TJMM
6 (2014), No. 1, 13-16

STARLIKENESS OF LIBERA OPERATOR ON CERTAIN CONCAVE UNIVALENT FUNCTIONS

IBTISAM AL-DAWISH AND MASLINA DARUS

Abstract. Let $C_0(\alpha)$ denote the class of concave univalent functions defined in the open unit disk $\mathbb{D}$. Each function $f \in C_0(\alpha)$ maps the unit disk $\mathbb{D}$ onto the complement of an unbounded convex set. In this paper, we shall prove that the Libera operator $F(z) = \frac{2}{z}\int_0^z f(t)dt$ is starlike by finding certain conditions on $Re\, 2f'$.

1. Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let A denote the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{n=\infty} a_n z^n \quad (z \in \mathbb{D})$$

We say that a function $f \in \mathcal{H}$ is subordinate to a function $F \in \mathcal{H}$ in $\mathbb{D}$, and write $f(z) \prec F(z)$ (or simply $f \prec F$), if and only if there exists a Schwarz function ω, analytic in $\mathbb{D}$ with $(|\omega(z)| \le |z|, z \in \mathbb{D})$ such that $f(z) = F(\omega(z))$ for $z \in \mathbb{D}$.

In particular, if F is univalent in $\mathbb{D}$, we have the following equivalence

$$f(z) \prec F(z) \Longleftrightarrow f(0) = F(0) \,\text{and}\, f(\mathbb{D}) \subset F(\mathbb{D}).$$

Denote by S^* and C, the class of starlike functions and convex functions respectively, which are analytically defined as follows:

$$S^* := \left\{ f : f \in A \text{ and } \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}, z \in \mathbb{D} \right\} = \left\{ f : f \in A \text{ and } Re\left[\frac{zf'(z)}{f(z)}\right] > 0, z \in \mathbb{D} \right\}$$

$$C := \left\{ f : f \in A \text{ and } \frac{zf''(z)}{f(z)} \prec \frac{2z}{1-z}, z \in \mathbb{D} \right\} = \left\{ f : f \in A \text{ and } Re\left[\frac{zf''(z)}{f(z)}\right] > -1, z \in \mathbb{D} \right\}$$

Let $\mathfrak{D}_\alpha, 0 < \alpha \le 1$ denote the class

$$\begin{aligned} \mathfrak{D}_\alpha : &= \left\{ q \in \mathcal{H} : q\,(0) = 1, q \prec \frac{1 + (2\alpha - 1)z}{1 - z}, z \in \ \mathbb{D} \right\} \\ &= \{ q \in \mathcal{H} : q\,(0) = 1, Re\, q\,(z) > 1 - \alpha \ for\ z \in \mathbb{D} \} \end{aligned} \tag{1}$$

For functions f, g of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n$$

by $f * g$ we denote the Hadamard product (or convolution) of f and g, defined by

$$(f * g)\,(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in \mathbb{D}).$$

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. Unit disk, Analytic function, Concave function, Starlike function, Convex function, Hadamard product (or convolution), Subordination.

The work here is supported by GUP-2012-023.

To establish our results we need the following theorems and lemma.

Theorem 1 ([8]). *Let $F, G \in \mathcal{H}$ be any convex univalent functions in $\mathbb{D}$. If $f \prec F$ and $g \prec G$, then*

$$f * g \prec F * G \text{ in } \mathbb{D}.$$

Theorem 2 ([1]). *If $f \in C$ and $g \in C$, then $f * g \in C$.*

The problems on subordination and convolution were studied by Ruscheweyh in [6, 7].

Definition 1 ([2]). *The Libera operator is defined by $F(z) = \frac{2}{z}\int_0^z f(t)dt$.*

A function $f : \mathbb{D} \to \mathbb{C}$ is said to belong to the family $C_0(\alpha)$ if f satisfies the following conditions:

(1) f is analytic in $\mathbb{D}$ with the standard normalization $f(0) = f'(0) - 1 = 0$. In addition it satisfies $f(1) = \infty$.
(2) f maps $\mathbb{D}$ conformally onto a set whose complement with respect to $\mathbb{C}$ is convex.
(3) the opening angle of $f(\mathbb{D})$ at ∞ is less than or equal to $\pi\alpha$, $\alpha \in (1, 2]$.

This class has been extensively studied in the recent years and for a detailed discussion about concave functions, we refer to [3].

Theorem 3 ([4]). *Let $\alpha \in (1, 2]$. A function $f \in C_0(\alpha)$ if and only if there exist a starlike function $\phi \in \mathcal{S}^*$ such that $f(z) = \Lambda_\phi(z)$, where*

$$\Lambda_\phi(z) = \int_0^z \frac{1}{(1-t)^{\alpha+1}} \left(\frac{t}{\phi(t)}\right)^{\frac{\alpha-1}{2}} dt.$$

where the series expansion of $\Lambda_\phi(z)$ is $f(z) = z + \frac{c_1}{2}z^2 + \ldots$, where $c_1 = (\alpha+1) - \frac{\alpha-1}{2}\varphi_2, \ldots$.

Lemma 1 ([2]). *Suppose that the function $\Psi : \mathbb{C}^2 \times \mathbb{D} \longrightarrow \mathbb{C}$ satisfies the condition $Re\,[\Psi(ix, y; z)] \leq \delta$ for $x, y \in R$, $x, y \leq -\frac{1+x^2}{2}$ and all $z \in \mathbb{D}$. If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$ is analytic in $\mathbb{D}$ and*

$$Re[\Psi(p(z), zp'(z); z)] > \delta, \text{ for } z \in \mathbb{D}, \tag{2}$$

then $Re[p(z)] > 0$ in $\mathbb{D}$.

The main aim of this work is to prove that the Libera operator $F(z) = \frac{2}{z}\int_0^z f(t)dt$ is starlike by finding certain conditions on $Re\, 2f'$.

2. Main results

To prove the main results we follow similar method of [5]. Our first result is the following:

Lemma 2. *Let $f \in C_0(\alpha), \alpha > 0$. If $f' \in \mathfrak{D}_\alpha$ and $F(z) = \frac{2}{z}\int_0^z f(t)dt$, then*

$$|F'(z)| \leq 1 + 2\alpha(-z - log(1-z))/z \tag{3}$$

This result is sharp.

Proof. It is sufficies to prove that

$$F'(z) \prec 1 + 2\alpha \left\{ \frac{z}{2} + \frac{z^2}{3} + \frac{z^3}{4} + \ldots \right\}. \tag{4}$$

If $f' \in \mathfrak{D}_\alpha$, then

$$f'(z) \prec \mathfrak{d}_\alpha(z) = (1 - 2\alpha) + 2\alpha \frac{1}{1-z} \quad (z \in \mathbb{D}) \tag{5}$$

where $\mathfrak{d}_\alpha$ is the convex univalent function. Define the function by

$$k_1(z) = z + \frac{2}{3}z^2 + \frac{2}{4}z^2 + \ldots = \frac{\log(1-z)^{-2}}{z} - 2. \tag{6}$$

We note that k_1 is convex univalent and normalized function. Thus, $f \in C_0(\alpha)$ and we observe that

$$(f * k_1)(z) = \frac{2}{z}\int_0^z f(t)dt.$$

Also the function

$$k_2(z) = \frac{\log(1-z)^{-2}}{z} - 1 \tag{7}$$

is convex univalent function in $\mathbb{D}$. Thus $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$, we observe

$$(p * k_2)(z) = \frac{2}{z}\int_0^z p(t)dt - 1.$$

Hence from (5), $k_2 \prec k_2$, and by applying Theorem 1, we obtain

$$(f' * k_2)(z) \prec (p_\alpha * k_2)(z). \tag{8}$$

According to Theorem 2, we deduce the right side of (8) is the convex univalent function. Hence (8) gives

$$\frac{f(z)}{z} \prec \frac{1}{z}\int_0^z p_\alpha(t)dt. \tag{9}$$

So we obtain (4) or $|F'(z)| \le 1 + 2\alpha(-z - log(1-z))/z$. □

Theorem 4. *Let $f \in C_0(\alpha)$, $\alpha > 0$. If $f' \in \mathfrak{D}_\alpha$ and $F(z) = \frac{2}{z}\int_0^z f(t)dt$, then*

$$\frac{F(z)}{z} \prec \mathcal{G}(z) \quad (z \in D) \tag{10}$$

where

$$\mathcal{G}(z) = 1 + 2\alpha(\frac{Li_2(z) - z}{z}) \tag{11}$$

is the convex univalent function and

$$1 + 2\alpha\left(\frac{\pi^2}{12} - 1\right) \le Re\frac{F(z)}{z} \le 1 + 2\alpha\left(\frac{\pi^2}{6} - 1\right). \tag{12}$$

The subordination and inequalities are sharp.

Proof. If we let $k_2(z)$ is the same function in (7). Since $f \in C_0(\alpha)$, by Lemma 2, we have $F'(z) \prec 1 + 2\alpha\left\{\frac{z}{2} + \frac{z^2}{3} + \frac{z^3}{4} + \ldots\right\}$ and $k_2 \prec k_2$. We next employ Theorem 1 to obtain sharp result

$$\frac{F(z)}{z} \prec \mathcal{G}(z) \quad (z \in D)$$

where $\mathcal{G}(z)$ is the convex univalent function from Theorem 2 and given by

$$\mathcal{G}(z) = 1 + 2\alpha\left\{\frac{z}{2^2} + \frac{z^2}{3^2} + \frac{z^3}{4^2} + \ldots\right\} = 1 + 2\alpha(\frac{Li_2(z) - z}{z}).$$

Since $\mathcal{G}(z)$ is convex we also have

$$\mathcal{G}(-1) \le Re\frac{F(z)}{z} \le \mathcal{G}(1) \tag{13}$$

where $\mathcal{G}(1) = 1 + 2\alpha\Sigma_{n=1}^{\infty}\frac{1}{(n+1)^2} = 1 + 2\alpha(\frac{\pi^2}{6} - 1)$ and $\mathcal{G}(-1) = 1 + 2\alpha\Sigma_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^2} = 1 + 2\alpha(\frac{\pi^2}{12} - 1)$ So by (13) we obtain (12). □

Theorem 5. *Let $f \in C_0(\alpha)$, $\alpha > 0$. If $f' \in \mathfrak{D}_\alpha$ and $F(z) = \frac{2}{z}\int_0^z f(t)dt$, where*

$$\alpha \leq \frac{30}{36-\pi^2} \simeq 1.1476 \tag{14}$$

then $F(z) = \frac{2}{z}\int_0^z f(t)dt$, is the starlike function.

Proof. Let $p(z) = \frac{zF'(z)}{F(z)}$, $P(z) = \frac{F(z)}{z}$, then $p(z)$ and $P(z)$ are analytic in $\mathbb{D}$ and $p(0) = P(0) = 1$. If

$$\Psi[p(z), zp'(z); z] = P(z)[p^2(z) + zp'(z) + p(z)]$$

then $\Psi[p(z), zp'(z); z] = P(z)[p^2(z) + zp'(z) + p(z)] = 2F'(z) + zF''(z) = 2f'(z)$ and from (5) we deduce that

$$ReP(z)[p^2(z) + zp'(z)] \geq 2(1-\alpha). \tag{15}$$

However, $Re[\Psi(i\rho, \sigma; z)] = Re[P(z)(\sigma - \rho^2 + i\rho)]$ and for $\rho, \sigma \in R$, $\sigma \leq -\frac{1+\rho^2}{2}$. We want to show that

$$Re[\Psi(i\rho, \sigma; z)] \leq 2(1-\alpha). \tag{16}$$

By using these conditions on ρ and σ and by using (12) we deduce that

$$Re[\Psi(i\rho, \sigma; z)] = (\sigma - \rho^2)ReP(z) \leq (\sigma - \rho^2)\mathcal{G}(-1).$$

Then simplifies

$$(\sigma - \rho^2)\mathcal{G}(-1) \leq 2(1-\alpha).$$

We deduce that (14). Then conditions (15) and (16) hold, so Lemma 1 can be applied to obtain $Re\, p(z) > 0$ and $F(z)$ is a starlike function. □

References

[1] Duren, P.L., *Univalent Functions*, Springer-Verlag Inc., New York, 1983.
[2] Miller, S.S., Mocanu, P.T., *Differential Subordinations Theory and Applications*, Marcel Dekker Inc., 2000.
[3] Avkhadiev, F.G., Pommerenke, C., Wirths, K-J., *Sharp inequalities for the coefficient of concave schlicht functions*, Comment . Math. Helv. **81** (2006), 801-807.
[4] Bhowmik, B., Ponnusamy, S., Wirths, K-J., *Characterization and pre-Schwarzian norm estimate for concave univalent functions*, Monatsh. Math. **161** (2010) 59-75.
[5] Sokol, J., *On sufficient condition for starlikeness of certain integral of analytic function*, Jour. Math. Appl. **28** (2006), 127-130.
[6] Ruscheweyh, St., Sheil-Small, T., *Hadamard product of schlicht functions and the Poyla-Schoenberg conjecture*, Comm. Math. Helv. **48** (1973), 119-135.
[7] Ruscheweyh, St., *Convolution in Geometric Function Theory*, Les Presses de I'Univ. de Montreal, 1982.
[8] Ruscheweyh, St., Stankiewicz, J., *Subordination under convex univalent function*, Bull. Pol. Acad. Sci. Math. **33** (1985), 499-502.

Universiti Kebangsaan Malaysia
School of Mathematical Sciences
Faculty of Science and Technology
UKM Bangi 43600, Selangor, Malaysia
E-mail address: epdo04@hotmail.com

Universiti Kebangsaan Malaysia
School of Mathematical Sciences
Faculty of Science and Technology
UKM Bangi 43600, Selangor, Malaysia
E-mail address: maslina@ukm.edu.my