TJMM
5 (2013), No. 2, 149-156

OSCILLATION AND ASYMPTOTIC BEHAVIOR OF SECOND ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS OF MIXED ARGUMENTS

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Abstract. In this paper the authors present conditions for the oscillatory and asymptotic behavior of solutions of second order nonlinear neutral difference equations of mixed arguments of the form

$$\Delta\Big(r_n\Delta(x_n+ax_{n-\tau}+bx_{n+\sigma})\Big)+p_nx^{\alpha}_{n-k}+q_nx^{\beta}_{n+m}=0$$

where $n\in\mathbb{N}(n_0)$. Examples are provided to illustrate the main results.

1. Introduction

In this paper, we are concerned with the oscillatory and asymptotic behavior of solutions of second order difference equation of the form

$$\Delta\Big(r_n\Delta(x_n+ax_{n-\tau}+bx_{n+\sigma})\Big)+p_nx^{\alpha}_{n-k}+q_nx^{\beta}_{n+m}=0,\ n\in\mathbb{N}(n_0), \tag{1}$$

where $\mathbb{N}(n_0)=\{n_0,n_0+1,\cdots\}$, and n_0 is a positive integer, α and β are ratios of odd positive integers. Further we assume the following conditions without further mention:

(H_1) $\{r_n\}$ is a positive real sequence for all $n\in\mathbb{N}(n_0)$ and $\sum_{n=n_0}^{\infty}\frac{1}{r_n}<\infty$;
(H_2) a and b are nonnegative constants such that $0\le a+b<1$;
(H_3) $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences and not eventually zero for many values of n;
(H_4) τ, σ, k and m are nonnegative integers.

By a solution of equation (1), we mean a real sequence $\{x_n\}$ defined for all $n\ge n_0-\theta_1$, where $\theta_1=max\Big\{\tau,k\Big\}$ and satisfies equation (1) for all $n\ge n_0$. Let $\theta_2=max\Big\{\sigma,m\Big\}$. Clearly, if the initial condition $x_n=\phi_n$ for all $n\in[n_0-\theta_1,\ n_0+\theta_2-1]$ is given, then equation (1) has a unique solution satisfying the initial condition. A solution of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

From the review of literature, it is known that there has been a lot of interest in studying the oscillatory and asymptotic behavior of solutions of difference equations of second order, see for example, [1, 2, 3, 4, 6, 12, 15, 16, 17, 18, 19, 20, 22] and the references cited therein. Recently in [15, 16, 17], the authors considered equation of the type (1) and established some sufficient conditions for the oscillation of all solutions when $\sum_{n=n_0}^{\infty}\frac{1}{r_n}=\infty$.

However when $b=0$ and either $p_n=0$ or $q_n=0$ there are some results available in the literature dealing with the oscillatory behavior of equation (1) when $\sum_{n=n_0}^{\infty}\frac{1}{r_n}<\infty$, see [12, 18, 19, 20, 22] and the references cited therein.

2010 *Mathematics Subject Classification.* 39A11.
Key words and phrases. Second order, nonlinear, neutral type difference equation, oscillation.

Motivated from the above observation, in this paper we obtain some new sufficient conditions for the oscillation of solutions of equation (1) when $\sum_{n=n_0}^{\infty}\frac{1}{r_n}<\infty$. In Section 2, we present some preliminary lemmas and in Section 3, we obtain some new oscillation criteria for equation (1). Section 4 contains some examples to illustrate the main results. Thus the results of this paper are new and extend to those established in [12, 18, 19, 20, 22].

2. Some Preliminary Lemmas

Throughout this paper we use the following notation without further mention:

$$\begin{aligned} z_n &= x_n + a x_{n-\tau} + b x_{n+\sigma}, \\ P_n &= \min\{p_n,\ p_{n-\tau},\ p_{n+\sigma}\}, \\ Q_n &= \min\{q_n,\ q_{n-\tau},\ q_{n+\sigma}\}, \\ R_n &= \sum_{s=n_0}^{n-1}\frac{1}{r_s} \text{ and } P_n^* = P_n + Q_n. \end{aligned}$$

We begin with the following lemma.

Lemma 1. *Let $\{x_n\}$ be an eventually positive solution of equation* (1)*. Then one of the following two cases holds for all sufficiently large n:*

(I) $z_n>0,\ r_n\Delta z_n>0,\ \Delta(r_n\Delta z_n)\le 0$;
(II) $z_n>0,\ r_n\Delta z_n<0,\ \Delta(r_n\Delta z_n)\le 0$.

Proof. Assume that $x_{n-\theta_1}>0$ for $n\ge N\in\mathbb{N}(n_0)$. Then by the condition (H_3), we have $z_n>0$ and $\Delta(r_n\Delta z_n)\le 0$ for $n\ge N$. Hence $\{r_n\Delta z_n\}$ is eventually of one sign. This completes the proof. □

Lemma 2. *Let $\{x_n\}$ be an eventually positive solution of equation* (1)*, and the corresponding $\{z_n\}$ satisfies Case (II) of Lemma 1. If*

$$\sum_{n=n_0}^{\infty}\frac{1}{r_n}\sum_{s=n_0}^{n-1}(p_s+q_s)=\infty, \tag{2}$$

then $\lim_{n\to\infty}x_n=\lim_{n\to\infty}z_n=0$.

Proof. Let $\{x_n\}$ be an eventually positive solution of equation (1) for $n\ge n_1\in\mathbb{N}(n_0)$. Since $z_n>0$ and $\Delta z_n<0$, there is a finite constant ℓ such that $\lim_{n\to\infty}z_n=\ell$. We shall prove that $\ell=0$. Assume that $\ell>0$, then for any $\epsilon>0$ we have $\ell+\epsilon>z_n>\ell$ eventually for $n\ge n_1$. Choose $0<\epsilon<\frac{\ell(1-a-b)}{a+b}$ with $a+b<1$. It is easy to verify that

$$\begin{aligned} x_n &= z_n - a x_{n-\tau} - b x_{n+\sigma}, \text{ for } n\ge n_1 \\ &> \ell - a z_{n-\tau} - b z_{n+\sigma} > \ell-(a+b)(\ell+\epsilon) \\ &> k z_n, \text{ for } n\ge n_1 \end{aligned}$$

where $k=\frac{\ell-(a+b)(\ell+\epsilon)}{\ell+\epsilon}>0$. From the last inequality and equation (1) we have

$$\begin{aligned} \Delta(r_n\Delta z_n) &\le -p_n k^{\alpha} z_{n-k}^{\alpha} - q_n k^{\beta} z_{n+m}^{\beta} \\ &\le -k^{\alpha}\ell^{\alpha}p_n - k^{\beta}\ell^{\beta}q_n \\ &\le -d(p_n+q_n) \end{aligned}$$

where $d = \min\{k^{\alpha}\ell^{\alpha},\ k^{\beta}\ell^{\beta}\}$. Summing the last inequality from n_1 to $n-1$, we obtain

$$\begin{aligned} r_n\Delta z_n - r_{n_1}\Delta z_{n_1} &\leq -d\sum_{s=n_1}^{n-1}(p_s+q_s) \\ -\Delta z_n &\geq d\frac{1}{r_n}\sum_{s=n_1}^{n-1}(p_s+q_s). \end{aligned}$$

Summing the last inequality from n_1 to ∞, we obtain

$$z_{n_1} \geq d\sum_{n=n_1}^{\infty}\frac{1}{r_n}\sum_{s=n_1}^{n-1}(p_s+q_s).$$

This contradicts to (2). Thus $\ell = 0$. Moreover, the inequality $0 < x_n \leq z_n$ implies that $\lim_{n\to\infty} x_n = 0$. The proof is now complete. □

Lemma 3. *Let $A \geq 0, B \geq 0$ and $\alpha \geq 1$. Then*

$$A^{\alpha} + B^{\alpha} \geq \frac{1}{2^{\alpha-1}}(A+B)^{\alpha}. \tag{3}$$

Lemma 4. *Assume $A \geq 0, B \geq 0$ and $0 < \alpha \leq 1$. Then*

$$A^{\alpha} + B^{\alpha} \geq (A+B)^{\alpha}. \tag{4}$$

The proof of Lemmas 3 and 4 can be found in [10].

Lemma 5. *Let $\{x_n\}$ be a positive solution of equation (1) satisfies Case(I) of Lemma 2. Then*

$$z_n \geq R_n r_n \Delta z_n$$

for all $n \geq N$.

Proof. Since $\Delta(r_n\Delta z_n) \leq 0$ and $r_n\Delta z_n > 0$, we see that

$$z_n = z_N + \sum_{s=N}^{n-1}\Delta z_s \geq \sum_{s=N}^{n-1}\frac{1}{r_s}r_s\Delta z_s \geq r_n\Delta z_n\sum_{s=N}^{n-1}\frac{1}{r_s}.$$

The proof is now complete. □

3. Oscillation Results

In this section we obtain some sufficient conditions for the oscillation and asymptotic behavior of all solutions of equation (1). We begin with the following theorem.

Theorem 1. *Let the condition (3) holds. Assume that $1 \leq \alpha \leq \beta$, and*

$$\Delta w_n + \frac{P_n}{4^{\alpha-1}}\frac{R_{n-k}^{\alpha}\ w_{n-k+\tau}^{\alpha}}{(1+a^{\alpha}+\frac{b^{\alpha}}{2^{\alpha-1}})^{\alpha}} \leq 0,\ \ n \in \mathbb{N}(n_0) \tag{5}$$

has no eventually positive nonincreasing solutions, then every solution $\{x_n\}$ of equation (1) is either oscillatory or $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1). Without loss of generality, we may assume that $x_{n-\theta_1} > 0$ for all $n \geq N_1 \in \mathbb{N}(n_0)$, where N_1 is chosen so that both the cases of Lemma 1 hold for all $n \geq N_1$. From the equation (1) for all $n \geq N_1$, we have

$$\begin{aligned}&\Delta(r_n\Delta z_n) + p_n x^{\alpha}_{n-k} + q_n x^{\beta}_{n+m} + a^{\alpha}\Delta(r_{n-\tau}\Delta z_{n-\tau})\\&+a^{\alpha}p_{n-\tau}x^{\alpha}_{n-k-\tau} + a^{\alpha}q_{n-k}x^{\beta}_{n+m-\tau} + \frac{b^{\alpha}}{2^{\alpha-1}}\Delta(r_{n+\sigma}\Delta z_{n+\sigma})\\&+\frac{b^{\alpha}}{2^{\alpha-1}}p_{n+\sigma}x^{\alpha}_{n-k+\sigma} + \frac{b^{\alpha}}{2^{\alpha-1}}q_{n+\sigma}x^{\beta}_{n+m+\sigma} = 0,\ n \geq N_1.\end{aligned} \tag{6}$$

Since $\alpha \leq \beta,\ a < 1$ and $b < 1$, we have

$$\begin{aligned}&\Delta(r_n\Delta z_n) + a^{\alpha}\Delta(r_{n-\tau}\Delta z_{n-\tau}) + \frac{b^{\alpha}}{2^{\alpha-1}}\Delta(r_{n+\sigma}\Delta z_{n+\sigma})\\&+P_n\Big[x^{\alpha}_{n-k} + a^{\alpha}x^{\alpha}_{n-k-\tau} + \frac{b^{\alpha}}{2^{\alpha-1}}x^{\alpha}_{n-k+\sigma}\Big]\\&+Q_n\Big[x^{\beta}_{n+m} + a^{\beta}x^{\beta}_{n+m-\tau} + \frac{b^{\beta}}{2^{\beta-1}}x^{\beta}_{n+m+\sigma}\Big] \leq 0,\ n \geq N_1.\end{aligned}$$

Using the inequality (3), we obtain for $n \geq N_1$

$$\Delta(r_n\Delta z_n) + a^{\alpha}\Delta(r_{n-\tau}\Delta z_{n-\tau}) + \frac{b^{\alpha}}{2^{\alpha-1}}\Delta(r_{n+\sigma}\Delta z_{n+\sigma}) + \frac{P_n}{4^{\alpha-1}}z^{\alpha}_{n-k} + \frac{Q_n}{4^{\beta-1}}z^{\beta}_{n+m} \leq 0 \tag{7}$$

or

$$\Delta(r_n\Delta z_n + a^{\alpha}r_{n-\tau}\Delta z_{n-\tau} + \frac{b^{\alpha}}{2^{\alpha-1}}r_{n+\sigma}\Delta z_{n+\sigma}) + \frac{P_n}{4^{\alpha-1}}z^{\alpha}_{n-k} \leq 0,\ n \geq N_1. \tag{8}$$

Next we consider the two cases of Lemma 1

Case (I). Assume that Case (I) of Lemma 1 holds for all $n \geq N_1$. From (8), and Lemma 5, we obtain

$$\Delta(y_n + a^{\alpha}y_{n-\tau} + \frac{b^{\alpha}}{2^{\alpha-1}}y_{n+\sigma}) + \frac{P_n}{4^{\alpha-1}}R^{\alpha}_{n-k}y^{\alpha}_{n-k} \leq 0 \tag{9}$$

where $y_n = r_n\Delta z_n$ and $y_n > 0$ is nonincreasing for all $n \geq N_1$. Now we define, $w_n = y_n + a^{\alpha}y_{n-\tau} + \frac{b^{\alpha}}{2^{\alpha-1}}y_{n+\sigma}$, then $w_n > 0$, $n \geq N_1$, and

$$w_n \leq (1 + a^{\alpha} + \frac{b^{\alpha}}{2^{\alpha-1}})y_{n-\tau},\ n \geq N_1.$$

Using the last inequality in (9), we obtain

$$\Delta w_n + \frac{P_n}{4^{\alpha-1}}\frac{R^{\alpha}_{n-k}\ w^{\alpha}_{n+\tau-k}}{(1 + a^{\alpha} + \frac{b^{\alpha}}{2^{\alpha-1}})^{\alpha}} \leq 0,\ n \geq N_1.$$

Thus $\{w_n\}$ is an eventually positive nonincreasing solution of (5) which is a contradiction.

Case (II). Assume that Case (II) of Lemma 1 holds. Then by Lemma 2, we prove that $\lim_{n\to\infty} x_n = 0$. This completes the proof. □

Corollary 1. *Assume $\alpha = 1$ and $\beta \geq 1$ and $k - \tau \geq 1$ hold. If the condition (3) and*

$$\lim_{n\to\infty}\inf \sum_{s=n+\tau-k}^{n-1} P_s R_{s-k} > M\Big(\frac{k-\tau}{1+k-\tau}\Big)^{1+k-\tau} \tag{10}$$

where $M = (1 + a + b)$ are satisfied, then every solution $\{x_n\}$ of equation (1) is either oscillatory or $\lim_{n\to\infty} x_n = 0$.

Proof. The proof follows from Theorem 1 and [13, Theorem 2]. □

Corollary 2. *Assume* $\alpha > 1$, $k - \tau \geq 1$ *and condition* (3) *hold. If there exists a* $\lambda > \frac{1}{k-\tau} log\alpha$ *such that*

$$\lim_{n\to\infty} \inf \left[P_n R^{\alpha}_{n-k} exp(-e^{\lambda n}) \right] > 0, \ n \in N \tag{11}$$

then every solution $\{x_n\}$ *of equation* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

Proof. The proof follows from Theorem 1 and [21, Theorem 2]. □

Theorem 2. *Let condition* (3) *holds. Assume that* $\alpha > \beta \geq 1$, *and*

$$\Delta u_n + \frac{P_n}{4^{\alpha-1}} \frac{R^{\alpha}_{n-k} \ u^{\alpha}_{n+\tau-k}}{(1 + a^{\beta} + \frac{b^{\beta}}{2^{\beta-1}})^{\alpha}} \leq 0, \ n \in N \tag{12}$$

has no eventually positive nonincreasing solutions, then every solution $\{x_n\}$ *of equation* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1). Without loss of generality, we may assume that there is a $N \in \mathbb{N}(n_0)$ such that $x_n > 0$ and $x_{n-\theta_1} > 0$ for all $n \geq N$. Then by Lemma 1, there are two cases for $\{z_n\}$. Assume that Case (I) holds for all $n \geq n_2 \geq N$. Proceeding as in Theorem 1 we obtain

$$\Delta(y_n + a^{\beta} y_{n-\tau} + \frac{b^{\beta}}{2^{\beta-1}} y_{n+\sigma}) + \frac{P_n}{4^{\alpha-1}} R^{\alpha}_{n-k} y^{\alpha}_{n-k} \leq 0, \ n \geq n_2. \tag{13}$$

Now we define $u_n = y_n + a^{\beta} y_{n-\tau} + \frac{b^{\beta}}{2^{\beta-1}} y_{n+\sigma}$, then $u_n > 0$, and

$$u_n \leq (1 + a^{\alpha} + \frac{b^{\alpha}}{2^{\alpha-1}}) y_{n-\tau}, \ n \geq n_2.$$

Using the last inequality in (13), we obtain

$$\Delta u_n + \frac{P_n}{4^{\alpha-1}} \frac{R^{\alpha}_{n-k} \ u^{\alpha}_{n+\tau-k}}{(1 + a^{\beta} + \frac{b^{\beta}}{2^{\beta-1}})^{\alpha}} \leq 0, \ n \geq n_2.$$

Thus $\{u_n\}$ is an eventually positive nonincreasing solution of (12) which is a contradiction. Case (II) follows from Theorem 1 (Case (II)). This completes the proof. □

Theorem 3. *Let condition* (3) *holds. Assume that* $0 < \alpha < \beta < 1$, *and*

$$\Delta v_n + \frac{P_n \ R^{\alpha}_{n-k} \ v^{\alpha}_{n+\tau-k}}{(1 + a^{\alpha} + b^{\alpha})^{\alpha}} \leq 0, \ n \in \mathbb{N}(n_0) \tag{14}$$

has no eventually positive nonincreasing solutions, then every solution $\{x_n\}$ *of equation* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

Proof. The proof is similar to that of Theorem 1 by using Lemma 4 instead of Lemma 3, and hence the details are omitted. □

Theorem 4. *Let condition* (3) *holds. Assume that* $0 < \beta < \alpha < 1$, *and*

$$\Delta h_n + \frac{P_n \ R^{\alpha}_{n-k} \ h^{\alpha}_{n+\tau-k}}{(1 + a^{\beta} + b^{\beta})^{\alpha}} \leq 0, \ n \in \mathbb{N}(n_0) \tag{15}$$

has no eventually positive nonincreasing solutions, then every solution $\{x_n\}$ *of equation* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

Proof. The proof is similar to that of Theorem 2 by using Lemma 4 instead of Lemma 3, and hence the details are omitted. □

Corollary 3. *Let condition* (3) *hold. If* $0 < \alpha < 1,\ \tau < k$ *and*

$$\sum_{n=n_0}^{\infty} P_n R_{n-k}^{\alpha} = \infty \tag{16}$$

then every solution $\{x_n\}$ *of equation* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

Proof. The proof follows from Theorem 2 and [21, Theorem 1]. □

Theorem 5. *Let condition* (3) *holds. Assume that* $1 \le \alpha = \beta,\ k \ge \tau$ *and*

$$\Delta w_n + \frac{P_n^*\ R_{n-k}^{\alpha}\ w_{n+\tau-k}^{\alpha}}{4^{\alpha-1}(1 + a^{\alpha} + \frac{b^{\alpha}}{2^{\alpha-1}})^{\alpha}} \le 0,\ n \in \mathbb{N}(n_0) \tag{17}$$

has no eventually positive nonincreasing solutions, then every solution $\{x_n\}$ *of equation* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

Proof. Proceeding as in the proof of Theorem 1, we obtain (7), that is,

$$\Delta(r_n\Delta z_n)+a^{\alpha}\Delta(r_{n-\tau}\Delta z_{n-\tau})+\frac{b^{\alpha}}{2^{\alpha-1}}\Delta(r_{n+\sigma}\Delta z_{n+\sigma})+\frac{P_n}{4^{\alpha-1}}z_{n-k}^{\alpha}+\frac{Q_n}{4^{\alpha-1}}z_{n+m}^{\beta} \le 0, n \ge n_1$$

Case (I). For this case $\{z_n\}$ is increasing and we have from the last inequality

$$\Delta(r_n\Delta z_n + a^{\alpha}r_{n-\tau}\Delta z_{n-\tau} + \frac{b^{\alpha}}{2^{\alpha-1}}r_{n+\sigma}\Delta z_{n+\sigma}) + \frac{P_n^*}{4^{\alpha-1}}z_{n-k}^{\alpha} \le 0,\ \text{for } n \ge N. \tag{18}$$

The rest of the proof is similar to that of Theorem 1 of Case (I).
Case (II). The proof follows from Case (II) of Theorem 1. The proof is complete. □

Corollary 4. *Assume* $\alpha = 1$, *and* $k - \tau \ge 1$ *hold. If the condition* (3) *and*

$$\lim_{n\to\infty}\inf \sum_{s=n+\tau-k}^{n-1} P_s^* R_{s-k} > M\Big(\frac{k-\tau}{1+k-\tau}\Big)^{1+k-\tau},\ n \in \mathbb{N}(n_0) \tag{19}$$

where $M = (1 + a + b)$ *are satisfied, then every solution* $\{x_n\}$ *of equation* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

Proof. The proof follows from Theorem 5 and [13, Theorem 2]. □

Corollary 5. *Assume* $\alpha > 1$, $k \ge \tau$ *and condition* (3) *hold. If there exists a* $\lambda > \frac{1}{k-\tau} log\alpha$ *such that*

$$\lim_{n\to\infty}\inf \Big[P_n^* R_{n-k}^{\alpha} exp(-e^{\lambda n})\Big] > 0, \tag{20}$$

then every solution $\{x_n\}$ *of equation* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

Proof. The proof follows from Theorem 5 and [21, Theorem 2]. □

Theorem 6. *Let condition* (3) *holds. Assume that* $0 < \alpha = \beta < 1$, $k > \tau$, *and*

$$\Delta w_n + \frac{P_n^*\ R_{n-k}^{\alpha}\ w_{n+\tau-k}^{\alpha}}{(1 + a^{\alpha} + b^{\alpha})^{\alpha}} \le 0,\ n \in \mathbb{N}(n_0) \tag{21}$$

has no eventually positive nonincreasing solutions, then every solution $\{x_n\}$ *of equation* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

Proof. The proof is similar to that of Theorem 5 by using Lemma 4 instead of Lemma 3, and hence the details are omitted. □

Corollary 6. *Assume* $0 < \alpha < 1$ *and* $k - \tau < 0$ *holds. If condition* (3) *and*

$$\sum_{s=n_0}^{\infty} P_s^* R_{s-k}^{\alpha} = \infty \tag{22}$$

then every solution $\{x_n\}$ *of equation* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

Proof. The proof follows from Theorem 6 and [21, Theorem 1]. □

4. Examples

In this section we present some examples to illustrate the main results.

Example 1. *Consider the difference equation*

$$\Delta\Big(n(n+1)\Delta\Big(x_n + \frac{1}{4}x_{n-2} + \frac{1}{2}x_{n+3}\Big)\Big) + \frac{p}{n}x_{n-4} + \frac{q}{n}x_{n+3} = 0,\ n \geq 1. \tag{23}$$

Here $r_n = n(n+1)$, $a = \frac{1}{4}$, $b = \frac{1}{2}$, $p_n = \frac{p}{n} > 0$, $q_n = \frac{q}{n} > 0$, $\tau = 2$, $\sigma = 3$, $k = 4$, $m = 3$ *and* $\alpha = \beta = 1$. *Then* $R_n = \frac{n-1}{n}$, $P_n = \frac{p}{n+3}$, $Q_n = \frac{q}{n+3}$ *and* $P_n^* = \frac{p+q}{n+3}$. *Since*

$$\lim_{n\to\infty}\inf \sum_{s=n+\tau-k}^{n-1} P_s^* R_{s-k} = 2(p+q)$$

then by Corollary 4 we conclude that the solution $\{x_n\}$ *of equation* (23) *is either oscillatory or* $\lim_{n\to\infty} = 0$ *if* $p + q > \frac{7}{27}$.

Example 2. *Consider the difference equation*

$$\Delta\Big((n+4)(n+5)\Delta\Big(x_n + \frac{1}{2}x_{n-2} + \frac{1}{4}x_{n+3}\Big)\Big) + p x_{n-4}^{1/3} + q x_{n+3}^{1/5} = 0,\ n \geq 1. \tag{24}$$

Here $r_n = (n+4)(n+5)$, $a = \frac{1}{2}$, $b = \frac{1}{4}$, $p_n = p$, $q_n = q$, $\tau = 2$, $\sigma = 3$, $k = 4$, $m = 3$ *and* $\alpha = \frac{1}{3}$, $\beta = \frac{1}{5}$. *Then* $R_n = \frac{n-1}{n}$, $P_n = p$, $Q_n = q$. *Since*

$$\sum_{s=n_0}^{\infty} P_s R_{s-k}^{\alpha} = \sum_{n=6}^{\infty} p\Big(\frac{n-5}{n}\Big)^{1/3} = \infty.$$

then by Corollary 3 we conclude that the solution $\{x_n\}$ *of equation* (24) *is either oscillatory or* $\lim_{n\to\infty} = 0$.

Example 3. *Consider the difference equation*

$$\Delta\Big(2^{n+1}\Delta(x_n + \frac{1}{3}x_{n-2} + \frac{1}{5}x_{n+3})\Big) + e^{e^n}x_{n-3}^3 + 3^n x_{n+3}^5 = 0,\ n \geq 1. \tag{25}$$

Here $r_n = 2^{n+1}$, $a = \frac{1}{3}$, $b = \frac{1}{5}$, $p_n = e^{e^n}$, $q_n = 3^n$, $\tau = 2$, $\sigma = 3$, $k = 3$, $m = 3$ *and* $\alpha = 3$, $\beta = 5$. *Then* $R_n = \frac{2^{n-1}-1}{2^n}$, $P_n = \frac{e^{e^n}}{n+3}$, $Q_n = \frac{3^n}{n+3}$ *and* $P_n^* = \frac{e^{e^n}+3^n}{n+3}$. *Since*

$$\begin{aligned}\lim_{n\to\infty}\inf \Big[P_n^* R_{n-k}^{\alpha} exp(-e^{\lambda n})\Big] &= \lim_{n\to\infty}\inf \Big[e^{e^n}\Big(\frac{2^{n-1}-1}{2^n}\Big)^3 e^{-e^n}\Big] \\ &= \lim_{n\to\infty}\inf \Big(\frac{2^{n-5}-1}{2^n-4}\Big) = \frac{1}{2} > 0\end{aligned}$$

then by Corollary 5 we conclude that the solution $\{x_n\}$ *of equation* (25) *is either oscillatory or* $\lim_{n\to\infty} = 0$.

We conclude this paper with the following remark.

Remark 1. *It would be interesting to extend the results of this paper to the equation*

$$\Delta\Big(r_n\Delta(x_n + a_n x_{n-\tau} + b_n x_{n+\sigma})\Big) = p_n x^{\alpha}_{n-k} + q_n x^{\beta}_{n+m},\ n \geq n_0.$$

with $\sum_{n=n_0}^{\infty} \frac{1}{r_n} < \infty$.

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