} \leq 1.$$

Proof. From [1, p.806], we have

$$M(r) \leq \mu(r)(2r)^{n\rho_2+\varepsilon}$$

or

$$\log M(r) \leq \log \mu(r) + (n\rho_2 + \varepsilon)\log(2r)$$

or

$$\frac{\log M(r)}{\log \mu(r)} \leq \left[1 + \frac{(n\rho_2+\varepsilon)\log(2)}{\log \mu(r)} + \frac{(n\rho_2+\varepsilon)\log(r)}{\log \mu(r)}\right].$$

Proceeding to limits as $r \to \infty$ on both sides we get

$$\lim_{r\to\infty} \frac{\log M(r)}{\log \mu(r)} \leq 1.$$

Hence the Theorem 2 is proved. □

Next we prove

Theorem 3. *Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be an entire monogenic function whose Taylor's series representation is given by $g(z) = \sum_{|\boldsymbol{m}|=0}^{\infty} a_{\boldsymbol{m}} V_{\boldsymbol{m}}(z)$. Also if $0 < \rho \leq \rho_2 < \infty$, then $\sigma \leq \omega_1$ and $\varpi \leq \omega_2$.*

Proof. From Theorem 2, we have

$$\log M(r) \leq \log \mu(r) \left[1 + \frac{(n\rho_2+\varepsilon)\log(2)}{\log \mu(r)} + \frac{(n\rho_2+\varepsilon)\log(r)}{\log \mu(r)}\right]$$

or

$$\frac{\log M(r)}{r^\rho} \leq \frac{\log \mu(r)}{r^\rho} \left[1 + \frac{(n\rho_2+\varepsilon)\log(2)}{\log \mu(r)} + \frac{(n\rho_2+\varepsilon)\log(r)}{\log \mu(r)}\right].$$

Proceeding to limits as $r \to \infty$ on both sides we get

$$\sigma \leq \omega_1$$

and

$$\varpi \leq \omega_2.$$

Hence the Theorem 3 is proved. □

Next we prove

Theorem 4. *Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be an entire monogenic function whose Taylor's series representation is given by $g(z) = \sum_{|\boldsymbol{m}|=0}^{\infty} a_{\boldsymbol{m}} V_{\boldsymbol{m}}(z)$. Then following inequalities hold*

$$\lim_{r\to\infty} \sup \frac{\log \mu(r)}{|\nu(r)|} \geq \frac{1}{\lambda_1}$$

and

$$\lim_{r\to\infty} \inf \frac{\log \mu(r)}{|\nu(r)|} \leq \frac{1}{\rho_1}.$$

Proof. Let

$$\lim_{r\to\infty} \sup \frac{\log \mu(r)}{|\nu(r)|} = A.$$

Then for $\varepsilon > 0$ and $r > r_0(\varepsilon)$, we have

$$\log \mu(r) < (A+\varepsilon)|\nu(r)|. \tag{6}$$

Now from ([1, p.806], we have

$$\frac{\mu'(r)}{\mu(r)} = \frac{|\nu(r)|}{r}. \tag{7}$$

Now from (6) and (7), we get

$$\log \mu(r) < (A+\varepsilon)\frac{\mu'(r)}{\mu(r)}r$$

or

$$\frac{\mu'(r)}{\mu(r)\log \mu(r)} > \frac{1}{(A+\varepsilon)r}$$

or

$$\log\log \mu(r) > \frac{1}{(A+\varepsilon)}\log r + O(1)$$

or

$$\frac{\log\log \mu(r)}{\log r} > \frac{1}{(A+\varepsilon)} + o(1).$$

Proceeding to limits as $r \to \infty$ and taking *inf* on both sides we get

$$\lambda_1 \geq \frac{1}{A}.$$

Hence the first part of theorem is proved. To prove second part let us assume that

$$\lim_{r\to\infty}\inf \frac{\log \mu(r)}{|\nu(r)|} = B.$$

Then for $\varepsilon > 0$ and $r > r_0(\varepsilon)$, we have

$$\log \mu(r) > (B-\varepsilon)|\nu(r)|. \tag{8}$$

Now from (7) and (8), we get

$$\log \mu(r) > (B-\varepsilon)\frac{\mu'(r)}{\mu(r)}r$$

or

$$\frac{\mu'(r)}{\mu(r)\log \mu(r)} < \frac{1}{(B-\varepsilon)r}$$

or

$$\log\log \mu(r) < \frac{1}{(B-\varepsilon)}\log r + O(1)$$

or

$$\frac{\log\log \mu(r)}{\log r} < \frac{1}{(B-\varepsilon)} + o(1).$$

Proceeding to limits as $r \to \infty$ and taking *sup* on both sides we get

$$\rho_1 \leq \frac{1}{B}.$$

Hence the second part of Theorem 4 is proved. □

Next we prove

Theorem 5. *Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be an entire monogenic function whose Taylor's series representation is given by $g(z) = \sum_{|\mathbf{m}|=0}^{\infty} a_{\mathbf{m}} V_{\mathbf{m}}(z)$. If the order of g is $\rho (0 < \rho < \infty)$, then following inequalities hold*

$$\tau_2 \leq \frac{\tau_1}{e}e^{\tau_2/\tau_1} \leq \tau_2\omega_1 \leq \tau_1,$$

$$\tau_2 \leq \rho\omega_2 \leq \tau_2(1+\log\frac{\tau_1}{\tau_2}) \leq \tau_1$$

and

$$\tau_1 + \tau_2 \leq e\tau_2\omega_1.$$

Proof. From [1, p.806], for $r \geq r_0$ and $k \geq 1$, we have

$$\log \mu(kr) = O(1) + \int_{r_0}^{r} \frac{|\nu(t)|}{t} dt + \int_{r}^{kr} \frac{|\nu(t)|}{t} dt \tag{9}$$

or

$$\log \mu(kr) \geq O(1) + \frac{(\tau_2 - \varepsilon) r^{\rho}}{\rho} + |\nu(r)| \log k.$$

Dividing both sides by $(kr)^{\rho}$, we get

$$\frac{\log \mu(kr)}{(kr)^{\rho}} \geq o(1) + \frac{(\tau_2 - \varepsilon)}{\rho k^{\rho}} + \frac{|\nu(r)|}{r^{\rho}} \frac{\log k}{k^{\rho}}. \tag{10}$$

Proceeding to limits as $r \to \infty$ and taking *sup* on both sides, we get

$$\omega_1 \geq \frac{\tau_2 + \rho \tau_1 \log k}{\rho k^{\rho}}. \tag{11}$$

Also proceeding to limits as $r \to \infty$ and taking *inf* on both sides of (10), we get

$$\omega_2 \geq \frac{\tau_2 (1 + \rho \log k)}{\rho k^{\rho}}. \tag{12}$$

Now taking $k = \exp[(\tau_1 - \tau_2)/(\rho \tau_2)]$ in (11), we get

$$e \rho \omega_1 \geq \tau_1 e^{\tau_2/\tau_1}.$$

Since $\exp(t) \geq et$ for all $t \geq 0$. Therefore finally, we get

$$e \rho \omega_1 \geq \tau_1 e^{\tau_2/\tau_1} \geq e \tau_2. \tag{13}$$

Also taking $k = 1$ in (12), we get

$$\omega_2 \geq \frac{\tau_2}{\rho}. \tag{14}$$

Now again from (9), we have

$$\log \mu(kr) \leq O(1) + \frac{(\tau_1 + \varepsilon) r^{\rho}}{\rho} + |\nu(kr)| \log k.$$

Dividing both sides by $(kr)^{\rho}$, we get

$$\frac{\log \mu(kr)}{(kr)^{\rho}} \leq o(1) + \frac{(\tau_1 + \varepsilon)}{\rho k^{\rho}} + \frac{|\nu(kr)|}{(kr)^{\rho}} \log k. \tag{15}$$

So here we get

$$\omega_1 \leq \frac{\tau_1 (1 + \rho k^{\rho} \log k)}{\rho k^{\rho}} \tag{16}$$

and

$$\omega_2 \leq \frac{\tau_1 + \rho \tau_2 k^{\rho} \log k}{\rho k^{\rho}}. \tag{17}$$

Now taking $k = 1$ in (16), we get

$$\omega_1 \leq \frac{\tau_1}{\rho}. \tag{18}$$

Also taking $k = (\tau_1/\tau_2)^{1/\rho}$ in (17), we get

$$\rho \omega_2 \leq \tau_2 \left(1 + \log \frac{\tau_1}{\tau_2} \right).$$

Since $\log(1 + t) \leq t$ for all $t \geq 0$. Therefore finally we get

$$\rho \omega_2 \leq \tau_2 \left(1 + \log \frac{\tau_1}{\tau_2} \right) \leq \tau_1. \tag{19}$$

Now from (13), (14), (18) and (19), we get

$$\tau_2 \leq \frac{\tau_1}{e} e^{\tau_2/\tau_1} \leq \tau_2\omega_1 \leq \tau_1 \tag{20}$$

and

$$\tau_2 \leq \rho\omega_2 \leq \tau_2(1 + \log\frac{\tau_1}{\tau_2}) \leq \tau_1.$$

From (20), we have

$$\frac{\tau_1}{e} e^{\tau_2/\tau_1} \leq \tau_2\omega_1$$

or

$$\tau_1 \left[1 + \frac{\tau_2}{\tau_1} + \ldots\right] \leq e\tau_2\omega_1$$

or

$$\tau_1 \left[1 + \frac{\tau_2}{\tau_1}\right] \leq e\tau_2\omega_1$$

or

$$\tau_1 + \tau_2 \leq e\tau_2\omega_1.$$

Hence the Theorem 5 is proved. □

Next we prove

Theorem 6. *Let $g : \mathbb{R}^{n+1} \to Cl_{0n}$ be an entire monogenic function whose Taylor's series representation is given by $g(z) = \sum_{|\boldsymbol{m}|=0}^{\infty} a_{\boldsymbol{m}} V_{\boldsymbol{m}}(z)$. If the order of g is $\rho (0 < \rho < \infty)$, then*

$$\tau_1 + \rho\omega_2 \leq e\rho\omega_1$$

and

$$e\rho\omega_2 \leq \rho\omega_1 + e\tau_2.$$

Proof. From [1, p.806] for $r \geq r_0$ and $k \geq 1$, we have

$$\log\mu(kr) = \log\mu(r) + \int_r^{kr} \frac{|\nu(t)|}{t} dt \tag{21}$$

or

$$\log\mu(kr) > (\omega_2 - \varepsilon)r^{\rho} + |\nu(r)| \log k.$$

Dividing both sides by $(kr)^{\rho}$, we get

$$\frac{\log\mu(kr)}{(kr)^{\rho}} > \frac{(\omega_2 - \varepsilon)}{k^{\rho}} + \frac{|\nu(r)|}{r^{\rho}} \frac{\log k}{k^{\rho}}.$$

Proceeding to limits as $r \to \infty$ and taking *sup* on both sides, we get

$$\omega_1 \geq \frac{\omega_2}{k^{\rho}} + \frac{\tau_1 \log k}{k^{\rho}}.$$

Now taking $k = e^{1/\rho}$ in above inequality, we get

$$\omega_1 \geq \frac{\omega_2}{e} + \frac{\tau_1}{\rho e}. \tag{22}$$

Now again from (21), we have

$$\log\mu(kr) < (\omega_1 + \varepsilon)r^{\rho} + |\nu(kr)| \log k.$$

Dividing both sides by $(kr)^{\rho}$, we get

$$\frac{\log\mu(kr)}{(kr)^{\rho}} < \frac{(\omega_1 + \varepsilon)}{k^{\rho}} + \frac{|\nu(kr)|}{(kr)^{\rho}} \log k.$$

Proceeding to limits as $r \to \infty$ and taking *inf* on both sides, we get

$$\omega_2 \leq \frac{\omega_1}{k^\rho} + \tau_2 \log k.$$

Now again taking $k = e^{1/\rho}$ in above inequality, we get

$$\omega_2 \leq \frac{\omega_1}{e} + \frac{\tau_2}{\rho}. \tag{23}$$

Now from (22) and (23), we get

$$\tau_1 + \rho\omega_2 \leq e\rho\omega_1.$$

and

$$e\rho\omega_2 \leq \rho\omega_1 + e\tau_2.$$

Hence the Theorem 6 is proved. □

Note: Similar results were obtained for entire functions of one variable by Shah ([4], [5], [6] and [7]).

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Jaypee University of Information Technology
Department of Mathematics
Samirpur - 177601, Himachal Pradesh, INDIA.
E-mail address: sus83dma@gmail.com

Jaypee Institute of Information Technology
Department of Mathematics
A-10, Sector-62, Noida-201309, Uttar Pradesh, INDIA.
E-mail address: gs91490@gmail.com