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THE OPTIMALITY CONDITIONS FOR THE TOLL ROAD BILEVEL PROGRAMMING PROBLEMS

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Abstract. In a general traffic network, Di Wu, Yafeng Yin and Hai Yang (2011) proved that the level of service, represented by the volume-capacity (v/c) ratio, offered by a profit-maximizing private firm on a private toll road is independent of another competitor's choice of capacity and toll rate for another private toll road. However, the optimality condition for the toll road bilevel programming is not discussed in Di Wu, Yafeng Yin and Hai Yang (2011). In this paper, the Abadie constraint qualifications and KKT conditions for the toll road bilevel programming is proposed and analyzed, which is the base and prerequisite of the main results in Di Wu, Yafeng Yin and Hai Yang (2011).

1. Introduction

The private provision of roads in an oligpolistic market is very interesting problem, which is assumed that some or all of the roads are built and then operated by individual private firms. Each private firm only controls one road. For a private road, the concession period is pre-determined to be the road life and the firm simultaneously decides the capacity to construct and the toll rate to charge given its belief on other firms choice in order to maximize profit. These private firms compete for the same travel demand of multiple origin-destination (OD) pairs and the underlying flow distribution is assumed to be in user equilibrium. Xiao et. al (2007) studied both toll and capacity competition in a network with one OD pair connected by parallel links, and discovered the property of constant v/c ratio. Yang et. al (2009) observed the same property in their numerical example on a simple but more general network. Di Wu, Yafeng Yin and Hai Yang (2011) prove that in a general nework, the level of of service, represented by the volume-capacity(v/c) ratio, offered by a profit-maximizing private firm on a private toll road is independent of another competitor's choice of capacity and toll rate for another private toll road, and is same as the v/c ratio in the socially optimal provision of roads. However, the optimality condition for the toll road bilevel programming is not discussed in Di Wu, Yafeng Yin and Hai Yang (2011). The contribution of this paper is to propose and analyze the Abadie constraint qualifications and KKT conditions for the toll road bilevel programming, which is the base and prerequisite of the main results in Di Wu, Yafeng Yin and Hai Yang (2011).

2. A toll road bilevel programming problem

Consider a general traffic network $G(N, A)$, where N is the set of nodes and A the set of links in the network. Let W be the set of OD pairs and K the set of paths connecting all OD pairs. Let d_w denote the travel demand for OD pair, $w \in W$, f_k the flow on route $k \in K$ and v_a the flow on link $a \in A$.

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It is assumed that there are multiple private toll roads in the network with each being controlled by one individual private firm. The firm simultaneously decides the road capacity to built and the toll charge to collect from road users. We denote the capacity and toll as c_a and τ_a respectively for link $a \in L$, where L is the set of private toll roads and $L \subsetneqq A$. We further assume that travelers will choose routes with minimum generalized travel cost, which includes travel time and toll charge. Consequently, the network flow distribution is in user equilibrium.

It is assumed that each firm attempts to choose both capacity and toll to maximize its profit and it can accurately predict the choices of other firms. Consequently, Nash equilibrium will be achieved in such an ologopolistic market where no private firm is able to further increase its profit by unilaterally changing its choice on capacity or toll. In the following, we focus on the decision made by a private firm for a particular toll road, say link a, in the network. The profit-maximizing choice of toll and capacity can be obtained by solving a bi-level optimization problem below, which is shorted by BLPP

$$\begin{aligned} &\min I(c_a) - \tau_a v_a \\ &s.t. \quad \tau_a, c_a \geqslant 0 \end{aligned} \tag{1}$$

where v_a is obtained by solving the following problem

$$\begin{aligned} \min_{f,v} \sum_{j\in A} \int_0^{v_j} t_j(\overline{W}, c_j) d\overline{W} + \sum_{j\in L} v_j \tau_j \\ s.t. \sum_{k\in K} \bigwedge_k^w f_k = d_w, \ \forall w \in W \\ f_k \geq 0, \ \forall k \in K \\ v_j = \sum_{k\in K} f_j \Delta_{jk}, \ \forall j \in A \end{aligned} \tag{2}$$

In the above, $I(c_a)$ is the amortized cost for constructing and maintaining capacity c_a; f and v are the vectors of path and link flows respectively; $t_j(c_j, v_j)$ is travel cost function of link j, $\bigwedge_k^w$ is OD path index,and when path k connects OD pair w; otherwise. Similarly, $\triangle_{jk}$ is a link-path index and $\triangle_{jk} = 1$ when path k uses link j; otherwise $\triangle_{jk} = 0$.

3. Main results

Denote by

$$Y = \left\{ v \in R^m : \sum_{k\in K} \bigwedge_k^w f_k = d_w, \forall w \in W, -f_k \leqslant 0, \forall k \in K, \sum_{k\in K} f_j \triangle_{jk} = v_j, \forall v_j \in A \right\}$$

the feasible region of the lower-level program. Let be a local optimal solution of BLPP,

$$\psi(\tau, c, v, v') := \max_{v'\in Y} \sigma(\tau, c, v, v')$$

where $\sigma(\tau, c, v, v') := \sum_{j\in A} \int_0^{v_j} t_j(\overline{W}, c_j) d\overline{W} + \sum_{j\in L} v_j \tau_j - \left(\sum_{j\in A} \int_0^{v'_j} t_j(\overline{W}, c_j) d\overline{W} + \sum_{j\in L} v'_j \tau_j \right)$. Let $S(\tau, c)$ be the set of solutions of lower-level programming problem.

Lemma 1. *Let $v \in S(\tau, c)$. Then the solution set of $\psi(\tau, c, v)$ is given by $S(\tau, c)$.*

Proof. Because $v \in S(\tau, c)$, it is easy to see that $\sigma(\tau, c, v, v') = 0$ for any $v' \in S(\tau, c)$. Hence,to prove that $S(\tau, c)$ is the set of solutions, it suffices to prove that $\sigma(\tau, c, v, v') \leq 0$, for any $v' \in Y$. On the contrary, suppose that for some $v' \in Y$, then

$$\sum_{j\in A}\int_0^{v_j} t_j(\overline{w}, c_j)d\overline{w} + \sum_{j\in L} v_j\tau_j > \left(\sum_{j\in A}\int_0^{v'_j} t_j(\overline{w}, c_j)d\overline{w} + \sum_{j\in L} v'_j\tau_j\right),$$

which contradicts the fact that $v \in S(\tau, c)$, and hence the solution set of the problem is given by $S(\tau, c)$. By virtue of Lemma 1, the local optimal solution (τ, c, v) of BLPP may become a local optimal solution of the following problem

$$\begin{aligned} (SP)_W \quad & \min I(c_a) - \tau_a v_a \\ s.t. \quad & \tau_a, c_a \geqslant 0 \\ & \psi(\tau, c, v) \leq 0 \\ \sum_{k\in K} \bigwedge_k^w f_k &= d_w, \forall w \in W \\ f_k &\geq 0,\ \forall k \in K \\ v_j &= \sum_{k\in K} f_j \triangle_{jk} \end{aligned} \tag{3}$$

□

Definition 1. *Let M be a closed subset in R^n. The contingent cone of M at $\overline{x}$ is closed cone defined by*

$$T(\overline{x}, M) := \{v \in X : \exists t_n \downarrow 0, v_n \to v, s.t. \overline{x} + t_n v_n \in M, \forall n \in N\}.$$

Definition 2. *We define the nonsmooth linearization cone of feasible set $\mathscr{F}$ of the BLPP at the local optimal solution (τ, c, v) as the set:*

$$L((\tau, c, v), \mathscr{F}) := \left\{u : w^T u \leq 0, w \in \partial^0\psi(\tau, c, v), u \geq 0\right\}$$

where a^T denotes the transpose of vector a, $\partial^0\psi$ denote the Clark generalized gradient.

Definition 3. *We define the extended linearization cone of feasible set $\mathscr{F}$ of the BLPP at the local optimal solution (τ, c, v) as the set:*

$$L'((\tau, c, v), \mathscr{F})) := \left\{u : w^T u \leq 0, w \in W, u \geq 0\right\},$$

where a^T denotes the transpose of vector a.

Definition 4 (N. Abadie CQ)**.** *Let $(\tau, c, v) \in \mathscr{F}$, we say that the nonsmooth Abadie constraint qualification holds, if $L((\tau, c, v), \mathscr{F}) \subseteq L((\tau, c, v), \mathscr{F})$.*

Lemma 2. *Let (τ, c, v) be a local solution of the BLPP. If the nonsmooth Abadie CQ holds at (τ, c, v), then*

$$0 \in [-v_a, \partial I/\partial c_a, -\tau_a, 0, \ldots, 0]^T + clcone\left(\partial^0\psi(\tau, c, v) \bigcup [0, 0, \ldots, 0, -1, \ldots, -1]\right),$$

where cone A denotes the convex cone generated by set A.

Proof. It is well known that because (τ, c, v) is a local minimizers for BLPP,

$$[-v_a, \partial I/\partial c_a, -\tau_a, 0, \ldots, 0]^T u \geq 0, \forall u \in T((\tau, c, v), \mathscr{F}).$$

Now suppose that the nonsmooth Abadie CQ holds at (τ, c, v). Then,

$$[-v_a, \partial I/\partial c_a, -\tau_a, 0, \ldots, 0]^T u \geq 0, \forall u \in L((\tau, c, v), \mathscr{F})$$

Consequently, $[-v_a, \partial I/\partial c_a, -\tau_a, 0, \ldots, 0]^T u \geq 0$, whenever $\max_{\alpha \in C} \alpha^T u \geq 0$, where, C denotes the convex cone generated by $\partial^0 \psi(\tau, c, v) \bigcup [0, 0, \ldots, 0, -1, \ldots, -1]$. Thus,the function $u \to [-v_a, \partial I/\partial c_a, -\tau_a, 0, \ldots, 0]^T u + \delta_{C^0}(v)$ attains its minimium at 0, where

$$C^0 = \left\{ w \in R^{n+m} : w^T u \geq 0, \forall u \in C \right\}$$

is the polar cone of C, and δ_{C^0} is the indicator function of set C^0. So, one has

$$0 \in [-v_a, \partial I/\partial c_a, -\tau_a, 0, \ldots, 0]^T + \delta_{C^0}(0),$$

Because $\delta_{C^0}(0) = C^{00} = clC$, the above inclusion is the same as

$$0 \in [-v_a, \partial I/\partial c_a, -\tau_a, 0, \ldots, 0]^T + clC.$$

□

Theorem 1. *Let (τ, c, v) be a local optimal solution of BLPP with $S(\tau, c, v) \neq \emptyset$. Then, under the N. Abadie CQ, (τ, c, v) satisfies KKT condition.*

Proof. By the virtue of Lemma 1, the local optimal solution of BLPP is a feasible solution of problem $(SP)_W$. The desired result follows from replacing the liniearization cone $L((\tau, c, v), \mathscr{F})$ by the extended liniearization $L'((\tau, c, v), \mathscr{F})$ in the proof of Lemma 2. □

4. Conclusions

We have proved that in a general network under N. Abadie CQ, the KKT condition holds, which is the base of of the main results in Di Wu, Yafeng Yin and Hai Yang (2011). Our future study will find some new necessary optimality conditions for the toll road bilevel programming problems.

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