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NONUNIFORM POLYNOMIAL TRICHOTOMY OF EVOLUTION OPERATORS IN BANACH SPACES

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Abstract. In this paper we define the notions of nonuniform polynomial trichotomy and polynomial trichotomy and we emphasize connections between these concepts. The approach is motivated by various examples.

1. Introduction

The exponential dichotomy property has gained prominence since the appearance of two fundamental monographs due to J.L. Massera and J.J. Schaffer [7], respectively J.L. Daleckii and M.G. Krein [6].

Diverse and important concepts of exponential dichotomy were studied by C. Chicone and Y. Latushkin in [4], S.N. Chow and H. Leiva in [5], R.S. Saker and G.R. Sell in [12]. For other papers about exponential dichotomies we refer to [1], [9] and the reference therein.

A generalization of the concept of dichotomy can by considered the concept of trichotomy, introduced by R.S. Sacker and G.R. Sell [11] in 1976.

The main idea in the study of trichotomy is to obtain at a certain moment a decomposition of the state space in three subspaces: the stable, the unstable and the central manifold.

Recently, the case of exponential trichotomy was studied by C. Stoica and M. Megan in [13], respectively L.H. Popescu and T. Vesselenyi in [10]. The case of polynomial dichotomy was also studied by L. Barreira, M. Fan, C. Valls and J. Zhang in [2], and A. Bento and C. Silva in [3], who obtained characterizations in discrete and continuous time for evolution operators in Banach spaces. Other results for the case of polynomial stability where due to M. Megan, T. Ceaușu and M.L. Rămneanțu in [8].

In this paper we investigate two nonuniform polynomial trichotomy concepts for the general case of evolution operators in Banach spaces. Our approach is based on the extension of the techniques for exponential trichotomy to the case of polynomial trichotomy.

The main objective is to give integral characterizations for nonuniform polynomial trichotomies. Some simple example are included to illustrate the connections between the trichotomy concepts considered in this paper.

2. Notations. Definitions

Let us denote by X a real or complex Banach space and by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X.

We consider the sets $\Delta = \{(t, s) \in \mathbb{R}_+^2 : t \geq s\}$ and $T = \{(t, s, r) \in \mathbb{R}_+^3 : t \geq s \geq r\}$

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Definition 1. *A mapping* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ *is said to be an evolution operator on* X *if the following relations hold:*

$e_1)$ $\Phi(t,t) = I$ *(the identity operator), for all* $t \geq 0$*;*
$e_2)$ $\Phi(t,s)\Phi(s,r) = \Phi(t,r)$*, for all* $(t,s,r) \in T$.

Definition 2. *An evolution operator* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ *is said to be strongly measurable, if for all* $(s,x) \in \mathbb{R}_+ \times X$ *the mapping defined by* $t \longmapsto \|\Phi(t,s)x\|$ *is measurable on* $[s,\infty)$.

Definition 3. *A strongly measurable application* $P : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ *is said to be a projection family on* X *if*

$$P^2(t) = P(t), \text{ for all } t \geq 0.$$

Definition 4. *Three projection families* $P_1,\ P_2,\ P_3 : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ *are said to be compatible with the evolution operator* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ *if*

(i) $P_1(t) + P_2(t) + P_3(t) = I$*, for all* $t \geq 0$*;*
(ii) $P_i(t)P_j(t) = 0$*, for all* $t \geq 0$*,* $i,j \in \{1,2,3\}$*,* $i \neq j$*;*
(iii) $\Phi(t,r)P_k(r) = P_k(t)\Phi(t,r)$*, for all* $(t,r) \in \Delta$ *and* $k \in \{1,2,3\}$.

In what follows, if $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ is an evolution operator on X and $P_1,\ P_2,\ P_3 : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ are projection families compatible with Φ, then we will denote by

$$\Phi_k(t,r) = \Phi(t,r)P_k(r), \text{ for all } (t,r) \in \Delta \text{ and all } k \in \{1,2,3\}.$$

3. Exponential trichotomy concepts

Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be an evolution operator and $P_1,\ P_2,\ P_3 : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ be the projection families compatible with Φ.

Definition 5. *An evolution operator* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ *is said to be uniformly trichotomic (and denote u.t) if there exists* $N \geq 1$ *such that :*
(ut_1)

$$\|\Phi_1(t,r)x\| \leq N\|\Phi_1(s,r)x\|; \tag{1}$$

(ut_2)

$$\|\Phi_2(s,r)x\| \leq N\|\Phi_2(t,r)x\|; \tag{2}$$

(ut_3)

$$\|\Phi_3(s,r)x\| \leq N\|\Phi_3(t,r)x\|, \tag{3}$$

for all $(t,s,r,x) \in T \times X$.

Definition 6. *The evolution operator* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ *is said to be nonuniformly exponentially trichotomic (and denote n.e.t) if there exists a constant* $\alpha > 0$ *and a nondecreasing function* $N : \mathbb{R}_+ \longrightarrow [1,\infty)$ *such that*
$(n.e.t_1)$

$$e^{\alpha(t-s)}\|\Phi_1(t,r)x\| \leq N(s)\|\Phi_1(s,r)x\|; \tag{4}$$

$(u.e.t_2)$

$$e^{\alpha(t-s)}\|\Phi_2(s,r)x\| \leq N(t)\|\Phi_2(t,r)x\|; \tag{5}$$

$(n.e.t_3)$

$$e^{-\alpha(t-s)}\|\Phi_3(t,r)x\| \leq N(s)\|\Phi_3(s,r)x\|, \tag{6}$$

for all $(t,s,r,x) \in T \times X$.

Definition 7. *The evolution operator* $\Phi : \Delta \longrightarrow \mathcal{B}(\mathcal{X})$ *is said to be exponentially trichotomic (and denote e.t) if there are some constants* $N \geq 1$, $\alpha > 0$ *and* $\beta \geq 0$ *such that*

$(e.t_1)$

$$e^{\alpha(t-s)}\|\Phi_1(t,r)x\| \leq Ne^{\beta\, s}\|\Phi_1(s,r)x\|; \tag{7}$$

$(e.t_2)$

$$e^{\alpha(t-s)}\|\Phi_2(s,r)x\| \leq Ne^{\beta\, t}\|\Phi_2(t,r)x\|; \tag{8}$$

$(e.t_3)$

$$e^{-\alpha(t-s)}\|\Phi_3(t,r)x\| \leq Ne^{\beta\, s}\|\Phi_3(s,r)x\|, \tag{9}$$

for all $(t,s,r,x) \in T \times X$.

The constant $\alpha > 0$ is called *the exponential trichotomy constant* of Φ.

4. Polynomial trichotomy concepts

Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be an evolution operator and $P_1,\ P_2,\ P_3 : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ be the projector families compatible with the evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$.

Definition 8. *The evolution operator* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ *is said to be nonuniformly polynomially trichotomic (and denote n.p.t) if there is a nondecreasing function* $N : \mathbb{R}_+ \longrightarrow [1,\infty)$ *and a constant* $\alpha > 1$ *such that:*

$(n.p.t_1)$

$$\left(\frac{t+1}{s+1}\right)^{\alpha}\|\Phi_1(t,r)x\| \leq N(s)\|\Phi_1(s,r)x\|; \tag{10}$$

$(n.p.t_2)$

$$\left(\frac{t+1}{s+1}\right)^{\alpha}\|\Phi_2(s,r)x\| \leq N(t)\|\Phi_2(t,r)x\|; \tag{11}$$

$(n.p.t_3)$

$$\left(\frac{s+1}{t+1}\right)^{\alpha}\|\Phi_3(t,r)x\| \leq N(s)\|\Phi_3(s,r)x\|, \tag{12}$$

for all $(t,s,r,x) \in T \times X$.

The constant $\alpha > 1$ is called *the polynomial trichotomy constant* of Φ.

In particular, when the function N is constant, we say that Φ is *uniformly polynomially trichotomic.*

Proposition 1. *If the uniform trichotomic evolution operator* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ *is nonuniformly exponentially trichotomic with the exponenetial trichotomy constant* $\alpha > 1$*, then it is nonuniformly polynomially trichotomic.*

Proof. According to the hypothesis there exist $\alpha > 1$ and a nondecreasing function $N : \mathbb{R}_+ \longrightarrow [1,\infty)$ such that

$$\left(\frac{t+1}{s+1}\right)^{\alpha}\|\Phi_1(t,r)x\| \leq \left(\frac{e^t}{e^s}\right)^{\alpha}\|\Phi_1(t,r)x\| \leq N(s)\|\Phi_1(s,r)x\|,$$

$$\left(\frac{t+1}{s+1}\right)^{\alpha}\|\Phi_2(s,r)x\| \leq \left(\frac{e^t}{e^s}\right)^{\alpha}\|\Phi_2(s,r)x\| \leq N(t)\|\Phi_2(t,r)x\|,$$

and

$$\left(\frac{s+1}{t+1}\right)^{\alpha}\|\Phi_3(t,r)x\| \leq N\|\Phi_3(t,r)x\| \leq N(s)\|\Phi_3(s,r)x\|,$$

for all $(t,s,r,x) \in T \times X$. Hence Φ is n.p.t. $\square$

The converse implication of Proposition 1 is not true, in general. We present an example of evolution operator which is nonuniformly polynomially trichotmic, but is not nonuniformly exponentially trichotomic.

Example 1. *Let $X = \mathbb{R}^3$. The mapping $\Phi : \Delta \longrightarrow \mathcal{B}(X)$,*

$$\Phi(t,s)x = \left(\left(\frac{s+2}{t+2}\right)^2 x_1, \left(\frac{t+2}{s+2}\right)^2 x_2, \frac{t+1}{s+1}x_3\right)$$

is an evolution operator on X.

We consider the projectors P_1, P_2, $P_3 : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$, $P_1(s)x = (x_1, 0, 0)$, $P_2(s)x = (0, x_2, 0)$ and $P_3(s)x = (0, 0, x_3)$, for all $x = (x_1, x_2, x_3) \in X$, compatible with the evolution operator Φ. We obtain that

$$\left(\frac{t+1}{s+1}\right)^2 \|\Phi_1(t,s)x\| = \frac{(t+1)^2(s+2)^2}{(s+1)^2(t+2)^2}\|P_1(s)x\| \leq (s+1)^3\|P_1(s)x\|,$$

$$\left(\frac{t+1}{s+1}\right)^2 \|P_2(s)x\| \leq \frac{(t+2)^2(s+1)^3}{(s+2)^2}\|P_2(s)x\| = (s+1)^3\|\Phi_2(t,s)x\|$$

and

$$\left(\frac{s+1}{t+1}\right)^2 \|\Phi_3(t,s)x\| = \left(\frac{s+1}{t+1}\right)^2 \cdot \left(\frac{t+1}{s+1}\right)^2 \|P_3(s)x\| \leq N(s)\|P_3(s)\|,$$

for all $(t,s,x) \in \Delta \times \mathbb{R}^3$. It results that Φ is n.p.t.

If we suppose that Φ is n.e.t, then there exist a nondecreasing function $N : \mathbb{R}_+ \longrightarrow [1,\infty)$ and a constant $\alpha > 0$ such that

$$e^{\alpha(t-s)}\left(\frac{s+2}{t+2}\right)^2 \leq N(s)$$

for all $(t,s) \in \Delta$. If we consider $s = 0$ and $t \longrightarrow \infty$, we obtain a contradiction, which shows that Φ is not n.e.t.

Definition 9. *Three projection families P_1, P_2, $P_3 : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ are said to be* polynomially compatible with the evolution operator $\Phi : \Delta \longrightarrow \mathcal{B}(X)$, *if*

(c_1) *$P_1(t) + P_2(t) + P_3(t) = I$, for all $t \geq 0$;*

(c_2) *$P_i(t)P_j(t) = 0$, for all $t \geq 0$, $i,j \in \{1,2,3\}$, $i \neq j$;*

(c_3) *$\Phi(t,r)P_k(r) = P_k(t)\Phi(t,r)$, for all $(t,r) \in \Delta$ and all $k \in \{1,2,3\}$;*

(c_4) *there exist $M \geq 1$ and $\omega > 0$ such that*

$$\|\Phi_1(t,r)x\| \leq M\left(\frac{t+1}{s+1}\right)^\omega \|\Phi_1(s,r)x\|$$

and

$$\|\Phi_2(s,r)x\| \leq M\left(\frac{t+1}{s+1}\right)^\omega \|\Phi_2(t,r)x\|,$$

for all $(t,s,r,x) \in T \times X$.

Theorem 1. *Let $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ be a strongly measurable evolution operator on the Banach space X and let P_1, P_2, $P_3 : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ be three projector families polynomially compatible with Φ. The evolution operator Φ is nonuniformly polynomially trichotomic if and only if there exist a nondecreasing function $D : \mathbb{R}_+ \longrightarrow [1,\infty)$ and a constant $d > 0$ such that*

(a)

$$\int_s^t \frac{(\tau+1)^d}{(s+1)^{d+1}}\|\Phi_1(\tau,r)x\|\,d\tau \leq D(s)\|\Phi_1(s,r)x\|; \tag{13}$$

(b)
$$\int_s^t \frac{(t+1)^{d+1}}{(\tau+1)^d} \|\Phi_2(\tau,r)x\| \, d\tau \le D(t)\|\Phi_2(t,s)x\|; \tag{14}$$
(c)
$$\int_s^t \left(\frac{s+1}{\tau+1}\right)^{d+1} \|\Phi_3(\tau,r)x\| \, d\tau \le D(s)\|\Phi_3(s,r)x\|, \tag{15}$$
for all $(t,s,r,x) \in T \times X$.

Proof. Necessity. Let $(t,s,x) \in \Delta \times X$ and let $d \in (0, \alpha-1)$. We have

$$\int_s^t \frac{(\tau+1)^d}{(s+1)^{d+1}} \|\Phi_1(\tau,r)x\| d\tau \le N(s)(s+1)^{\alpha-d-1} \|\Phi_1(s,r)x\| \int_s^t (\tau+1)^{d-\alpha} d\tau \le \\ \le D(s)\|\Phi_1(s,r)x\|,$$

$$\int_s^t \frac{(t+1)^{d+1}}{(\tau+1)^d} \|\Phi_2(\tau,r)x\| d\tau \le N(t)(t+1)^{d-\alpha+1} \|\Phi_2(t,r)x\| \int_s^t (\tau+1)^{\alpha-d} d\tau \le \\ \le D(t)\|\Phi_2(t,r)x\|,$$

where $D(t) = \frac{N(t)}{\alpha-d-1}(t+1)^2$.
Using (12) and integrating on $[s,t]$ we obtain (15).
Sufficiency. Let $(t,s,x) \in \Delta \times X$. If $t \ge 2s+1$ we have that

$$\left(\frac{t+1}{s+1}\right)^{d+1} \|\Phi_1(t,r)x\| = \frac{2}{t+1} \int_{\frac{t-1}{2}}^t \frac{(\tau+1)^d}{(s+1)^{d+1}} \frac{(t+1)^{d+1}}{(\tau+1)^d} \|\Phi_1(t,r)x\| d\tau \le \\ \le 2\int_{\frac{t-1}{2}}^t \frac{(\tau+1)^d}{(s+1)^{d+1}} \left(\frac{t+1}{\tau+1}\right)^d \|\Phi_1(t,\tau)\| \|\Phi_1(\tau,r)x\| d\tau \le \\ \le 2^{d+\omega+1} M \int_s^t \frac{(\tau+1)^d}{(s+1)^{d+1}} \|\Phi_1(\tau,r)x\| d\tau \le 2^{d+\omega+1} M D(s) \|\Phi_1(s,r)x\|.$$

For $t \in [s, 2s+1)$ we have that

$$\left(\frac{t+1}{s+1}\right)^{d+1} \|\Phi_1(t,r)x\| \le 2^{d+1} M \left(\frac{t+1}{s+1}\right) \|\Phi_1(s,r)x\| \le 2^{d+\omega+1} M \|\Phi_1(s,r)x\|.$$

So

$$\|\Phi_1(t,r)x\| \le N(s) \left(\frac{s+1}{t+1}\right)^{d+1} \|\Phi_1(s,r)x\|, \tag{16}$$

where $N(s) = 2^{d+\omega+1} M D(s)$.
If (15) holds and $t \ge 2s+1$ we have that

$$\left(\frac{t+1}{s+1}\right)^{d+1} \|\Phi_2(s,r)x\| = \frac{1}{s+1} \int_s^{2s+1} \frac{(t+1)^{d+1}}{(\tau+1)^d} \frac{(\tau+1)^d}{(s+1)^{d+1}} \|\Phi_2(s,r)x\| d\tau \le \\ \le \frac{M}{s+1} \int_s^{2s+1} \frac{(t+1)^{d+1}}{(\tau+1)^d} \frac{(\tau+1)^d}{(s+1)^{d+1}} \left(\frac{\tau+1}{s+1}\right)^\omega \|\Phi_2(\tau,r)x\| d\tau \le \\ \le \frac{2^{d+\omega} M}{(s+1)^2} \int_s^t \frac{(t+1)^{d+1}}{(\tau+1)^d} \|\Phi_2(\tau,r)x\| d\tau \le 2^{d+\omega} M D(t) \|\Phi_2(t,s)x\|.$$

For $t \in [s, 2s+1)$ we have that

$$\left(\frac{t+1}{s+1}\right)^{d+1} \|\Phi_2(s,r)x\| \le 2^{d+1} M \left(\frac{t+1}{s+1}\right)^\omega \|\Phi_2(t,r)x\| \le 2^{d+\omega+1} M \|\Phi_2(t,r)x\|.$$

So

$$\|\Phi_2(s,r)x\| \le N(t) \left(\frac{s+1}{t+1}\right)^{d+1} \|\Phi_2(t,r)x\|. \tag{17}$$

Relation (12) follows from (15). Hence, according to Definition 8, Φ is n.p.t. □

Definition 10. *The evolution operator* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ *is said to be polynomially trichotomic (and denote p.t) if there are some constants* $N \geq 1$, $\alpha > 1$ *and* $\beta \geq 0$ *such that:*
$(p.t_1)$

$$\left(\frac{t+1}{s+1}\right)^{\alpha} \|\Phi_1(t,r)x\| \leq N(s+1)^{\beta}\|\Phi_1(s,r)x\|; \tag{18}$$

$(p.t_2)$

$$\left(\frac{t+1}{s+1}\right)^{\alpha} \|\Phi_2(s,r)x\| \leq N(t+1)^{\beta}\|\Phi_2(t,r)x\|; \tag{19}$$

$(p.t_3)$

$$\left(\frac{s+1}{t+1}\right)^{\alpha} \|\Phi_3(t,r)x\| \leq N(s+1)^{\beta}\|\Phi_3(s,r)x\|, \tag{20}$$

for all $(t,s,r,x) \in T \times X$.

Remark 1. *If the evolution operator* $\Phi : \Delta \longrightarrow \mathcal{B}(\mathcal{X})$ *is exponentially trichotomic, then* Φ *is polynomially trichotomic. The converse statement is not true, as shown in what follows.*

Example 2. *We consider* $X = \mathbb{R}^3$, *the projectors* P_1, P_2, P_3 *defined as Example 1 and the evolution operator* $\Phi : \Delta \longrightarrow \mathcal{B}(\mathbb{R}^3)$,

$$\Phi(t,s)x = \left(\frac{(s+1)^3+1}{(t+1)^3+1}x_1, \frac{(t+1)^3+1}{(s+1)^3+1}x_2, \frac{t+1}{s+1}x_3\right).$$

We have that

$$\left(\frac{t+1}{s+1}\right)^2 \|\Phi_1(t,s)x\| = \frac{(t+1)^2}{(s+1)^2}\frac{(s+1)^3+1}{(t+1)^3+1}\|P_1(s)x\| \leq 2(s+1)^4\|P_1(s)x\|,$$

$$2(t+1)^4\|\Phi_2(t,s)x\| = 2(t+1)^4\frac{(t+1)^3+1}{(s+1)^3+1}\|P_2(s)x\| \geq \left(\frac{t+1}{s+1}\right)^2 \|P_2(s)x\|,$$

and

$$\left(\frac{s+1}{t+1}\right)^2 \|\Phi_3(t,s)x\| = \left(\frac{s+1}{t+1}\right)^2 \frac{t+1}{s+1}\|P_3(s)\| \leq 2(s+1)^4\|P_3(s)\|,$$

for all $(t,s,x) \in \Delta \times \mathbb{R}^3$, *which shows that* Φ *is p.t.*
We assume that if Φ *is e.t, then there exist* $N \geq 1$, $\alpha > 0$ *and* $\beta \geq 0$ *such that*

$$e^{\alpha(t-s)}\frac{(s+1)^3+1}{(t+1)^3+1} \leq Ne^{\beta s},$$

for all $(t,s) \in \Delta$. *Therefore, for* $s = 0$ *and* $t \longrightarrow \infty$, *we obtain a contradiction, which shows that* Φ *is not e.t.*

Remark 2. *If the evolution operator* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ *is polynomially trichotomic, then* Φ *is nonuniformly polynomially trichotomic. The following example shows that the converse implication is not true.*

Example 3. *Let* $X = \mathbb{R}^3$, $P_1(s)x = (x_1,0,0)$, $P_2(s)x = (0,x_2,0)$ *and* $P_3(s)x = (0,0,x_3)$. *We consider the functions* $N : [0,\infty) \longrightarrow [1,\infty)$, $N(t) = (t+2)^2e^{t^2}$ *and* $u : [0,\infty) \longrightarrow [1,\infty)$

$$u(t) = \begin{cases} e^{n^2} & if \quad t = n \\ e^4 & if \quad t = n + \frac{1}{n^2} \\ 1 & if \quad t \neq n, t \neq n + \frac{1}{n^2}, \end{cases}$$

for every $n \in \mathbb{N}^*$. *Then* $\Phi : \Delta \longrightarrow \mathcal{B}(\mathbb{R}^3)$,

$$\Phi(t,s)x = \left(\frac{(s+2)^2u(s)}{(t+2)^2u(t)}x_1, \frac{(t+2)^2u(t)}{(s+2)^2u(s)}x_2, \frac{(t+1)^2u(t)}{(s+1)^2u(s)}x_3\right)$$

has the property

$$\begin{aligned}\left(\tfrac{t+1}{s+1}\right)^2 \|\Phi_1(t,s)x\| = \tfrac{(t+1)^2}{(s+1)^2}\tfrac{(s+2)^2u(s)}{(t+2)^2u(t)}\|P_1(s)x\| &\le (s+2)^2u(s)\|P_1(s)x\| \\ &\le N(s)\|P_1(s)x\|,\end{aligned}$$

$$N(t)\|\Phi_2(t,s)x\| = N(t)\frac{(t+2)^2u(t)}{(s+2)^2u(s)}\|P_2(s)x\| \ge \left(\frac{t+1}{s+1}\right)^2\|P_2(s)x\|$$

and

$$\left(\tfrac{s+1}{t+1}\right)^2 \|\Phi_3(t,s)x\| = \tfrac{(s+1)^2(t+1)^2u(t)}{(t+1)^2(s+1)^2u(s)}\|P_3(s)x\| \le u(t)\|P_3(s)x\| \le N(t)\|P_3(s)x\|,$$

for all $(t,s,x) \in \Delta \times X$. *This show that* Φ *is n.p.t.*
If we suppose that Φ *is p.t., then there are* $N \ge 1$, $\alpha > 1$ *and* $\beta \ge 0$ *such that*

$$\left(\frac{t+1}{s+1}\right)^{\alpha}\frac{(s+2)^2u(s)}{(t+2)^2u(t)} \le N(s+1)^{\beta},$$

for all $(t,s) \in \Delta$. *Then for* $s = n$ *and* $t = n + \frac{1}{n^2}$ *we obtain*

$$\left(\frac{1+1/n^3+1/n}{1+1/n}\right)^{\alpha}\left(\frac{1+2/n}{1+1/n^3+2/n}\right)^2 \le N\frac{n^{\beta}}{e^{n^2}}(1+1/n)^{\beta}e^4,$$

which for $n \longrightarrow \infty$ *gives a contradiction and hence* Φ *is not p.t.*

A characterization for polynomial trichotomy concept is given by

Corollary 1. *Let* $\Phi : \Delta \longrightarrow \mathcal{B}(X)$ *be a strongly measurable evolution operator on the Banach space* X *and let* $P_1, P_2, P_3 : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$ *be three projector families polynomial compatible with* Φ. *The evolution operator* Φ *is polynomially trichotomic if and only if there exist* $K \ge 1$, $\delta > 0$ *and* $\xi \ge 0$ *such that*
(a)

$$\int_s^t \frac{(\tau+1)^{\delta}}{(s+1)^{\delta+1}}\|\Phi_1(\tau,r)x\|\,d\tau \le K(s+1)^{\xi}\|\Phi_1(s,r)x\|; \tag{21}$$

(b)

$$\int_s^t \frac{(t+1)^{\delta+1}}{(\tau+1)^{\delta}}\|\Phi_2(\tau,r)x\|\,d\tau \le K(t+1)^{\xi}\|\Phi_2(t,s)x\|; \tag{22}$$

(c)

$$\int_s^t \left(\frac{s+1}{\tau+1}\right)^{\delta+1}\|\Phi_3(\tau,r)x\|\,d\tau \le K(s+1)^{\xi}\|\Phi_3(s,r)x\|, \tag{23}$$

for all $(t,s,r,x) \in T \times X$.

Proof. Necessity it results for $K = \frac{N}{\alpha-d-1}$, $\delta \in (0,\alpha-1)$ and $\xi = \beta+1$.
Sufficiency. It results from the proof of Theorem 1 for $D(t) = K(t+1)^{\xi}$. □

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