TJMM
5 (2013), No. 2, 129-136

SOME INTEGRAL INEQUALITIES FOR TWO KINDS OF CONVEXITIES VIA FRACTIONAL INTEGRALS

MUHAMMAD ASLAM NOOR AND MUHAMMAD UZAIR AWAN

Abstract. In this paper, we consider the classes of s-convex function of first and second kinds respectively. We derive several new Hermite-Hadamard type of inequalities for these classes via fractional integrals.

1. Introduction and preliminary results

In recent years a lot of attention has been given to theory of convex functions. As a result classical convexity has been generalized and extended in different ways using different and new ideas see [1, 4, 6, 9, 13, 14, 17, 18]. Breckner [1] introduced the notion of s-convex functions which is usually known as s-convex functions of second kind in the literature. Inspired by this Toader et al. [13] established an other generalization of classical convex functions, which is commonly known as s-convex function of first kind. For the applications and other aspects of s-convex functions, see [1, 5, 17, 18].

This ongoing research motivated researchers to study the generalizations of different types of integral inequalities using these new generalizations of classical convex functions see [2, 3, 5, 6, 9, 10, 11, 12, 15, 16, 17, 18, 19]. Sarikaya et al. [12] established Hermite-Hadamard inequalities via fractional integrals. Since then many researchers have derived several Hermite-Hadamard type of inequalities for different generalizations of classical convex functions via fractional integrals see [9, 11, 12, 15].

In this paper, we consider the classes of $s-$convex functions of first and second kind. We prove several new inequalities of Hermite-Hadamard type for $s-$convex functions of first and second kind via fractional integrals. The ideas used in this paper may stimulate future research. This is the main motivation of this paper.

Before we establish our main results, we recall some previously known concepts.

Definition 1 ([7, 8]). *Let $f \in L_1[a,b]$. Then Riemann-Liouville integrals $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a,$$

and

$$J^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b,$$

where

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}x^{\alpha-1}dx,$$

2010 *Mathematics Subject Classification.* 26A33, 26D15, 26A51.

Key words and phrases. Convex functions, s-convex functions, fractional integrals, Hermite-Hadamard inequality.

is the Gamma function.

The following result is very important and well known result in the literature.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

the above double inequality is known as Hermite-Hadamard inequality in the literature.

For various generalizations and extensions of Hermite-Hadamard type of inequalities, and their applications in pure and applied sciences, interested readers are referred to [2, 3, 5, 6, 9, 10, 11, 12, 15, 16, 17, 18, 19] and the references therein.

Let $\mathbb{R}$ be the set of real numbers. Throughout the paper $I = [a, b] \subset \mathbb{R}$ be the interval unless otherwise specified.

In this section, we prove following lemma which plays a key part in proving our next results.

Lemma 1. *Let $f : [a, b] \to \mathbb{R}$ be twice differentiable function on (a, b) with $a < b$. If $f'' \in L[a, b]$, then, we have the following equality for fractional integrals:*

$$\begin{aligned} &\tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right] - f\left(\tfrac{a+b}{2}\right) \\ &= \tfrac{(b-a)^2}{8(\alpha+1)}\int_0^1 (1-t)^{\alpha+1}\left[f''\left(\tfrac{1+t}{2}a + \tfrac{1-t}{2}b\right) + f''\left(\tfrac{1-t}{2}a + \tfrac{1+t}{2}b\right)\right]dt. \end{aligned}$$

Proof. Let

$$\begin{aligned} I &= \int_0^1 (1-t)^{\alpha+1}\left[f''\left(\tfrac{1+t}{2}a + \tfrac{1-t}{2}b\right) + f''\left(\tfrac{1-t}{2}a + \tfrac{1+t}{2}b\right)\right]dt \\ &= \int_0^1 (1-t)^{\alpha+1} f''\left(\tfrac{1+t}{2}a + \tfrac{1-t}{2}b\right)dt + \int_0^1 (1-t)^{\alpha+1} f''\left(\tfrac{1-t}{2}a + \tfrac{1+t}{2}b\right)dt = I_1 + I_2. \end{aligned} \tag{1}$$

Integrating I_1 as:

$$\begin{aligned} I_1 &= \int_0^1 (1-t)^{\alpha+1} f''\left(\tfrac{1+t}{2}a + \tfrac{1-t}{2}b\right)dt \\ &= \left|-\tfrac{2(1-t)^{\alpha+1}f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)}{b-a}\right|_0^1 - \tfrac{2(\alpha+1)}{b-a}\int_0^1 (1-t)^\alpha f'\left(\tfrac{1+t}{2}a + \tfrac{1-t}{2}b\right)dt \\ &= \tfrac{2}{b-a}f'\left(\tfrac{a+b}{2}\right) - \tfrac{2(\alpha+1)}{b-a}\left[\tfrac{2}{b-a}f\left(\tfrac{a+b}{2}\right) - \tfrac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)\right] \\ &= \tfrac{2}{b-a}f'\left(\tfrac{a+b}{2}\right) - \tfrac{4(\alpha+1)}{(b-a)^2}f\left(\tfrac{a+b}{2}\right) + \tfrac{2^{\alpha+2}\Gamma(\alpha+2)}{(b-a)^{\alpha+2}}J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a). \end{aligned} \tag{2}$$

Now integrating I_2 as:

$$\begin{aligned} I_2 &= \int_0^1 (1-t)^{\alpha+1} f''\left(\tfrac{1-t}{2}a + \tfrac{1+t}{2}b\right)dt \\ &= \left|\tfrac{2(1-t)^{\alpha+1}f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)}{b-a}\right|_0^1 + \tfrac{2(\alpha+1)}{b-a}\int_0^1 (1-t)^\alpha f'\left(\tfrac{1-t}{2}a + \tfrac{1+t}{2}b\right)dt \\ &= -\tfrac{2}{b-a}f'\left(\tfrac{a+b}{2}\right) + \tfrac{2(\alpha+1)}{b-a}\left[-\tfrac{2}{b-a}f'\left(\tfrac{a+b}{2}\right) + \tfrac{2^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right] \\ &= -\tfrac{2}{b-a}f'\left(\tfrac{a+b}{2}\right) - \tfrac{4(\alpha+1)}{(b-a)^2}f'\left(\tfrac{a+b}{2}\right) + \tfrac{2^{\alpha+2}\Gamma(\alpha+2)}{(b-a)^{\alpha+2}}J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b). \end{aligned} \tag{3}$$

Summation of (1), (2) and (3) and then multiplying both sides by $\frac{(b-a)^2}{8}$ completes the proof. □

We would like to point out that for $\alpha = 1$ our Lemma 1 reduces to following result.

Lemma 2. *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable function on (a,b) with $a < b$. If $f'' \in L[a,b]$, then, we have the following equality for fractional integrals:*

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{16}\int_0^1 (1-t)^2\left[f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt.$$

2. Inequalities for s-convexity of second kind

In this section, we establish some inequalities for s-convexity of second kind. First we give the definition of s-convex functions, which is mainly due to Breckner [1].

Definition 2 ([1])**.** *A function $f : I \to (0,\infty)$ is said to be s-convex function in the second kind, if*

$$f((1-t)x+ty) \le (1-t)^s f(x) + t^s f(y), \quad \forall x,y \in I, t\in[0,1], s\in(0,1]. \tag{4}$$

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable function on (a,b) with $a < b$. If $f'' \in L[a,b]$ and $|f''|$ is $s-$convex function of second kind, then, we have the following inequality for fractional integrals:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|\\ &\le \frac{(b-a)^2}{2^{s+3}(\alpha+1)}\left\{\mathcal{C}(s,\alpha,t)+\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right\}\left[|f''(a)|+|f''(b)|\right],\end{aligned}$$

where $\mathcal{C}(s,\alpha,t) = \int_0^1 (1-t)^{\alpha+1}(1+t)^s dt$.

Proof. Using Lemma 1 and the fact that $|f''|$ is $s-$convex function of second kind, we have

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|\\
&=\left|\frac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}\left[f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt\right|\\
&\le\left|\frac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right|+\left|\frac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right|\\
&\le\frac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt+\frac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt\\
&\le\frac{(b-a)^2}{8(\alpha+1)}\left[\int_0^1(1-t)^{\alpha+1}\left[\left(\frac{1+t}{2}\right)^s|f''(a)|+\left(\frac{1-t}{2}\right)^s|f''(b)|\right]dt\right.\\
&\qquad\left.+\int_0^1(1-t)^{\alpha+1}\left[\left(\frac{1-t}{2}\right)^s|f''(a)|+\left(\frac{1+t}{2}\right)^s|f''(b)|\right]dt\right]\\
&=\frac{(b-a)^2}{2^{s+3}(\alpha+1)}\left[|f''(a)|\int_0^1(1-t)^{\alpha+1}(1+t)^s dt+|f''(b)|\int_0^1(1-t)^{\alpha+1}(1-t)^s dt\right.\\
&\qquad\left.+|f''(a)|\int_0^1(1-t)^{\alpha+1}(1-t)^s dt+|f''(b)|\int_0^1(1+t)^{\alpha+1}(1-t)^s dt\right]\\
&=\frac{(b-a)^2}{2^{s+3}(\alpha+1)}\left[|f''(a)|\mathcal{C}(s,\alpha,t)+|f''(b)|\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}+|f''(a)|\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}+|f''(b)|\mathcal{C}(s,\alpha,t)\right]\\
&=\frac{(b-a)^2}{2^{s+3}(\alpha+1)}\left\{\mathcal{C}(s,\alpha,t)+\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right\}\left[|f''(a)|+|f''(b)|\right].\end{aligned}$$

This completes the proof. □

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable function on (a,b) with $a<b$. If $f'' \in L[a,b]$ and $|f''|^q$ is $s-$convex function of second kind, then, we have the following inequality for fractional integrals:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|\\ &\leq \frac{(b-a)^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left\{|f''(a)|^q\left(2^{s+1}-1\right)+|f''(b)|^q\right\}^{\frac{1}{q}}\right.\\ &\qquad\left.+\left\{|f''(a)|^q+|f''(b)|^q\left(2^{s+1}-1\right)\right\}^{\frac{1}{q}}\right].\end{aligned}$$

Proof. Using Lemma 1, Holder's inequality and the fact that $|f''|^q$ is s-convex function of second kind, we have

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|\\ &=\left|\frac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}\left[f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt\right|\\ &\leq\left|\frac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right|+\left|\frac{b-a}{4}\int_0^1(1-t)^{\alpha+1}f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right|\\ &\leq\frac{(b-a)^2}{8(\alpha+1)}\left\{\left(\int_0^1(1-t)^{p(\alpha+1)}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\ &\qquad\left.+\left(\int_0^1(1-t)^{p(\alpha+1)}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\ &\leq\frac{(b-a)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left\{\left(|f''(a)|^q\int_0^1\left(\frac{1+t}{2}\right)^s dt+|f''(b)|^q\int_0^1\left(\frac{1-t}{2}\right)^s dt\right)^{\frac{1}{q}}\right.\\ &\qquad\left.+\left(|f''(a)|^q\int_0^1\left(\frac{1-t}{2}\right)^s dt+|f''(b)|^q\int_0^1\left(\frac{1+t}{2}\right)^s dt\right)^{\frac{1}{q}}\right\}\\ &=\frac{(b-a)^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left\{|f''(a)|^q\left(2^{s+1}-1\right)+|f''(b)|^q\right\}^{\frac{1}{q}}\right.\\ &\qquad\left.+\left\{|f''(a)|^q+|f''(b)|^q\left(2^{s+1}-1\right)\right\}^{\frac{1}{q}}\right].\end{aligned}$$

This completes the proof. □

Theorem 4. *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable function on (a,b) with $a<b$. If $f'' \in L[a,b]$ and $|f''|^q$ is $s-$convex function of second kind, then, we have the following inequality for fractional integrals:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|\\ &\leq\frac{(b-a)^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left\{\left(|f''(a)|^q\left(\mathcal{C}(s,\alpha,t)\right)+|f''(b)|^q\left(\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right)\right)^{\frac{1}{q}}\right.\\ &\qquad\left.+\left(|f''(a)|^q\left(\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right)+|f''(b)|^q\left(\mathcal{C}(s,\alpha,t)\right)\right)^{\frac{1}{q}}\right\},\end{aligned}$$

where $\mathcal{C}(s,\alpha,t)=\int_0^1(1-t)^{\alpha+1}(1+t)^s dt.$

Proof. Using Lemma 1, power-mean inequality and the fact that $|f''|^q$ is $s-$convex function of second kind, we have

$$\begin{aligned}
&\left|\tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\tfrac{a+b}{2}\right)\right|\\
&=\left|\tfrac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}\left[f''\left(\tfrac{1+t}{2}a+\tfrac{1-t}{2}b\right)+f'\left(\tfrac{1-t}{2}a+\tfrac{1+t}{2}b\right)\right]dt\right|\\
&\le\left|\tfrac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}f''\left(\tfrac{1+t}{2}a+\tfrac{1-t}{2}b\right)dt\right|+\left|\tfrac{b-a}{4}\int_0^1(1-t)^{\alpha+1}f''\left(\tfrac{1-t}{2}a+\tfrac{1+t}{2}b\right)dt\right|\\
&\le\tfrac{(b-a)^2}{8(\alpha+1)}\left\{\left(\int_0^1(1-t)^{(\alpha+1)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{(\alpha+1)}\left|f''\left(\tfrac{1+t}{2}a+\tfrac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(\int_0^1(1-t)^{(\alpha+1)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{(\alpha+1)}\left|f''\left(\tfrac{1-t}{2}a+\tfrac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
&\le\tfrac{(b-a)^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\tfrac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left\{\left(|f''(a)|^q\left(\mathcal{C}(s,\alpha,t)\right)+|f''(b)|^q\left(\tfrac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right)\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\left(|f''(a)|^q\left(\tfrac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right)+|f''(b)|^q\left(\mathcal{C}(s,\alpha,t)\right)\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

This completes the proof. □

Remark 1. *We would like to remark here that for $\alpha = 1$ and $s = 1$ in above results, we have several new results.*

3. Inequalities for s-convex functions of first kind

In this section, we prove some Hermite-Hadamard type of integral inequalities for s-convex functions of first kind which is mainly due to Toader et al. [13].

Definition 3 ([13])**.** *A function $f : I \to \mathbb{R}$ is said to be s-convex function in the first kind, if*

$$f((1-t)x+ty)\le(1-t^s)f(x)+t^sf(y),\ \forall x,y\in I, t\in[0,1], s\in(0,1]. \tag{5}$$

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be twice differentiable function on (a,b) with $a < b$. If $f'' \in L[a,b]$ and $|f''|$ is $s-$convex function of first kind, then, we have the following inequality for fractional integrals:*

$$\left|\tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\tfrac{a+b}{2}\right)\right|\le\tfrac{(b-a)^2}{8(\alpha+1)(\alpha+2)}\left[|f''(a)|+|f''(b)|\right].$$

Proof. Using Lemma 1 and the fact that $|f''|$ is $s-$convex function of first kind, we have

$$\begin{aligned}
&\left|\tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\tfrac{a+b}{2}\right)\right|\\
&=\left|\tfrac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}\left[f''\left(\tfrac{1+t}{2}a+\tfrac{1-t}{2}b\right)+f'\left(\tfrac{1-t}{2}a+\tfrac{1+t}{2}b\right)\right]dt\right|\\
&\le\left|\tfrac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}f''\left(\tfrac{1+t}{2}a+\tfrac{1-t}{2}b\right)dt\right|+\left|\tfrac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}f''\left(\tfrac{1-t}{2}a+\tfrac{1+t}{2}b\right)dt\right|\\
&\le\tfrac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}\left|f''\left(\tfrac{1+t}{2}a+\tfrac{1-t}{2}b\right)\right|dt+\tfrac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}\left|f''\left(\tfrac{1-t}{2}a+\tfrac{1+t}{2}b\right)\right|dt\\
&\le\tfrac{(b-a)^2}{8(\alpha+1)}\left[\int_0^1(1-t)^{\alpha+1}\left[\left\{1-\left(\tfrac{1-t}{2}\right)^s\right\}|f''(a)|+\left(\tfrac{1-t}{2}\right)^s|f''(b)|\right]dt\right.\\
&\quad\left.+\int_0^1(1-t)^{\alpha+1}\left[\left(\tfrac{1-t}{2}\right)^s|f''(a)|+\left\{1-\left(\tfrac{1-t}{2}\right)^s\right\}|f''(b)|\right]dt\right]
\end{aligned}$$

$$= \frac{(b-a)^2}{8(\alpha+1)(\alpha+2)} \left[|f''(a)| + |f''(b)|\right].$$

This completes the proof. □

Theorem 6. *Let $f : [a, b] \to \mathbb{R}$ be twice differentiable function on (a, b) with $a < b$. If $f'' \in L[a, b]$ and $|f''|^q$ is $s-$convex function of first kind, then, we have the following inequality for fractional integrals:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b)\right] - f\left(\frac{a+b}{2}\right)\right| \\
\leq &\frac{(b-a)^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left\{|f''(a)|^q\left(2^s(s+1)-1\right) + |f''(b)|^q\right\}^{\frac{1}{q}}\right. \\
&\left. + \left\{|f''(a)|^q + |f''(b)|^q\left(2^s(s+1)-1\right)\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. Using Lemma 1, Holder's inequality and the fact that $|f''|^q$ is $s-$convex function of second kind, we have

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b)\right] - f\left(\frac{a+b}{2}\right)\right| \\
= &\left|\frac{(b-a)^2}{8(\alpha+1)}\int_0^1 (1-t)^{\alpha+1}\left[f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) + f'\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right]dt\right| \\
\leq &\left|\frac{(b-a)^2}{8(\alpha+1)}\int_0^1 (1-t)^{\alpha+1} f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt\right| + \left|\frac{b-a}{4}\int_0^1 (1-t)^{\alpha+1} f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)dt\right| \\
\leq &\frac{(b-a)^2}{8(\alpha+1)}\left\{\left(\int_0^1 (1-t)^{p(\alpha+1)}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_0^1 (1-t)^{p(\alpha+1)}dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq &\frac{(b-a)^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left\{\left(|f''(a)|^q\int_0^1\left\{1-\left(\frac{1-t}{2}\right)^s\right\}dt + |f''(b)|^q\int_0^1\left(\frac{1-t}{2}\right)^s dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(|f''(a)|^q\int_0^1\left(\frac{1-t}{2}\right)^s dt + |f''(b)|^q\int_0^1\left\{1-\left(\frac{1-t}{2}\right)^s\right\}dt\right)^{\frac{1}{q}}\right\} \\
= &\frac{(b-a)^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left[\left\{|f''(a)|^q\left(2^s(s+1)-1\right) + |f''(b)|^q\right\}^{\frac{1}{q}}\right. \\
&\left. + \left\{|f''(a)|^q + |f''(b)|^q\left(2^s(s+1)-1\right)\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Theorem 7. *Let $f : [a, b] \to \mathbb{R}$ be twice differentiable function on (a, b) with $a < b$. If $f'' \in L[a, b]$ and $|f''|^q$ is $s-$convex function of first kind, then, we have the following inequality for fractional integrals:*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b)\right] - f\left(\frac{a+b}{2}\right)\right| \\
\leq &\frac{(b-a)^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left\{\left(|f''(a)|^q\left(\frac{1}{\alpha+1} - \frac{1}{2^s}\left(\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right)\right) + |f''(b)|^q\frac{1}{2^s}\left(\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right)\right)^{\frac{1}{q}}\right. \\
&\left. + \left(|f''(a)|^q\frac{1}{2^s}\left(\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right) + |f''(b)|^q\left(\frac{1}{\alpha+1} - \frac{1}{2^s}\left(\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right)\right)\right)^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. Using Lemma 1, power-mean inequality and the fact that $|f''|^q$ is $s-$convex function of second kind, we have

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$=\left|\frac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}\left[f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)+f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt\right|$$

$$\leq\left|\frac{(b-a)^2}{8(\alpha+1)}\int_0^1(1-t)^{\alpha+1}f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right|+\left|\frac{b-a}{4}\int_0^1(1-t)^{\alpha+1}f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right|$$

$$\leq\frac{(b-a)^2}{8(\alpha+1)}\left\{\left(\int_0^1(1-t)^{(\alpha+1)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{(\alpha+1)}\left|f''\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1(1-t)^{(\alpha+1)}dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)^{(\alpha+1)}\left|f''\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq\frac{(b-a)^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left\{\left(\int_0^1(1-t)^{(\alpha+1)}\left[\left\{1-\left(\frac{1-t}{2}\right)^s\right\}|f''(a)|^q+\left(\frac{1-t}{2}\right)|f''(b)|^q\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1(1-t)^{(\alpha+1)}\left[\left\{1-\left(\frac{1-t}{2}\right)^s\right\}|f''(b)|^q+\left(\frac{1-t}{2}\right)|f''(a)|^q\right]dt\right)^{\frac{1}{q}}\right\}$$

$$=\frac{(b-a)^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left\{\left(|f''(a)|^q\left(\frac{1}{\alpha+1}-\frac{1}{2^s}\left(\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right)\right)+|f''(b)|^q\frac{1}{2^s}\left(\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(|f''(a)|^q\frac{1}{2^s}\left(\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right)+|f''(b)|^q\left(\frac{1}{\alpha+1}-\frac{1}{2^s}\left(\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right)\right)\right)^{\frac{1}{q}}\right\}.$$

This completes the proof. □

References

[1] Breckner, W.W., *Stetigkeitsaussagen fur eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen*, Publ. Inst. Math. **23** (1978), 13-20.

[2] Cristescu, G., *Hadamard type inequalities for φ-convex functions*, Annals of the University of Oradea, Fascicle of Management and Technological Engineering, CD-Rom Edition, III (XIII), 2004.

[3] Cristescu, G., *Improved integral inequalities for products of convex functions*, Journal of Inequalities in Pure and Applied Mathematics **6** (2005), No. 2.

[4] Cristescu, G. and Lupsa, L., *Non-connected Convexities and Applications*, Kluwer Academic Publishers, Dordrecht, Holland, 2002.

[5] Dragomir, S.S. and Pearce, C.E.M., *Selected topics on Hermite-Hadamard inequalities and applications*, Victoria University, Australia, 2000.

[6] Dragomir, S.S., Pečarić, J. and Persson, L.E., *Some inequalities of Hadamard type*, Soochow J. Math **21** (1995), 335-341.

[7] Katugampola, U.N., *New approach to a generalized fractional integral*, Appl. Math. Comput. **218** (2011), No. 3, 860-865.

[8] Kilbas, A., Srivastava, H.M. and Trujillo, J.J., *Theory and applications of fractional differential equations*, Elsevier B.V., Amsterdam, Netherlands, 2006.

[9] Noor, M.A., Awan, M.U. and Noor, K.I., *On some inequalities for relative semi-convex functions*, Journal of Inequalities and Applications, 2013.

[10] Noor, M.A., Noor, K.I. and Awan, M.U., *Hermite-Hadamard inequalities for relative semi-convex functions and applications*, Filomat, in press.

[11] Ozdemir, M.E., Yildiz, C. and Kavurmaci, H., *Fractional integral inequalities for different functions*, available online at: arXiv:1202.2431v1, 2012.

[12] Sarikaya, M.Z., Set, E., Yaldiz, H. and Basak, N., *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling **57** (2013), 2403-2407.

[13] Toader, G. and Toader, S., *A hierarchy of logarithmic convexity of functions*, Annals of the T. Popoviciu Seminar of Functional Equations, Approximation and Convexity **7** (2009), ISSN 1584-4536, 147-154.

[14] Varosanec, S., *On h-convexity*, J. Math. Anal. Appl. **326** (2007), 303-311.

[15] Wang, J., Li, X., Feckan, M. and Zhou, Y., *Hermite-Hadamard-type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity*, Applicable Analysis, 2012, doi:10.1080/00036811.2012.727986.

[16] Xi, B.-Y, Wang, S.-H and Qi, F., *Some inequalities of Hermite-Hadamard type for functions whose 3rd derivatives are $P-$convex*, Applied Mathematics **3** (2012), 1898-1902.

[17] Shuang, Y., Yin, H-P. and Qi, F., *Hermite-Hadamard type integral inequalities for geometric-arithamatically s-convex functions*, Analysis **33** (2013), 197-208.

[18] Zhang, T-Y., Ji, A-P. and Qi, F., On integral inequalities of Hermite-Hadamard type for s-geometrically convex functions, Abstract and Applied Analysis, 2012, doi:10.1155/2012/560586.

[19] Zhang, T-Y., Ji, A-P. and Qi, F., *Some inequalities of Hermite-Hadamard type for GA-convex functions with applications to means*, Le Matematiche **LXVIII** (2013), Fasc. I, 229-239, doi:10.4418/2013.68.1.17.

COMSATS Institute of Information Technology
Department of Mathematics
Park Road, Islamabad, Pakistan.
E-mail address: noormaslam@hotmail.com

COMSATS Institute of Information Technology
Department of Mathematics
Park Road, Islamabad, Pakistan.
E-mail address: awan.uzair@gmail.com