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A DIFFERENTIAL SANDWICH-TYPE RESULT USING AN EXTENDED GENERALIZED SĂLĂGEAN OPERATOR AND EXTENDED RUSCHEWEYH OPERATOR

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Abstract. The purpose of this paper is to introduce sufficient conditions for strong differential subordination and superordination involving the extended derivative operator $DR_{\lambda}^{m,n}$ and also to obtain sandwich-type result.

1. Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let

$$\mathcal{A}_{n\zeta}^{*} = \left\{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\right\},$$

with $\mathcal{A}_{1\zeta}^{*} = \mathcal{A}_{\zeta}^{*}$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$, and

$$\mathcal{H}^{*}[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\},$$

for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

Generalizing the notion of differential subordinations, J.A. Antonino and S. Romaguera have introduced in [16] the notion of strong differential subordinations, which was developed by G.I. Oros and Gh. Oros in [17].

Definition 1 ([17]). *Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z,\zeta) = H(w(z),\zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec H(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1 ([17]). *(i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 1 is equivalent to $f(0,\zeta) = H(0,\zeta)$, for all $\zeta \in \overline{U}$, and $f(U \times \overline{U}) \subset H(U \times \overline{U})$.*

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, the strong subordination becomes the usual notion of subordination.

As a dual notion of strong differential subordination G.I. Oros has introduced and developed the notion of strong differential superordinations in [18].

Definition 2 ([18]). *Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly superordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $H(z,\zeta) = f(w(z),\zeta)$, for all $\zeta \in \overline{U}$. In such a case we write $H(z,\zeta) \prec\prec f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

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Remark 2 ([18]). *(i) Since* $f(z,\zeta)$ *is analytic in* $U\times\overline{U}$*, for all* $\zeta\in\overline{U}$*, and univalent in* U*, for all* $\zeta\in\overline{U}$*, Definition 2 is equivalent to* $H(0,\zeta)=f(0,\zeta)$*, for all* $\zeta\in\overline{U}$*, and* $H\left(U\times\overline{U}\right)\subset f\left(U\times\overline{U}\right)$.

(ii) If $H(z,\zeta)\equiv H(z)$ *and* $f(z,\zeta)\equiv f(z)$*, the strong superordination becomes the usual notion of superordination.*

Definition 3. *We denote by* Q^* *the set of functions that are analytic and injective on* $\overline{U}\times\overline{U}\backslash E(f,\zeta)$*, where* $E(f,\zeta)=\{y\in\partial U:\lim_{z\to y}f(z,\zeta)=\infty\}$*, and are such that* $f'_z(y,\zeta)\neq 0$ *for* $y\in\partial U\times\overline{U}\backslash E(f,\zeta)$*. The subclass of* Q^* *for which* $f(0,\zeta)=a$ *is denoted by* $Q^*(a)$.

For two functions $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)z^j$ and $g(z,\zeta)=z+\sum_{j=2}^{\infty}b_j(\zeta)z^j$ analytic in $U\times\overline{U}$, the Hadamard product (or convolution) of $f(z,\zeta)$ and $g(z,\zeta)$, written as $(f*g)(z,\zeta)$ is defined by

$$f(z,\zeta)*g(z,\zeta)=(f*g)(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)b_j(\zeta)z^j.$$

Definition 4 ([1]). *For* $f\in\mathcal{A}^*_\zeta$*,* $\lambda\geq 0$ *and* $m\in\mathbb{N}$*, the extended generalized Sălăgean operator* D^m_λ *is defined by* $D^m_\lambda:\mathcal{A}^*_\zeta\to\mathcal{A}^*_\zeta$,

$$\begin{aligned}
D^0_\lambda f(z,\zeta) &= f(z,\zeta)\\
D^1_\lambda f(z,\zeta) &= (1-\lambda)f(z,\zeta)+\lambda z f'_z(z,\zeta)=D_\lambda f(z,\zeta)\\
&\ldots\\
D^{m+1}_\lambda f(z,\zeta) &= (1-\lambda)D^m_\lambda f(z,\zeta)+\lambda z\left(D^m_\lambda f(z,\zeta)\right)'_z=D_\lambda\left(D^m_\lambda f(z,\zeta)\right),
\end{aligned}$$

for $z\in U$*,* $\zeta\in\overline{U}$.

Remark 3. *If* $f\in\mathcal{A}^*_\zeta$ *and* $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)z^j$*, then*

$$D^m_\lambda f(z,\zeta)=z+\sum_{j=2}^{\infty}\left[1+(j-1)\lambda\right]^m a_j(\zeta)z^j,$$

for $z\in U$*,* $\zeta\in\overline{U}$.

Definition 5 ([2]). *For* $f\in\mathcal{A}^*_\zeta$*,* $m\in\mathbb{N}$*, the extended Ruscheweyh derivative* R^m *is defined by* $R^m:\mathcal{A}^*_\zeta\to\mathcal{A}^*_\zeta$,

$$\begin{aligned}
R^0 f(z,\zeta) &= f(z,\zeta)\\
R^1 f(z,\zeta) &= z f'_z(z,\zeta)\\
&\ldots\\
(m+1)R^{m+1}f(z,\zeta) &= z\left(R^m f(z,\zeta)\right)'_z+mR^m f(z,\zeta),\ z\in U,\ \zeta\in\overline{U}.
\end{aligned}$$

Remark 4. *If* $f\in\mathcal{A}^*_\zeta$*,* $f(z,\zeta)=z+\sum_{j=2}^{\infty}a_j(\zeta)z^j$*, then*

$$R^m f(z,\zeta)=z+\sum_{j=2}^{\infty}\frac{(m+j-1)!}{m!\,(j-1)!}a_j(\zeta)z^j,\ z\in U,\ \zeta\in\overline{U}.$$

Extending the results from [10] to the class $\mathcal{A}^*_\zeta$ we obtain:

Definition 6 ([11]). *Let* $\lambda\geq 0$ *and* $n,m\in\mathbb{N}$*. Denote by* $DR^{m,n}_\lambda:\mathcal{A}^*_\zeta\to\mathcal{A}^*_\zeta$ *the operator given by the Hadamard product of the extended generalized Sălăgean operator* D^m_λ *and the extended Ruscheweyh operator* R^n,

$$DR^{m,n}_\lambda f(z,\zeta)=\left(D^m_\lambda*R^n\right)f(z,\zeta),$$

for any $z\in U$*,* $\zeta\in\overline{U}$*, and each nonnegative integers* m,n.

Remark 5. *If $f \in \mathcal{A}^*_\zeta$ and $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, then*

$$DR_\lambda^{m,n} f(z,\zeta) = z + \sum_{j=2}^{\infty} [1 + (j-1)\lambda]^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2(\zeta) z^j,$$

for $z \in U$, $\zeta \in \overline{U}$.

Remark 6. *For $m = n$ we obtain the operator DR_λ^m studied in* [12, 13, 14, 15, 3, 4, 5].

For $\lambda = 1$, $m = n$, we obtain the Hadamard product SR^n [6] *of the Sălăgean operator S^n and Ruscheweyh derivative R^n, which was studied in* [7, 8, 9].

Using simple computation one obtains the next result.

Proposition 1. *For $m, n \in \mathbb{N}$ and $\lambda \geq 0$ we have*

$$DR_\lambda^{m+1,n} f(z,\zeta) = (1-\lambda) DR_\lambda^{m,n} f(z,\zeta) + \lambda z \left(DR_\lambda^{m,n} f(z,\zeta)\right)'_z \tag{1}$$

and

$$z \left(DR_\lambda^{m,n} f(z,\zeta)\right)'_z = (n+1) DR_\lambda^{m,n+1} f(z,\zeta) - n DR_\lambda^{m,n} f(z,\zeta). \tag{2}$$

The main object of the present paper is to find sufficient condition for certain normalized analytic functions to satisfy

$$q_1(z,\zeta) \prec\prec \frac{z DR_\lambda^{m+1,n} f(z,\zeta)}{\left(DR_\lambda^{m,n} f(z,\zeta)\right)^2} \prec\prec q_2(z,\zeta),$$

where q_1 and q_2 are given convex and univalent functions in $U \times \overline{U}$ such that $q_1(z,\zeta) \neq 0$ and $q_2(z,\zeta) \neq 0$, for all $z \in U$, $\xi \in \overline{U}$.

In order to prove our strong differential subordination and strong differential superordination results, we make use of the following known results.

Lemma 1. *Let the function q be univalent in $U \times \overline{U}$ and θ and ϕ be analytic in a domain D containing $q\left(U \times \overline{U}\right)$ with $\phi(w) \neq 0$ when $w \in q\left(U \times \overline{U}\right)$. Set $Q(z,\zeta) = z q'_z(z,\zeta) \phi(q(z,\zeta))$ and $h(z,\zeta) = \theta(q(z,\zeta)) + Q(z,\zeta)$. Suppose that*

1. Q is starlike univalent in $U \times \overline{U}$ and

2. $\mathrm{Re}\left(\frac{z h'_z(z,\zeta)}{Q(z,\zeta)}\right) > 0$ for $z \in U$, $\zeta \in \overline{U}$.

If p is analytic with $p(0,\zeta) = q(0,\zeta)$, $p\left(U \times \overline{U}\right) \subseteq D$ and

$$\theta(p(z,\zeta)) + z p'_z(z,\zeta) \phi(p(z,\zeta)) \prec\prec \theta(q(z,\zeta)) + z q'_z(z,\zeta) \phi(q(z,\zeta)),$$

then $p(z,\zeta) \prec\prec q(z,\zeta)$ and q is the best dominant.

Lemma 2. *Let the function q be convex univalent in $U \times \overline{U}$ and ν and ϕ be analytic in a domain D containing $q\left(U \times \overline{U}\right)$. Suppose that*

1. $\mathrm{Re}\left(\frac{\nu'_z(q(z,\zeta))}{\phi(q(z,\zeta))}\right) > 0$ for $z \in U$, $\zeta \in \overline{U}$ and

2. $\psi(z,\zeta) = z q'_z(z,\zeta) \phi(q(z,\zeta))$ is starlike univalent in $U \times \overline{U}$.

If $p(z,\zeta) \in \mathcal{H}^*[q(0,\zeta), 1, \zeta] \cap Q^*$, with $p\left(U \times \overline{U}\right) \subseteq D$ and $\nu(p(z,\zeta)) + z p'_z(z) \phi(p(z,\zeta))$ is univalent in $U \times \overline{U}$ and

$$\nu(q(z,\zeta)) + z q'_z(z,\zeta) \phi(q(z,\zeta)) \prec\prec \nu(p(z,\zeta)) + z p'_z(z,\zeta) \phi(p(z,\zeta)),$$

then $q(z,\zeta) \prec\prec p(z,\zeta)$ and q is the best subordinant.

2. Main results

Theorem 1. *Let* $\frac{z\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)'_{z}}{DR_{\lambda}^{m,n}f(z,\zeta)} \in \mathcal{H}^{*}\left(U\times\overline{U}\right),\ z\in U,\ \zeta\in\overline{U},\ f\in\mathcal{A}_{\zeta}^{*},\ m,n\in\mathbb{N},$ $\lambda\geq 0$ *and let the function* $q(z,\zeta)$ *be convex and univalent in* $U\times\overline{U}$ *such that* $q(0,\zeta)=1$. *Assume that*

$$\operatorname{Re}\left(1+\frac{\alpha}{\beta}q(z,\zeta)-\frac{zq'_{z}(z,\zeta)}{q(z,\zeta)}+\frac{zq''_{z^{2}}(z,\zeta)}{q'_{z}(z,\zeta)}\right)>0,\ z\in U,\ \zeta\in\overline{U}, \tag{3}$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $z\in U$, $\zeta\in\overline{U}$, *and*

$$\psi_{\lambda}^{m,n}(\alpha,\beta;z,\zeta):= \beta(n+1)\frac{\left(DR_{\lambda}^{m,n+1}f(z,\zeta)\right)'_{z}}{\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)'_{z}}+(\alpha-\beta)\frac{z\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)'_{z}}{DR_{\lambda}^{m,n}f(z,\zeta)}-\beta n. \tag{4}$$

If q *satisfies the following strong differential subordination*

$$\psi_{\lambda}^{m,n}(\alpha,\beta,\mu;z,\zeta)\prec\prec\alpha q(z,\zeta)+\frac{\beta zq'_{z}(z,\zeta)}{q(z,\zeta)}, \tag{5}$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$ *then*

$$\frac{DR_{\lambda}^{m+1,n}f(z,\zeta)}{DR_{\lambda}^{m,n}f(z,\zeta)}\prec\prec q(z,\zeta),\ z\in U,\ \zeta\in\overline{U}, \tag{6}$$

and q *is the best dominant.*

Proof. Our aim is to apply Lemma 1. Let the function p be defined by $p(z,\zeta):=\frac{z\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)'_{z}}{DR_{\lambda}^{m,n}f(z,\zeta)}$, $z\in U$, $\zeta\in\overline{U}$, $z\neq 0$, $f\in\mathcal{A}_{\zeta}^{*}$. The function p is analytic in $U\times\overline{U}$ and $p(0,\zeta)=1$.

Differentiating this function, with respect to z, we get

$$zp'_{z}(z,\zeta)=\frac{z\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)'_{z}}{DR_{\lambda}^{m,n}f(z,\zeta)}+\frac{z^{2}\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)''_{z^{2}}}{DR_{\lambda}^{m,n}f(z,\zeta)}-\left[\frac{z\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)'_{z}}{DR_{\lambda}^{m,n}f(z,\zeta)}\right]^{2}$$

By using the identity (2), we obtain

$$\frac{zp'_{z}(z,\zeta)}{p(z,\zeta)}=(n+1)\frac{\left(DR_{\lambda}^{m,n+1}f(z,\zeta)\right)'_{z}}{\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)'_{z}}-\frac{z\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)'_{z}}{DR_{\lambda}^{m,n}f(z,\zeta)}-n \tag{7}$$

By setting $\theta(w):=\alpha w$ and $\phi(w):=\frac{\beta}{w}$, $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$ it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z,\zeta)=zq'_{z}(z,\zeta)\phi(q(z,\zeta))=\frac{\beta zq'_{z}(z,\zeta)}{q(z,\zeta)}$, we find that $Q(z,\zeta)$ is starlike univalent in $U\times\overline{U}$.

Let $h(z,\zeta)=\theta(q(z,\zeta))+Q(z,\zeta)=\alpha q(z,\zeta)+\frac{\beta zq'_{z}(z,\zeta)}{q(z,\zeta)}$, $z\in U$, $\zeta\in\overline{U}$.

If we derive the function Q, with respect to z, perform calculations, we have

$$\operatorname{Re}\left(\frac{zh'_{z}(z,\zeta)}{Q(z,\zeta)}\right)=\operatorname{Re}\left(1+\frac{\alpha}{\beta}q,\zeta-\frac{zq'_{z}(z,\zeta)}{q(z,\zeta)}+\frac{zq''_{z^{2}}(z,\zeta)}{q'_{z}(z,\zeta)}\right)>0.$$

By using (7), we obtain

$$\alpha p(z,\zeta)+\beta\frac{zp'_z(z,\zeta)}{p(z,\zeta)}=$$

$$\alpha\frac{z\left(DR_\lambda^{m,n}f(z,\zeta)\right)'_z}{DR_\lambda^{m,n}f(z,\zeta)}+\beta\left[(n+1)\frac{\left(DR_\lambda^{m,n+1}f(z,\zeta)\right)'_z}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)'_z}-\frac{z\left(DR_\lambda^{m,n}f(z,\zeta)\right)'_z}{DR_\lambda^{m,n}f(z,\zeta)}-n\right]=$$

$$\beta(n+1)\frac{\left(DR_\lambda^{m,n+1}f(z,\zeta)\right)'_z}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)'_z}+(\alpha-\beta)\frac{z\left(DR_\lambda^{m,n}f(z,\zeta)\right)'_z}{DR_\lambda^{m,n}f(z,\zeta)}-\beta n.$$

By using (5), we have $\alpha p(z,\zeta)+\beta\frac{zp'_z(z,\zeta)}{p(z,\zeta)}\prec\prec\alpha q(z,\zeta)+\beta\frac{zq'_z(z,\zeta)}{q(z,\zeta)}$.

Therefore, the conditions of Lemma 1 are met, so we have $p(z,\zeta)\prec\prec q(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\frac{z\left(DR_\lambda^{m,n}f(z,\zeta)\right)'_z}{DR_\lambda^{m,n}f(z,\zeta)}\prec\prec q(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, and q is the best dominant. □

Corollary 1. *Let $q(z)=\frac{\zeta+Az}{\zeta+Bz}$, $-1\le B<A\le 1$, $m,n\in\mathbb{N}$, $\lambda\ge 0$, $z\in U$. Assume that* (3) *holds. If $f\in\mathcal{A}^*_\zeta$ and*

$$\psi_\lambda^{m,n}(\alpha,\beta;z,\zeta)\prec\prec\alpha\frac{\zeta+Az}{\zeta+Bz}+\beta\frac{\zeta(A-B)z}{(\zeta+Az)(\zeta+Bz)},$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\ne 0$, $-1\le B<A\le 1$, where $\psi_\lambda^{m,n}$ is defined in (4)*, then*

$$\frac{z\left(DR_\lambda^{m,n}f(z,\zeta)\right)'_z}{DR_\lambda^{m,n}f(z,\zeta)}\prec\prec\frac{\zeta+Az}{\zeta+Bz}$$

and $\frac{\zeta+Az}{\zeta+Bz}$ is the best dominant.

Proof. For $q(z,\zeta)=\frac{\zeta+Az}{\zeta+Bz}$, $-1\le B<A\le 1$, in Theorem 1 we get the corollary. □

Corollary 2. *Let $q(z,\zeta)=\frac{\zeta+z}{\zeta-z}$, $m,n\in\mathbb{N}$, $\lambda\ge 0$, $z\in U$, $\zeta\in\overline{U}$. Assume that* (3) *holds. If $f\in\mathcal{A}^*_\zeta$ and*

$$\psi_\lambda^{m,n}(\alpha,\beta,\mu;z,\zeta)\prec\prec\alpha\frac{\zeta+z}{\zeta-z}+\beta\frac{2\zeta z}{\zeta^2-z^2},$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\ne 0$, where $\psi_\lambda^{m,n}$ is defined in (4)*, then*

$$\frac{z\left(DR_\lambda^{m,n}f(z,\zeta)\right)'_z}{DR_\lambda^{m,n}f(z,\zeta)}\prec\prec\frac{\zeta+z}{\zeta-z},$$

and $\frac{\zeta+z}{\zeta-z}$ is the best dominant.

Proof. Corollary follows by using Theorem 1 for $q(z,\zeta)=\frac{\zeta+z}{\zeta-z}$

Theorem 2. *Let q be convex and univalent in $U\times\overline{U}$, such that $q(0,\zeta)=1$, $m,n\in\mathbb{N}$, $\lambda\ge 0$. Assume that*

$$\operatorname{Re}\left(\frac{\alpha}{\beta}q(z,\zeta)\right)>0,\ \text{for } \alpha,\beta\in\mathbb{C},\mu\ne 0,\ z\in U,\ \zeta\in\overline{U}. \tag{8}$$

If $f\in\mathcal{A}^*_\zeta$, $\frac{z\left(DR_\lambda^{m,n}f(z,\zeta)\right)'_z}{DR_\lambda^{m,n}f(z,\zeta)}\in\mathcal{H}^*[q(0,\zeta),1,\zeta]\cap Q^*$ and $\psi_\lambda^{m,n}(\alpha,\beta,\mu;z,\zeta)$ is univalent in $U\times\overline{U}$, where $\psi_\lambda^{m,n}(\alpha,\beta;z,\zeta)$ is as defined in (4)*, then*

$$\alpha q(z,\zeta)+\beta\frac{zq'_z(z,\zeta)}{p(z,\zeta)}\prec\prec\psi_\lambda^{m,n}(\alpha,\beta,\mu;z,\zeta),\ z\in U,\ \zeta\in\overline{U}, \tag{9}$$

implies

$$q(z,\zeta) \prec\prec \frac{z\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)_z'}{DR_{\lambda}^{m,n}f(z,\zeta)},\ z\in U,\ \zeta\in\overline{U}, \tag{10}$$

and q is the best subordinant.

□

Proof. Let the function p be defined by $p(z,\zeta) := \frac{z\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)_z'}{DR_{\lambda}^{m,n}f(z,\zeta)}$, $z\in U$, $\zeta\in\overline{U}$, $z\neq 0$, $f\in\mathcal{A}_{\zeta}^*$.

By setting $\nu(w) := \alpha w$ and $\phi(w) := \frac{\beta}{w}$ it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$.
Since q is convex and univalent function, it follows that $\mathrm{Re}\left(\frac{\nu_z'(q(z,\zeta))}{\phi(q(z,\zeta))}\right) = \mathrm{Re}\left(\frac{\alpha}{\beta}q(z,\zeta)\right) > 0$, for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$.

By using (9) we obtain

$$\alpha q(z,\zeta) + \beta\frac{zq_z'(z,\zeta)}{q(z,\zeta)} \prec\prec \alpha p(z,\zeta) + \beta\frac{zp_z'(z,\zeta)}{p(z,\zeta)}.$$

Using Lemma 2, we have

$$q(z,\zeta) \prec\prec p(z,\zeta) = \frac{z\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)_z'}{DR_{\lambda}^{m,n}f(z,\zeta)},\ z\in U,\ \zeta\in\overline{U},$$

and q is the best subordinant. □

Corollary 3. *Let $q(z,\zeta) = \frac{\zeta+Az}{\zeta+Bz}$, $-1\leq B<A\leq 1$, $m,n\in\mathbb{N}$, $\lambda\geq 0$. Assume that* (8) *holds. If $f\in\mathcal{A}_{\zeta}^*$, $\frac{z\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)_z'}{DR_{\lambda}^{m,n}f(z,\zeta)} \in \mathcal{H}^*\left[q(0,\zeta),1,\zeta\right]\cap Q^*$ and*

$$\alpha\frac{\zeta+Az}{\zeta+Bz} + \beta\frac{(A-B)\zeta z}{(\zeta+Az)(\zeta+Bz)} \prec\prec \psi_{\lambda}^{m,n}(\alpha,\beta;z,\zeta),$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, $-1\leq B<A\leq 1$, where $\psi_{\lambda}^{m,n}$ is defined in (4)*, then*

$$\frac{\zeta+Az}{\zeta+Bz} \prec\prec \frac{z\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)_z'}{DR_{\lambda}^{m,n}f(z,\zeta)}$$

and $\frac{\zeta+Az}{\zeta+Bz}$ is the best subordinant.

Proof. For $q(z,\zeta) = \frac{\zeta+Az}{\zeta+Bz}$, $-1\leq B<A\leq 1$ in Theorem 2 we get the corollary. □

Corollary 4. *Let $q(z,\zeta) = \frac{\zeta+z}{\zeta-z}$, $m,n\in\mathbb{N}$, $\lambda\geq 0$. Assume that* (8) *holds. If $f\in\mathcal{A}_{\zeta}^*$, $\frac{z\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)_z'}{DR_{\lambda}^{m,n}f(z,\zeta)} \in \mathcal{H}^*\left[q(0,\zeta),1,\zeta\right]\cap Q^*$ and*

$$\alpha\frac{\zeta+z}{\zeta-z} + \beta\frac{2\zeta z}{\zeta^2-z^2} \prec\prec \psi_{\lambda}^{m,n}(\alpha,\beta;z,\zeta),$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\neq 0$, where $\psi_{\lambda}^{m,n}$ is defined in (4)*, then*

$$\frac{\zeta+z}{\zeta-z} \prec\prec \frac{z\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)_z'}{DR_{\lambda}^{m,n}f(z,\zeta)}$$

and $\frac{\zeta+z}{\zeta-z}$ is the best subordinant.

Proof. Corollary follows by using Theorem 2 for $q(z,\zeta) = \frac{\zeta+z}{\zeta-z}$. □

Combining Theorem 1 and Theorem 2, we state the following sandwich theorem.

Theorem 3. *Let q_1 and q_2 be analytic and univalent in $U \times \overline{U}$ such that $q_1(z,\zeta) \neq 0$ and $q_2(z,\zeta) \neq 0$, for all $z \in U$, $\zeta \in \overline{U}$, with $z(q_1)'_z(z,\zeta)$ and $z(q_2)'_z(z,z)$ being starlike univalent. Suppose that q_1 satisfies* (3) *and q_2 satisfies* (8). *If $f \in \mathcal{A}^*_\zeta$, $\frac{z\left(DR_\lambda^{m,n} f(z,\zeta)\right)'_z}{DR_\lambda^{m,n} f(z,\zeta)} \in \mathcal{H}^*[q(0,\zeta),1,\zeta] \cap Q^2$ and $\psi_\lambda^{m,n}(\alpha,\beta;z,\zeta)$ is as defined in* (4) *univalent in $U \times \overline{U}$, then*

$$\alpha q_1(z,\zeta) + \frac{\beta z(q_1)'_z(,\zeta)}{q_1(z,\zeta)} \prec\prec \psi_\lambda^{m,n}(\alpha,\beta;z,\zeta) \prec\prec \alpha q_2(z,\zeta) + \frac{\beta z(q_2)'_z(z,\zeta)}{q_2(z,\zeta)},$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, implies

$$q_1(z,\zeta) \prec \frac{z\left(DR_\lambda^{m,n} f(z,\zeta)\right)'_z}{DR_\lambda^{m,n} f(z,\zeta)} \prec\prec q_2(z,\zeta),$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

For $q_1(z,\zeta) = \frac{\zeta + A_1 z}{\zeta + B_1 z}$, $q_2(z,\zeta) = \frac{\zeta + A_2 z}{\zeta + B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 5. *Let $m,n \in \mathbb{N}$, $\lambda \geq 0$. Assume that* (3) *and* (8) *hold for $q_1(z,\zeta) = \frac{\zeta + A_1 z}{\zeta + B_1 z}$ and $q_2(z,\zeta) = \frac{\zeta + A_2 z}{\zeta + B_2 z}$, respectively. If $f \in \mathcal{A}^*_\zeta$, $\frac{z\left(DR_\lambda^{m,n} f(z,\zeta)\right)'_z}{DR_\lambda^{m,n} f(z,\zeta)} \in \mathcal{H}^*[q(0,\zeta),1,\zeta] \cap Q^*$ and*

$$\alpha \frac{\zeta + A_1 z}{\zeta + B_1 z} + \beta \frac{(A_1 - B_1)\zeta z}{(\zeta + A_1 z)(\zeta + B_1 z)} \prec\prec \psi_\lambda^{m,n}(\alpha,\beta;z,\zeta)$$
$$\prec\prec \alpha \frac{\zeta + A_2 z}{\zeta + B_2 z} + \beta \frac{(A_2 - B_2)\zeta z}{(\zeta + A_2 z)(\zeta + B_2 z)},$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_\lambda^{m,n}$ is defined in (4), *then*

$$\frac{\zeta + A_1 z}{\zeta + B_1 z} \prec\prec \frac{z\left(DR_\lambda^{m,n} f(z,\zeta)\right)'_z}{DR_\lambda^{m,n} f(z,\zeta)} \prec\prec \frac{\zeta + A_2 z}{\zeta + B_2 z},$$

hence $\frac{\zeta + A_1 z}{\zeta + B_1 z}$ and $\frac{\zeta + A_2 z}{\zeta + B_2 z}$ are the best subordinant and the best dominant, respectively.

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