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A NEW APPROACH TO ONE-DIMENSIONAL OSCILLATORS IN RELATIVISTIC QUANTUM MECHANICS

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ABSTRACT. A new approach to the one-dimensional oscillatory motion of a relativistic quantum particle with the spin $\frac{1}{2}$ is presented. It is based on a modified form of the Hamilton operator of the particle. As particular cases, the one-dimensional Dirac oscillator and the one-dimensional oscillator with equidistant energy levels are discussed. In this context, the first oscillator appears as an ancient representative, while the second oscillator is a new representative of an entire class of relativistic quantum one-dimensional oscillators.

1. Introduction

The one-dimensional oscillatory motion is one of the most important motions in both classical mechanics and quantum mechanics. Many one-dimensional oscillators are exactly solvable models. The one-dimensional oscillators have multiple applications to diverse branches of physics.

In the non-relativistic classical mechanics, the Hamilton function of a particle with the mass m_0, moving, along the axis Ox, in an external field in which it possesses the linear momentum p, the potential energy $U(x)$ and the total energy E has the form

$$H = \frac{p^2}{2m_0} + U(x) = E. \tag{1}$$

For

$$U(x) = \frac{1}{2} m_0 \omega_0^2 x^2, \tag{2}$$

the law of motion is

$$x = \sqrt{\frac{2E}{m_0 \omega_0^2}} \sin(\omega_0 t + \varphi_0) \tag{3}$$

and the period of motion is

$$T = \frac{2\pi}{\omega_0}, \tag{4}$$

where ω_0 is the pulsation of motion and φ_0 is a constant of integration [1]. This oscillatory motion is both harmonic and isochronous.

In the non-relativistic quantum mechanics, the Hamilton operator of a particle in one-dimensional motion is obtained from the Hamilton function of the corresponding classical particle by substituting hermitian operators for physical quantities. In the coordinate representation, the Hamilton operator associated with the Hamilton function (1) is

$$\hat{H} = -\frac{\hbar^2}{2m_0} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + U(x), \tag{5}$$

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where $\hbar = \frac{h}{2\pi}$ and h is the Planck constant.

For $U(x)$ of the form (2), the energy eigenfunctions are

$$\psi_n\left(x\right) = \sqrt{\frac{1}{\sqrt{\pi}2^n n!}\sqrt{\frac{m_0\omega_0}{\hbar}}}\exp\left(-\frac{m_0\omega_0}{2\hbar}x^2\right)H_n\left(\sqrt{\frac{m_0\omega_0}{\hbar}}x\right) \tag{6}$$

and the energy eigenvalues are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0, \tag{7}$$

in which $n = 0, 1, 2, \ldots$ and $H_n(x)$ are the Hermite polynomials [2].

The energy spectrum is discrete. The energy eigenvalues are equally spaced, the interval between two adjacent energy levels being equal to $\hbar\omega_0$. The energy of the ground state $n = 0$ has the non-null value $\frac{1}{2}\hbar\omega_0$.

A complete analysis of properties and applications of the non-relativistic one-dimensional harmonic oscillator would require an encyclopedia. Numerous aspects of this subject are exposed in the books [3, 4, 5].

In the relativistic classical mechanics, the Hamilton function of a particle with the rest mass m_0, moving, along the axis Ox, in an external field in which it possesses the linear momentum p, the potential energy $U(x)$ and the total energy E has the form

$$H = \sqrt{m_0^2c^4 + p^2c^2} + U\left(x\right) = E. \tag{8}$$

For $U(x)$ of the form (2), the law of motion is

$$\frac{2E\left(\alpha,k\right) - k'^2F\left(\alpha,k\right)}{k'} = \omega_0 t + \varphi_0, \tag{9}$$

in which $F(\alpha, k)$ and $E(\alpha, k)$ are, respectively, the elliptic integrals of the first and the second kinds in the Legendre normal form, of argument

$$\alpha = \arcsin\sqrt{\frac{m_0\omega_0^2}{2\left(E - m_0c^2\right)}}x, \tag{10}$$

of modulus

$$k = \sqrt{\frac{E - m_0c^2}{E + m_0c^2}} \tag{11}$$

and of complementary modulus

$$k' = \sqrt{1 - k^2} = \sqrt{\frac{2m_0c^2}{E + m_0c^2}}, \tag{12}$$

and φ_0 is a constant of integration, while the period of motion is

$$T = \frac{4}{\omega_0}\frac{2E\left(k\right) - k'^2K\left(k\right)}{k'}, \tag{13}$$

in which $K(k) = F(\frac{\pi}{2}, k)$ and $E(k) = E(\frac{\pi}{2}, k)$ are, respectively, the complete elliptic integrals of the first and the second kinds in the Legendre normal form [6].

This oscillatory motion is neither harmonic nor isochronous.

In the relativistic quantum mechanics, the Hamilton operator of a particle with the spin $\frac{1}{2}$ in one-dimensional motion is obtained from the Hamilton function of the corresponding classical particle by the Dirac method [7]. In the coordinate representation, the Hamilton operator associated with the Hamilton function (12) can be written as

$$\hat{H} = c\hat{\alpha}\left(-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right) + \hat{\beta}m_0c^2 + U\left(x\right). \tag{14}$$

The operators $\hat{\alpha}$ and $\hat{\beta}$ must satisfy the commutation relations

$$\hat{\alpha}^2 = \hat{\beta}^2 = \hat{I},\ \ \hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha} = \hat{O}, \tag{15}$$

where $\hat{I}$ is the unity operator and $\hat{O}$ is the null operator.

Commonly, the operators $\hat{\alpha}$, $\hat{\beta}$, $\hat{I}$ and $\hat{O}$ are represented by the following 2×2 matrices

$$\alpha = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix},\ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{16}$$

The wave function can be expressed in terms of its components as

$$\psi\left(x\right) = \begin{pmatrix} \psi_1\left(x\right) \\ \psi_2\left(x\right) \end{pmatrix}. \tag{17}$$

The Dirac equation splits into a system of two differential equations of the first order

$$\left.\begin{array}{l} c\hbar\frac{\mathrm{d}\psi_1}{\mathrm{d}x} + \left[U\left(x\right) - \left(E + m_0c^2\right)\right]\psi_2 = 0, \\ c\hbar\frac{\mathrm{d}\psi_2}{\mathrm{d}x} - \left[U\left(x\right) - \left(E - m_0c^2\right)\right]\psi_1 = 0 \end{array}\right\} \tag{18}$$

or into a system of two differential equations of the second order

$$\left.\begin{array}{l} c^2\hbar^2\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2} + \frac{c^2\hbar^2\frac{\mathrm{d}U(x)}{\mathrm{d}x}}{E+m_0c^2-U(x)}\frac{\mathrm{d}\psi_1}{\mathrm{d}x} + \left\{\left[E - U\left(x\right)\right]^2 - m_0^2c^4\right\}\psi_1 = 0, \\ c^2\hbar^2\frac{\mathrm{d}^2\psi_2}{\mathrm{d}x^2} + \frac{c^2\hbar^2\frac{\mathrm{d}U(x)}{\mathrm{d}x}}{E-m_0c^2-U(x)}\frac{\mathrm{d}\psi_2}{\mathrm{d}x} + \left\{\left[E - U\left(x\right)\right]^2 - m_0^2c^4\right\}\psi_2 = 0. \end{array}\right\} \tag{19}$$

It is very difficult to solve these equations for $U(x)$ of the form (2) [8, 9].

In the conventional approach, exactly solvable models of relativistic quantum one-dimensional oscillators are scarce.

In order to remove these difficulties, we shall introduce a new approach to the one-dimensional oscillators in the relativistic quantum mechanics. It represents a translation from the classical language into the quantum language of the approach introduced in [10] for the one-dimensional oscillators in the relativistic classical mechanics. This approach has some advantages compared to the conventional approach:

- We can predict the form of the function $U(x)$ corresponding to a predetermined energy spectrum.
- It allow us to treat the relativistic quantum one-dimensional oscillators as simply as the non-relativistic quantum one-dimensional oscillators.
- The number of exactly solvable models of relativistic quantum one-dimensional oscillators is increased.

The advantages of the new approach will be illustrated by the one-dimensional Dirac oscillator and the one-dimensional oscillator with equidistant energy levels.

Variants of the Dirac oscillator were discussed some years ago [11, 12, 13, 14]. The model was rediscovered in 1989 in the context of relativistic many body theories, as a system whose Hamiltonian is linear in both momentum and coordinates [15]. It has been widely studied in $(1 + 1)$, $(2 + 1)$ and $(3 + 1)$ dimensions. Some properties of the one-dimensional Dirac oscillator have been analysed in [16, 17, 18, 19, 20, 21, 22, 23, 24]. In our approach, the one-dimensional Dirac oscillator is not a singular oscillator, but a member of an entire class of oscillators.

Attempts to construct an one-dimensional oscillator with equidistant energy levels in the relativistic quantum mechanics have not succeeded in expressing the function $U(x)$

in a simple analytic form [25, 26, 27, 28]. In our approach, the function $U(x)$ is expressed in terms of elementary functions.

The one-dimensional Dirac oscillator and the one-dimensional oscillator with equidistant energy levels have potential applications to quantum chromodynamics, quantum optics, condensed-matter physics etc.

2. A New Approach To The One-Dimensional Oscillatory Motion of A Relativistic Quantum Particle with The Spin $\frac{1}{2}$

The form (8) of the Hamilton function for a relativistic particle assumes that, in the relativistic classical mechanics, there exist forces independent of velocities, which derive from simple potentials. But, in the relativistic theory, the dependence of forces on velocities is unavoidable. Because of this, in general, the Hamilton function of a relativistic particle cannot be divided into a term dependent on velocity and a term independent of velocity. The form

$$H = \sqrt{m_0^2 c^4 + p^2 c^2 + 2m_0 c^2 U\left(x\right)} = E \tag{20}$$

of the Hamilton function for a particle in a relativistic one-dimensional motion, which has these properties, has been proposed in [10]. It allows us to solve the direct and inverse problems of the relativistic one-dimensional oscillatory motion:

- the determination of the total-energy dependence of the period if the coordinate dependence of the potential energy is known (direct problem);
- the determination of the coordinate dependence of the potential energy if the total-energy dependence of the period is given (inverse problem).

In the relativistic quantum mechanics, the Hamilton operator of a particle with the spin $\frac{1}{2}$ in one-dimensional motion is obtained from the Hamilton function of the corresponding classical particle by the same method as in the conventional approach. In the coordinate representation, the Hamilton operator associated with the Hamilton function (20) is

$$\hat{H} = c\hat{\alpha}\left(-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x}\right) + \hat{\beta}m_0c^2 + \hat{\gamma}\sqrt{2m_0c^2U\left(x\right)}. \tag{21}$$

The operators $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ must satisfy the commutation relations

$$\left.\begin{array}{c} \hat{\alpha}^2 = \hat{\beta}^2 = \hat{\gamma}^2 = \hat{I}, \\ \hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha} = \hat{O},\ \hat{\beta}\hat{\gamma} + \hat{\gamma}\hat{\beta} = \hat{O},\ \hat{\gamma}\hat{\alpha} + \hat{\alpha}\hat{\gamma} = \hat{O}. \end{array}\right\} \tag{22}$$

The operator $\hat{\gamma}$ can be expressed with the aid of the operators $\hat{\alpha}$ and $\hat{\beta}$ as

$$\hat{\gamma} = -\mathrm{i}\hat{\alpha}\hat{\beta}. \tag{23}$$

Substituting (23) into (21), we obtain

$$\hat{H} = c\hat{\alpha}\left[-\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}x} - \mathrm{i}\hat{\beta}\sqrt{2m_0U\left(x\right)}\right] + \hat{\beta}m_0c^2. \tag{24}$$

For the present purposes, it is convenient to represent the operators $\hat{\alpha}$, $\hat{\beta}$, $\hat{I}$ and $\hat{O}$ by the 2×2 matrices (16) and the operator $\hat{\gamma}$ by the 2×2 matrix

$$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{25}$$

The wave function can be expressed in terms of its components as in Eq. (17).

The wave equation

$$\hat{H}\psi = E\psi \tag{26}$$

can be written as a system of two differential equations of the first order in which the unknown functions ψ_1 and ψ_2 are interconnected

$$\left.\begin{array}{l} c\hbar\frac{\mathrm{d}\psi_1}{\mathrm{d}x} + \sqrt{2m_0c^2U\left(x\right)}\psi_1 - \left(E + m_0c^2\right)\psi_2 = 0, \\ c\hbar\frac{\mathrm{d}\psi_2}{\mathrm{d}x} - \sqrt{2m_0c^2U\left(x\right)}\psi_2 + \left(E - m_0c^2\right)\psi_1 = 0 \end{array}\right\} \tag{27}$$

or as a system of two differential equations of the second order where the unknown functions ψ_1 and ψ_2 are separated

$$\left.\begin{array}{l} c^2\hbar^2\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x^2} + \left[E^2 - m_0^2c^4 + c\hbar\frac{\mathrm{d}\sqrt{2m_0c^2U(x)}}{\mathrm{d}x} - 2m_0c^2U\left(x\right)\right]\psi_1 = 0, \\ c^2\hbar^2\frac{\mathrm{d}^2\psi_2}{\mathrm{d}x^2} + \left[E^2 - m_0^2c^4 - c\hbar\frac{\mathrm{d}\sqrt{2m_0c^2U(x)}}{\mathrm{d}x} - 2m_0c^2U\left(x\right)\right]\psi_2 = 0. \end{array}\right\} \tag{28}$$

Because there is a close connection between the total-energy dependence of the period of a relativistic classical one-dimensional oscillator and the form of the energy spectrum of the corresponding relativistic quantum one-dimensional oscillator, we can predict the form of the function $U(x)$ corresponding to a given energy spectrum.

Eqs. (28) are equivalent to the Schrödinger equations of two particles with the same mass m_0, the potential energies $U\left(x\right) - \frac{\hbar}{\sqrt{2m_0}}\frac{\mathrm{d}\sqrt{U(x)}}{\mathrm{d}x}$ and $U\left(x\right) + \frac{\hbar}{\sqrt{2m_0}}\frac{\mathrm{d}\sqrt{U(x)}}{\mathrm{d}x}$, the same total energy $\frac{E^2-m_0^2c^4}{2m_0c^2}$ and the wave functions ψ_1 and ψ_2. They are much simpler than Eqs. (19).

3. One-Dimensional Dirac Oscillator

For $U(x)$ of the form (2), the law of motion and the period of motion corresponding to the Hamilton function (20) are, respectively,

$$x = \sqrt{\frac{E^2 - m_0^2c^4}{m_0^2c^2\omega_0^2}}\sin\left[\frac{m_0c^2}{E}\left(\omega_0 t + \varphi_0\right)\right] \tag{29}$$

and

$$T = \frac{2\pi}{\omega_0}\frac{E}{m_0c^2}, \tag{30}$$

where φ_0 is a constant of integration [10].

The oscillatory motion is harmonic and non-isochronous.

For the same form of $U(x)$, Eqs. (27) and (28) are the wave equations of the one-dimensional Dirac oscillator

$$\frac{\mathrm{d}\psi_{1,2}}{\mathrm{d}x} \pm \frac{m_0\omega_0}{\hbar}x\psi_{1,2} \mp \frac{E \pm m_0c^2}{c\hbar}\psi_{2,1} = 0 \tag{31}$$

and

$$\frac{\mathrm{d}^2\psi_{1,2}}{\mathrm{d}x^2} + \left(\frac{E^2 - m_0^2c^4}{c^2\hbar^2} \pm \frac{m_0\omega_0}{\hbar} - \frac{m_0^2\omega_0^2}{\hbar^2}x^2\right)\psi_{1,2} = 0. \tag{32}$$

This oscillator has the energy eigenfunctions

$$\left.\begin{array}{l} \psi_{1,\pm n}\left(x\right) = \sqrt{\frac{1}{\sqrt{\pi}2^n n!}\sqrt{\frac{m_0\omega_0}{\hbar}}\frac{E_{\pm n}+m_0c^2}{2E_{\pm n}}}e^{-\frac{m_0\omega_0}{2\hbar}x^2}H_n\left(\sqrt{\frac{m_0\omega_0}{\hbar}}x\right), \\ \psi_{2,\pm n}\left(x\right) = \pm\sqrt{\frac{1}{\sqrt{\pi}2^{n-1}(n-1)!}\sqrt{\frac{m_0\omega_0}{\hbar}}\frac{E_{\pm n}-m_0c^2}{2E_{\pm n}}}e^{-\frac{m_0\omega_0}{2\hbar}x^2}H_{n-1}\left(\sqrt{\frac{m_0\omega_0}{\hbar}}x\right) \end{array}\right\} \tag{33}$$

and the energy eigenvalues

$$E_{\pm n} = \pm\sqrt{m_0^2c^4 + 2nm_0c^2\hbar\omega_0}, \tag{34}$$

where $+n = 0, 1, 2, \ldots$ and $-n = -1, -2, \ldots$ [16, 17, 18].

The energy eigenfunctions (33) constitute an orthonormal, closed and complete system of functions on the interval $(-\infty, +\infty)$ [19].

They are similar to the energy eigenfunctions (6) of the non-relativistic one-dimensional harmonic oscillator.

4. One-Dimensional Oscillator with Equidistant Energy Levels

For $U(x)$ of the form

$$U\left(x\right) = \frac{1}{2} m_0 c^2 \tan^2 \frac{\omega_0 x}{c} \tag{35}$$

and the Hamilton function (20), the law of motion is

$$x = \frac{c}{\omega_0} \arcsin\left[\sqrt{\frac{E^2 - m_0^2 c^4}{E^2}} \sin\left(\omega_0 t + \varphi_0\right)\right], \tag{36}$$

where φ_0 is a constant of integration, and the period of motion has the form (4) [10].

The oscillatory motion is non-harmonic and isochronous.

For the same form of $U(x)$, Eqs. (27) and (28) become, respectively,

$$\frac{\mathrm{d}\psi_{1,2}}{\mathrm{d}x} \pm \frac{m_0 c}{\hbar} \tan \frac{\omega_0 x}{c} \psi_{1,2} \mp \frac{E \pm m_0 c^2}{c\hbar} \psi_{2,1} = 0 \tag{37}$$

and

$$\frac{\mathrm{d}^2\psi_{1,2}}{\mathrm{d}x^2} + \left(\frac{E^2 - m_0^2 c^4}{c^2 \hbar^2} \pm \frac{m_0 \omega_0}{\hbar} \frac{1}{\cos^2 \frac{\omega_0 x}{c}} - \frac{m_0^2 c^2}{\hbar^2} \tan^2 \frac{\omega_0 x}{c}\right) \psi_{1,2} = 0, \tag{38}$$

where $-\frac{\pi c}{2\omega_0} \leq x \leq +\frac{\pi c}{2\omega_0}$.

Because Eqs. (38) are invariant under inversion, the wave functions $\psi_{1,2}(x)$ must have a well-defined parity: $\psi_{1,2}\left(-x\right) = \pm\psi_{1,2}\left(x\right)$. For this reason, it is sufficient to find the wave functions $\psi_{1,2}(x)$ on the one of the intervals $\left(-\frac{\pi c}{2\omega_0}, 0\right)$ and $\left(0, +\frac{\pi c}{2\omega_0}\right)$. The continuity of these functions at the point $x = 0$ presupposes that their values $\psi_{1,2}(x)$ and the values of their one-sided limits $\psi_{1,2}(x+0)$ and $\psi_{1,2}(x-0)$ coincide at this point: $\psi_{1,2}(0) = \psi_{1,2}(+0) = \psi_{1,2}(-0)$.

Making the change of variables

$$\xi = \mathrm{i} \tan \frac{\omega_0 x}{c}, \tag{39}$$

we obtain

$$\left(\xi^2 - 1\right) \frac{\mathrm{d}\psi_{1,2}}{\mathrm{d}\xi} \pm \frac{m_0 c^2}{\hbar\omega_0} \xi \psi_{1,2} \mp \mathrm{i} \frac{E \pm m_0 c^2}{\hbar\omega_0} \psi_{2,1} = 0 \tag{40}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(\xi^2 - 1\right) \frac{\mathrm{d}\psi_{1,2}}{\mathrm{d}\xi}\right] - \left(\frac{m_0^2 c^4 \mp m_0 c^2 \hbar\omega_0}{\hbar^2 \omega_0^2} + \frac{E^2}{\hbar^2 \omega_0^2} \frac{1}{\xi^2 - 1}\right) \psi_{1,2} = 0. \tag{41}$$

Eqs. (40) and (41) are similar to the equations of the associated Legendre functions of the first kind $P_\nu^\mu\left(\xi\right)$ [29]:

$$\left.\begin{array}{c}
\left(\xi^2 - 1\right) \frac{\mathrm{d}P_{\nu-1}^\mu(\xi)}{\mathrm{d}\xi} + \nu\xi P_{\nu-1}^\mu\left(\xi\right) + \left(\mu - \nu\right) P_\nu^\mu\left(\xi\right) = 0, \\
\left(\xi^2 - 1\right) \frac{\mathrm{d}P_\nu^\mu(\xi)}{\mathrm{d}\xi} - \nu\xi P_\nu^\mu\left(\xi\right) + \left(\mu + \nu\right) P_{\nu-1}^\mu\left(\xi\right) = 0, \\
\left(\xi^2 - 1\right) \frac{\mathrm{d}P_\nu^\mu(\xi)}{\mathrm{d}\xi} + \left(\nu + 1\right) \xi P_\nu^\mu\left(\xi\right) + \left(\mu - \nu - 1\right) P_{\nu+1}^\mu\left(\xi\right) = 0, \\
\left(\xi^2 - 1\right) \frac{\mathrm{d}P_{\nu+1}^\mu(\xi)}{\mathrm{d}\xi} - \left(\nu + 1\right) \xi P_{\nu+1}^\mu\left(\xi\right) + \left(\mu + \nu + 1\right) P_\nu^\mu\left(\xi\right) = 0
\end{array}\right\} \tag{42}$$

and

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(\xi^2-1\right)\frac{\mathrm{d}P^{\mu}_{\nu-1}(\xi)}{\mathrm{d}\xi}\right]-\left[(\nu-1)\,\nu+\frac{\mu^2}{\xi^2-1}\right]P^{\mu}_{\nu-1}(\xi)=0,\\ \frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(\xi^2-1\right)\frac{\mathrm{d}P^{\mu}_{\nu}(\xi)}{\mathrm{d}\xi}\right]-\left[\nu\,(\nu+1)+\frac{\mu^2}{\xi^2-1}\right]P^{\mu}_{\nu}(\xi)=0,\\ \frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(\xi^2-1\right)\frac{\mathrm{d}P^{\mu}_{\nu+1}(\xi)}{\mathrm{d}\xi}\right]-\left[(\nu+1)\,(\nu+2)+\frac{\mu^2}{\xi^2-1}\right]P^{\mu}_{\nu+1}(\xi)=0. \end{aligned}\right\} \tag{43}$$

Comparing Eqs. (40) – (43), we have

$$\left.\begin{aligned} \mu=+\frac{E}{\hbar\omega_0},\nu=\frac{m_0c^2}{\hbar\omega_0}-1,\psi_1(\xi)\sim\sqrt{\frac{E+m_0c^2}{E-m_0c^2}}P^{\mu}_{\nu}(\xi),\psi_2(\xi)\sim\mathrm{i}\sqrt{\frac{E-m_0c^2}{E+m_0c^2}}P^{\mu}_{\nu+1}(\xi);\\ \mu=-\frac{E}{\hbar\omega_0},\nu=\frac{m_0c^2}{\hbar\omega_0}-1,\psi_1(\xi)\sim P^{\mu}_{\nu}(\xi),\psi_2(\xi)\sim-\mathrm{i}P^{\mu}_{\nu+1}(\xi);\\ \mu=+\frac{E}{\hbar\omega_0},\nu=-\frac{m_0c^2}{\hbar\omega_0},\psi_1(\xi)\sim\sqrt{\frac{E+m_0c^2}{E-m_0c^2}}P^{\mu}_{\nu}(\xi),\psi_2(\xi)\sim\mathrm{i}\sqrt{\frac{E-m_0c^2}{E+m_0c^2}}P^{\mu}_{\nu-1}(\xi);\\ \mu=-\frac{E}{\hbar\omega_0},\nu=-\frac{m_0c^2}{\hbar\omega_0},\psi_1(\xi)\sim P^{\mu}_{\nu}(\xi),\psi_2(\xi)\sim-\mathrm{i}P^{\mu}_{\nu-1}(\xi). \end{aligned}\right\} \tag{44}$$

The associated Legendre functions of the first kind of the imaginary variable $\xi=\mathrm{i}\cot\varphi$ can be expressed in terms of hypergeometric functions as

$$\begin{aligned} &P^{\mu}_{\nu}(\mathrm{i}\cot\varphi)=\\ &=\frac{1}{\sqrt{2\pi}}\frac{\Gamma(-\nu-1/2)}{\Gamma(-\mu-\nu)}e^{-\mathrm{i}(\nu+1)\pi/2}\sqrt{\sin\varphi}\left(\tan\frac{\varphi}{2}\right)^{\nu+1/2}F\left(\frac{1}{2}+\mu,\frac{1}{2}-\mu;\frac{3}{2}+\nu;\sin^2\frac{\varphi}{2}\right)\\ &+\frac{1}{\sqrt{2\pi}}\frac{\Gamma(\nu+1/2)}{\Gamma(-\mu+\nu+1)}e^{\mathrm{i}\nu\pi/2}\sqrt{\sin\varphi}\left(\cot\frac{\varphi}{2}\right)^{\nu+1/2}F\left(\frac{1}{2}+\mu,\frac{1}{2}-\mu;\frac{1}{2}-\nu;\sin^2\frac{\varphi}{2}\right), \end{aligned} \tag{45}$$

where $\Gamma(z)$ is the Euler integral of the second kind, $2\nu\neq\pm1,\pm3,\pm5,\ldots$ and $0\leq\varphi\leq\frac{\pi}{2}$ [29].

The associated Legendre function of the first kind $P^{\mu}_{\nu}(\mathrm{i}\cot\varphi)$ is bounded for $\varphi\to0$ if

$$\nu+\frac{1}{2}>0 \tag{46}$$

and

$$\mu-\nu-1=n, \tag{47}$$

with $n=0,1,2,\ldots$, when it is reduced to its first term, or if

$$\nu+\frac{1}{2}<0 \tag{48}$$

and

$$\mu+\nu=n, \tag{49}$$

with $n=0,1,2,\ldots$, when it is reduced to its second term.

Taking into account that [29] the hypergeometric function $F(\alpha,\beta;\gamma;u)$ is symmetric with respect to the parameters α and β, the hypergeometric function $F(\alpha,\beta;\gamma;u)$ is connected with the hypergeometric function $F(\gamma-\alpha,\gamma-\beta;\gamma;u)$ by the relation

$$F(\alpha,\beta;\gamma;u)=(1-u)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma;u) \tag{50}$$

and the hypergeometric function $F(n+2\lambda,-n;\lambda+\frac{1}{2};\frac{1-u}{2})$ is expressed in terms of the Gegenbauer polynomials $C^{\lambda}_{n}(u)$ as

$$F\left(n+2\lambda,-n;\lambda+\frac{1}{2};\frac{1-u}{2}\right)=\frac{n!\Gamma(2\lambda)}{\Gamma(n+2\lambda)}C^{\lambda}_{n}(u), \tag{51}$$

we can rewrite the associated Legendre functions of the first kind $P^{\mu}_{\nu}(\mathrm{i}\cot\varphi)$, with $\varphi=\frac{\pi}{2}-\frac{\omega_0x}{c}$, in the two cases, respectively, in the forms

$$\begin{aligned} &P^{n+\nu+1}_{\nu}\left(\mathrm{i}\tan\frac{\omega_0x}{c}\right)=\\ &=\frac{1}{\sqrt{2\pi}}\frac{n!\Gamma(-\nu-1/2)\Gamma(2\nu+2)}{2^{\nu+1/2}\Gamma(-n-2\nu-1)\Gamma(n+2\nu+2)}e^{-\mathrm{i}(\nu+1)\pi/2}\left(\cos\frac{\omega_0x}{c}\right)^{\nu+1}C^{\nu+1}_{n}\left(\sin\frac{\omega_0x}{c}\right) \end{aligned} \tag{52}$$

and

$$P_\nu^{n-\nu}\left(\mathrm{i}\tan\frac{\omega_0 x}{c}\right)= \\ =\frac{1}{\sqrt{2\pi}}\frac{n!\Gamma(\nu+1/2)\Gamma(-2\nu)}{2^{-\nu-1/2}\Gamma(-n+2\nu+1)\Gamma(n-2\nu)}e^{\mathrm{i}\nu\pi/2}\left(\cos\frac{\omega_0 x}{c}\right)^{-\nu}C_n^{-\nu}\left(\sin\frac{\omega_0 x}{c}\right). \tag{53}$$

The functions

$$f_n(u)=2^\lambda\Gamma(\lambda)\sqrt{\frac{n!\,(n+\lambda)}{2\pi\Gamma(n+2\lambda)}}\left(1-u^2\right)^{(2\lambda-1)/4}C_n^\lambda(u), \tag{54}$$

formed with the aid of the Gegenbauer polynomials $C_n^\lambda(u)$ with $\lambda\geq\frac{1}{2}$, have the following properties.

They possess the values

$$f_n(0)=2^{n+\lambda}\sqrt{\frac{n+\lambda}{2n!\Gamma(n+2\lambda)}}\frac{\Gamma(n/2+\lambda)}{\Gamma((1-n)/2)} \tag{55}$$

and

$$f_n(\pm 1)=0 \tag{56}$$

and a well-defined parity:

$$f_n(-u)=(-1)^n f_n(u). \tag{57}$$

These functions satisfy the orthonormality relations

$$\int_{-1}^{+1} f_m(u)\,f_n(u)\,\mathrm{d}u=\delta_{mn} \tag{58}$$

and the closure relation

$$\sum_{n=0}^{\infty} f_n(u)\,f_n(u')=\delta(u-u'). \tag{59}$$

They form a complete system of functions, so that any sufficiently smooth function $f(u)$ can be expanded in a generalized Fourier series relative to the system of functions $f_n(u)$ with its sum converging to $f(u)$ almost everywhere:

$$f(u)=\sum_{n=0}^{\infty} c_n f_n(u), \tag{60}$$

where the Fourier coefficients c_n have the expressions

$$c_n=\int_{-1}^{+1} f_n(u)\,f(u). \tag{61}$$

These coefficients obey the Parseval relation

$$\sum_{n=0}^{\infty} c_n^2=\int_{-1}^{+1}[f(u)]^2\,\mathrm{d}u. \tag{62}$$

The properties (58) – (62) show that the functions $f_n(u)$ constitute an orthonormal, closed and complete system of functions on the interval $(-1,+1)$.

It follows from these properties that the associated Legendre functions of the first kind $P_\nu^{n+\nu+1}\left(\mathrm{i}\tan\frac{\omega_0 x}{c}\right)$ and $P_\nu^{n-\nu}\left(\mathrm{i}\tan\frac{\omega_0 x}{c}\right)$ form two orthogonal, closed and complete systems of functions on the interval $\left(-\frac{\pi c}{2\omega_0},+\frac{\pi c}{2\omega_0}\right)$ with the norms, respectively,

$$\left\|P_\nu^{n+\nu+1}\right\|=\sqrt{\int_{-\pi c/(2\omega_0)}^{+\pi c/(2\omega_0)}\left|P_\nu^{n+\nu+1}\left(\mathrm{i}\tan\frac{\omega_0 x}{c}\right)\right|^2\mathrm{d}x}= \\ =\frac{\Gamma(-\nu-1/2)\Gamma(2\nu+2)}{2^{2\nu+3/2}\Gamma(-n-2\nu-1)\Gamma(\nu+1)}\sqrt{\frac{n!}{(n+\nu+1)\Gamma(n+2\nu+2)}\frac{c}{\omega_0}} \tag{63}$$

and

$$\begin{array}{c}\|P_{\nu}^{n-\nu}\| = \sqrt{\int_{-\pi c/(2\omega_0)}^{+\pi c/(2\omega_0)} \left|P_{\nu}^{n-\nu}\left(\mathrm{i}\tan\frac{\omega_0 x}{c}\right)\right|^2 \mathrm{d}x} = \\ = \frac{\Gamma(\nu+1/2)\Gamma(-2\nu)}{2^{-2\nu-1/2}\Gamma(-n+2\nu+1)\Gamma(-\nu)} \sqrt{\frac{n!}{(n-\nu)\Gamma(n-2\nu)}\frac{c}{\omega_0}}.\end{array} \tag{64}$$

Eqs. (44), (46) – (49) and (52) – (64) allow us to determine the energy eigenfunctions and the energy eigenvalues of the oscillator, as well as their conditions of applicability.

They define the energy eigenfunctions on the interval $\left(-\frac{\pi c}{2\omega_0}, 0\right)$. Taking into account the conditions $\psi_{1,2}(-x) = \pm\psi_{1,2}(x)$ and $\psi_{1,2}(0) = \psi_{1,2}(+0) = \psi_{1,2}(-0)$ imposed upon the energy eigenfunctions and the properties of the functions $f_n(u)$, we can extend the functions $\psi_{1,2}(x)$ in the same forms to the interval $\left(0, +\frac{\pi c}{2\omega_0}\right)$.

The energy eigenfunctions $\psi_1(x)$ and $\psi_2(x)$ can be normalized in the sense of

$$\int_{-\pi c/(2\omega_0)}^{+\pi c/(2\omega_0)} \left[|\psi_1(x)|^2 + |\psi_2(x)|^2\right] \mathrm{d}x = 1 \tag{65}$$

with the aid of the relations (63) and (64).

Substituting the relations (46) and (47) into the first two ranks of Eqs. (44) or the relations (48) and (49) into the last two ranks of Eqs. (44), we find that, for

$$\hbar\omega_0 < 2m_0c^2, \tag{66}$$

the energy eigenfunctions are

$$\left.\begin{array}{c}\psi_{1,\pm n}(x) = 2^{\nu+1}\Gamma(\nu+1)\sqrt{\frac{n!(n+\nu+1)}{2\pi\Gamma(n+2\nu+2)}}\times \\ \times e^{-\mathrm{i}(\nu+1)\pi/2}\sqrt{\frac{\omega_0}{c}\frac{E_{\pm n}+m_0c^2}{2E_{\pm n}}}\left(\cos\frac{\omega_0 x}{c}\right)^{\nu+1} C_n^{\nu+1}\left(\sin\frac{\omega_0 x}{c}\right), \\ \psi_{2,\pm n}(x) = \pm\mathrm{i}2^{\nu+2}\Gamma(\nu+2)\sqrt{\frac{(n-1)!(n+\nu+1)}{2\pi\Gamma(n+2\nu+3)}}\times \\ \times e^{-\mathrm{i}(\nu+2)\pi/2}\sqrt{\frac{\omega_0}{c}\frac{E_{\pm n}-m_0c^2}{2E_{\pm n}}}\left(\cos\frac{\omega_0 x}{c}\right)^{\nu+2} C_{n-1}^{\nu+2}\left(\sin\frac{\omega_0 x}{c}\right) \\ \left(\nu = \frac{m_0c^2}{\hbar\omega_0} - 1\right)\end{array}\right\} \tag{67}$$

or

$$\left.\begin{array}{c}\psi_{1,\pm n}(x) = 2^{-\nu}\Gamma(-\nu)\sqrt{\frac{n!(n-\nu)}{2\pi\Gamma(n-2\nu)}}\times \\ \times e^{\mathrm{i}\nu\pi/2}\sqrt{\frac{\omega_0}{c}\frac{E_{\pm n}+m_0c^2}{2E_{\pm n}}}\left(\cos\frac{\omega_0 x}{c}\right)^{-\nu} C_n^{-\nu}\left(\sin\frac{\omega_0 x}{c}\right), \\ \psi_{2,\pm n}(x) = \pm\mathrm{i}2^{-\nu+1}\Gamma(-\nu+1)\sqrt{\frac{(n-1)!(n-\nu)}{2\pi\Gamma(n-2\nu+1)}}\times \\ \times e^{\mathrm{i}(\nu-1)\pi/2}\sqrt{\frac{\omega_0}{c}\frac{E_{\pm n}-m_0c^2}{2E_{\pm n}}}\left(\cos\frac{\omega_0 x}{c}\right)^{-\nu+1} C_{n-1}^{-\nu+1}\left(\sin\frac{\omega_0 x}{c}\right) \\ \left(\nu = -\frac{m_0c^2}{\hbar\omega_0}\right)\end{array}\right\} \tag{68}$$

and the energy eigenvalues are

$$E_{\pm n} = \pm\left(m_0c^2 + n\hbar\omega_0\right). \tag{69}$$

The properties of the functions $\psi_{1,\pm n}(x)$ and $\psi_{2,\pm n}(x)$ are derived from the properties of the functions $f_n(u)$.

It is easy to see that $\psi_{1,-0}(x) = 0$ and $\psi_{2,\pm 0}(x) = 0$. Hence, the energy eigenvalue E_{-0} is forbidden.

The energy eigenfunctions (67) and (68) have the values

$$\psi_{1,\pm n}\left(\pm\frac{\pi c}{2\omega_0}\right) = 0, \qquad \psi_{2,\pm n}\left(\pm\frac{\pi c}{2\omega_0}\right) = 0 \tag{70}$$

and the parities

$$\psi_{1,\pm n}(-x) = (-1)^n \psi_{1,\pm n}(x), \qquad \psi_{2,\pm n}(-x) = (-1)^{n-1} \psi_{2,\pm n}(x). \tag{71}$$

They satisfy the orthonormality relations

$$\left.\begin{array}{r} \int_{-\pi c/(2\omega_0)}^{+\pi c/(2\omega_0)} \left[\psi_{1,\pm m}^*(x)\,\psi_{1,\pm n}(x) + \psi_{2,\pm m}^*(x)\,\psi_{2,\pm n}(x)\right] \mathrm{d}x = \delta_{mn}, \\ \int_{-\pi c/(2\omega_0)}^{+\pi c/(2\omega_0)} \left[\psi_{1,\pm m}^*(x)\,\psi_{1,\mp n}(x) + \psi_{2,\pm m}^*(x)\,\psi_{2,\mp n}(x)\right] \mathrm{d}x = 0 \end{array}\right\} \tag{72}$$

and the closure relation

$$\begin{array}{r} \sum_{n=0}^{\infty} \left[\begin{pmatrix} \psi_{1,+n}^*(x) \\ \psi_{2,+n}^*(x) \end{pmatrix} \begin{pmatrix} \psi_{1,+n}(x') & \psi_{2,+n}(x') \end{pmatrix} + \right. \\ \left. + \begin{pmatrix} \psi_{1,-n}^*(x) \\ \psi_{2,-n}^*(x) \end{pmatrix} \begin{pmatrix} \psi_{1,-n}(x') & \psi_{2,-n}(x') \end{pmatrix}\right] = I\delta(x - x'). \end{array} \tag{73}$$

The components $\psi_1(x)$ and $\psi_2(x)$ of an arbitrary wave function can be expanded in two series of energy eigenfunctions $\psi_{1,\pm n}(x)$ and $\psi_{2,\pm n}(x)$, respectively, as

$$\begin{array}{l} \psi_1(x) = \sum_{n=0}^{\infty} \left[c_{1,+n}\psi_{1,+n}(x) + c_{1,-n}\psi_{1,-n}(x)\right], \\ \psi_2(x) = \sum_{n=0}^{\infty} \left[c_{2,+n}\psi_{2,+n}(x) + c_{2,-n}\psi_{2,-n}(x)\right], \end{array} \tag{74}$$

with the Fourier coefficients $c_{1,\pm n}$ and $c_{2,\pm n}$ given by the formulas

$$\begin{array}{l} c_{1,\pm n} = \int_{-\pi c/(2\omega_0)}^{+\pi c/(2\omega_0)} \psi_{1,\pm n}^*(x)\,\psi_1(x)\,\mathrm{d}x, \\ c_{2,\pm n} = \int_{-\pi c/(2\omega_0)}^{+\pi c/(2\omega_0)} \psi_{2,\pm n}^*(x)\,\psi_2(x)\,\mathrm{d}x. \end{array} \tag{75}$$

One can see that $c_{1,-0} = 0$ and $c_{2,\pm 0} = 0$.

If the components $\psi_1(x)$ and $\psi_2(x)$ of the wave function are normalized as in Eq. (65), these coefficients verify the Parseval relation

$$\sum_{n=0}^{\infty} \left(|c_{1,+n}|^2 + |c_{1,-n}|^2 + |c_{2,+n}|^2 + |c_{2,-n}|^2\right) = 1. \tag{76}$$

The substitutions of the relations (48) and (49) into the first two ranks of Eqs. (44) or of the relations (46) and (47) into the last two ranks of Eqs. (44) show that, for

$$\hbar\omega_0 > 2m_0c^2, \tag{77}$$

the energy eigenfunctions $\psi_{1,\pm n}(x)$ are bounded, the energy eigenfunctions $\psi_{2,\pm n}(x)$ are unbounded and the energy eigenvalues are

$$E_{\pm n} = \pm\left[-m_0c^2 + (n+1)\,\hbar\omega_0\right]. \tag{78}$$

For

$$\hbar\omega_0 = 2m_0c^2, \tag{79}$$

the expressions (69) and (78) take a form identical with (7),

$$E_{\pm n} = \pm\left(n + \frac{1}{2}\right)\hbar\omega_0. \tag{80}$$

The energy eigenvalues (69), (78) and (80) of this oscillator are equally spaced with an interval between two adjacent energy levels equal to $\hbar\omega_0$. They are similar to the energy eigenvalues (7) of the non-relativistic one-dimensional harmonic oscillator. The energy levels E_{+n} corresponds to the particle states, while the energy levels E_{-n} are associated with the antiparticle states. When the interval $\hbar\omega_0$ between two adjacent energy levels is greater than the threshold energy $2m_0c^2$ $\left(\hbar\omega_0 > 2m_0c^2\right)$ particle-antiparticle pairs are created and annihilated.

5. Conclusions

We have presented a new approach to the one-dimensional oscillatory motion of a relativistic quantum particle with the spin $\frac{1}{2}$. It is based on a modified form of the Hamilton operator of the particle. This form has allowed us to solve the wave equation of some relativistic quantum one-dimensional oscillators in a simple way.

We have applied this approach to the one-dimensional Dirac oscillator and the one-dimensional oscillator with equidistant energy levels. In this context, the first oscillator appears as an ancient member, whereas the second oscillator represents a new member of an entire class of relativistic quantum one-dimensional oscillators.

There is a close connection between the properties of these oscillators and the properties of the non-relativistic quantum one-dimensional harmonic oscillator. On the one hand, both the energy eigenfunctions of the one-dimensional Dirac oscillator and the energy eigenfunctions of the non-relativistic quantum one-dimensional harmonic oscillator are expressed in terms of the Hermite polynomials. This property is related to the harmonic laws of motion of the corresponding classical oscillators. On the other hand, both the energy eigenvalues of the one-dimensional oscillator with equidistant energy levels and the energy eigenvalues of the non-relativistic quantum one-dimensional harmonic oscillator are equally spaced. This property is a quantum manifestation of the isochronous characters of the corresponding classical oscillators.

Our approach can also be used for other relativistic quantum systems.

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