TJMM
5 (2013), No. 2, 121-127

NUMERICAL SOLUTION FOR PARABOLIC EQUATION WITH NONLOCAL CONDITIONS

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Abstract. In this paper, we study a parabolic equation with purely nonlocal (integral) conditions. We present a numerical approximate solution by Laplace transform method. Some experimental numerical results using the proposed numerical procedure are discussed.

1. Introduction

In the recent years, evolution problems with nonlocal (integral) conditions have received an increasing attention. The physical significance of integral conditions (mean, total flux, total energy, total mass, moments,...) has served as a fundamental reason for the interest carried to this type of problems.

In the rectangular domain $D = \{(x,t) : 0 < x < 1, 0 < t \leq T\}$, we consider parabolic equation

$$\frac{\partial v}{\partial t} - \alpha \frac{\partial^2 v}{\partial x^2} = g\left(x,t\right), \quad 0 < x < 1, \quad 0 < t \leq T, \tag{1}$$

subject to the initial conditions

$$v(x,0) = \Phi\left(x\right), \quad 0 < x < 1, \tag{2}$$

and the nonlocal boundary conditions

$$\int_0^1 v\left(x,t\right) dx = E(t), \quad 0 < t \leq T, \tag{3}$$

$$\int_0^1 xv\left(x,t\right) dx = M(t), \quad 0 < t \leq T. \tag{4}$$

where g, Φ, Ψ, E and M are known functions, α and T are known positive constants. Introducing a new unknown function

$$u\left(x,t\right) = v(x,t) - u_1(x,t), \tag{5}$$

where

$$u_1(x,t) = E(t) + 6\left(3x^2 - 2x\right).\left(2M\left(t\right) - E(t)\right). \tag{6}$$

Problems (1)–(4) with inhomogeneous integral conditions (3), (4) can be equivalently reduced to the problem of finding a function u satisfying:

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = f\left(x,t\right), \quad 0 \leq x \leq 1, \quad 0 < t \leq T \tag{7}$$

$$u(x,0) = \varphi\left(x\right), \quad 0 \leq x \leq 1, \tag{8}$$

$$\int_0^1 u\left(x,t\right) dx = 0, \quad 0 < t \leq T, \tag{9}$$

2010 *Mathematics Subject Classification.* 35K10, 49Mxx, 35B45.
Key words and phrases. Parabolic equation, nonlocal conditions, a priori estimates, numerical method.

$$\int_0^1 xu(x,t)\,dx = 0, \quad 0 < t \leq T, \tag{10}$$

where

$$f(x,t) = g(x,t) - \left(\frac{\partial u_1}{\partial t} - \alpha\frac{\partial^2 u_1}{\partial x^2}\right), \tag{11}$$

and

$$\varphi(x) = \Phi(x) - u_1(x,0). \tag{12}$$

Hance, instead of looking for v, we simply look for u. The solution of problem (1)–(4) will be obtained by the relations (5), (6).

2. Preliminaries and Notations

Definition 1. *We denote by $B_2^m(0,1)$, $m \geq 1$ so called Bouziani space, it is a completion of the space $C_0(0,1)$, of continuous functions with compact support.*

Proposition 1. *The space $B_2^m(0,1)$, $m \geq 1$ is a Hilbert space for the scalar product*

$$(u,w)_{B_2^m(0,1)} = \int_0^1 \Im_x^m u.\Im_x^m w dx; \text{ for } m \geq 1, \tag{13}$$

with the associated norm

$$\|u\|_{B_2^m(0,1)} = \left(\int_0^1 (\Im_x^m u)^2\,dx\right)^{1/2} = \|\Im_x^m u\|;\ \text{for } m \geq 1, \tag{14}$$

where

$$\Im_x^m u = \int_0^x \frac{(x-\xi)^{m-1}}{(m-1)!} u(\xi,t)\,d\xi;\ \text{for } m \geq 1 \tag{15}$$

Lemma 1. *For all $m \in \mathbb{N}^*$ we have*

$$\|u\|^2_{B_2^m(0,1)} \leq \frac{1}{2}\|u\|^2_{B_2^{m-1}(0,1)} \tag{16}$$

Corollary 1. *For all $m \in \mathbb{N}^*$ we have*

$$\|u\|^2_{B_2^m(0,1)} \leq \left(\frac{1}{2}\right)^m \|u\|^2_{L^2(0,1)} \tag{17}$$

Lemma 2 (Gronwall Lemma)**.** *Let $f_1(t), f_2(t) \geq 0$ be two integrable functions on $[0,T]$, $f_2(t)$ is nondecreasing. If*

$$f_1(\tau) \leq f_2(\tau) + c\int_0^\tau f_1(t)\,dt, \quad \forall\tau \in [0,T] \tag{18}$$

where $c \in \mathbb{R}^+$, then

$$f_1(t) \leq f_2(t)\exp(ct), \quad \forall t \in [0,T] \tag{19}$$

Corollary 2. *For every $u \in L^2(0,l)$, from which we deduce the continuity of the embedding $L^2(0,1) \longrightarrow B_2^m(0,1)$, for $m \geq 1$.*

3. Uniqueness and Continuous dependence of the Solution

Theorem 1. *If $u(x,t)$ is a solution of problem (7)–(10) and $f \in C\left(\overline{Q}\right)$, then we have a priori estimates*

$$\|u(.,\tau)\|^2_{L^2(0,1)} \le c_1 \left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\, dt + \|\varphi\|^2_{L^2(0,1)} \right) \tag{20}$$

$$\left\| \frac{\partial u(.,\tau)}{\partial t} \right\|^2_{L^2\left(0,T;\ B_2^1(0,1)\right)} \le c_2 \left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\, dt + \|\varphi\|^2_{L^2(0,1)} \right) \tag{21}$$

where $c_1 = \frac{\max(1,\alpha)}{\alpha}$, $c_2 = \max(1,\alpha)$ *and* $0 \le \tau \le T$.

Proof. Taking the scalar product in $B_2^1(0,1)$ of equation (7) and $\frac{\partial u}{\partial t}$, and integrating over $(0,\tau)$, we have

$$\begin{aligned} \int_0^\tau \left(\tfrac{\partial u(.,t)}{\partial t}, \tfrac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt - \alpha \int_0^\tau \left(\tfrac{\partial^2 u(.,t)}{\partial x^2}, \tfrac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt \\ = \int_0^\tau \left(f(.,t), \tfrac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt. \end{aligned} \tag{22}$$

The integration by parts on the left-hand side of (22) we obtain

$$\begin{aligned} \int_0^\tau \left\| \tfrac{\partial u(.,t)}{\partial t} \right\|^2_{B_2^1(0,1)} dt + \tfrac{\alpha}{2} \|u(.,\tau)\|^2_{L^2(0,1)} - \tfrac{\alpha}{2} \|\varphi\|^2_{L^2(0,1)} \\ = \int_0^\tau \left(f(.,t), \tfrac{\partial u(.,t)}{\partial t} \right)_{B_2^1(0,1)} dt. \end{aligned} \tag{23}$$

by the **Cauchy inequality**, the right-hand side of (23) is bounded by

$$\frac{1}{2} \int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\, dt + \frac{1}{2} \int_0^\tau \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{B_2^1(0,1)} dt. \tag{24}$$

Substitution of (24) into (23) yield

$$\int_0^\tau \left\| \frac{\partial u(.,t)}{\partial t} \right\|^2_{B_2^2(0,1)} dt + \alpha \|u(.,\tau)\|^2_{L^2(0,1)} \le \int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\, dt + \alpha \|\varphi\|^2_{L^2(0,1)}. \tag{25}$$

From (25), we obtain estimates (20) and (21). □

4. Method of the Solution

Laplace transform is an efficient method for solving many differential equations and partial differential equations,The main difficulty with Laplace transform method is in inverting the Laplace domain solution into the real domain. In this section we shall apply the Laplace transform technique to find solutions of partial differential equations, we have The Laplace transform

$$V(x,s) = \mathcal{L}\{v(x,t); t \longrightarrow s\} = \int_0^\infty v(x,t) \exp(-st)\, dt, \tag{26}$$

where s is positive réel parameter. Taking the Laplace transforms on both sides of (1), we have

$$sV(x,s) - \alpha \frac{d^2}{dx^2} [V(x,s)] = G(x,s) + \Phi(x), \tag{27}$$

where $G(x,s) = \mathcal{L}\{g(x,t); t \longrightarrow s\}$. Similarly,we have

$$\int_0^1 V(x,s)\, dx = A(s), \tag{28}$$

$$\int_0^1 xV\left(x,s\right)dx = B(s). \tag{29}$$

where $A\left(s\right) = \mathcal{L}\left\{E(t);t \longrightarrow s\right\}$ and $B\left(s\right) = \mathcal{L}\left\{M(t);t \longrightarrow s\right\}$. Thus, considered equation is reduced in boundary value problem governed by second order inhomogeneous ordinary differential equation. We obtain a general solution of (27) as

$$\begin{array}{c} V\left(x,s\right) = -\sqrt{\frac{\alpha}{s}}\int_0^x \left[G\left(\tau,s\right)+\Phi\left(\tau\right)\right]\sinh\left(\sqrt{\frac{s}{\alpha}}\left[x-\tau\right]\right)d\tau + \\ C_1\left(s\right)\exp\left(-\sqrt{\frac{s}{\alpha}}x\right) + C_2\left(s\right)\exp\left(\sqrt{\frac{s}{\alpha}}x\right) \end{array} \tag{30}$$

where C_1 and C_2 are arbitrary functions of s. Substitution of (30) into (28)–(29), we have

$$\begin{array}{c} C_1\left(s\right)\int_0^1 \exp\left(-\sqrt{\frac{s}{\alpha}}x\right)dx + C_2\left(s\right)\int_0^1 \exp\left(\sqrt{\frac{s}{\alpha}}x\right)dx \\ = \sqrt{\frac{\alpha}{s}}\int_0^1 \left[\left[F\left(\tau,s\right)+\varphi\left(\tau\right)\right]\int_\tau^1 \sinh\left(\sqrt{\frac{s}{\alpha}}\left[x-\tau\right]\right)dx\right]d\tau + A(s), \end{array} \tag{31}$$

$$\begin{array}{c} C_1\left(s\right)\int_0^1 x\exp\left(-\sqrt{\frac{s}{\alpha}}x\right)dx + C_2\left(s\right)\int_0^1 x\exp\left(\sqrt{\frac{s}{\alpha}}x\right)dx \\ = \sqrt{\frac{\alpha}{s}}\int_0^1 \left[\left[G\left(\tau,s\right)+\Phi\left(\tau\right)\right]\int_\tau^1 x\sinh\left(\sqrt{\frac{s}{\alpha}}\left[x-\tau\right]\right)dx\right]d\tau + B(s). \end{array} \tag{32}$$

where

$$\begin{pmatrix} C_1\left(s\right) \\ C_2\left(s\right) \end{pmatrix} = \begin{pmatrix} a_{11}\left(s\right) & a_{12}\left(s\right) \\ a_{21}\left(s\right) & a_{22}\left(s\right) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1\left(s\right) \\ b_2\left(s\right) \end{pmatrix}, \tag{33}$$

and

$$\begin{array}{c} a_{11}\left(s\right) = \int_0^1 \exp\left(-\sqrt{\frac{s}{\alpha}}x\right)dx, \\ a_{12}\left(s\right) = \int_0^1 \exp\left(\sqrt{\frac{s}{\alpha}}x\right)dx, \\ a_{21}\left(s\right) = \int_0^1 x\exp\left(-\sqrt{\frac{s}{\alpha}}x\right)dx, \\ a_{22}\left(s\right) = \int_0^1 x\exp\left(\sqrt{\frac{s}{\alpha}}x\right)dx, \\ b_1\left(s\right) = \sqrt{\frac{\alpha}{s}}\int_0^1 \left[\left[G\left(\tau,s\right)+\Phi\left(\tau\right)\right]\int_\tau^1 \sinh\left(\sqrt{\frac{s}{\alpha}}\left[x-\tau\right]\right)dx\right]d\tau + A(s), \\ b_2\left(s\right) = \sqrt{\frac{\alpha}{s}}\int_0^1 \left[\left[G\left(\tau,s\right)+\Phi\left(\tau\right)\right]\int_\tau^1 x\sinh\left(\sqrt{\frac{s}{\alpha}}\left[x-\tau\right]\right)dx\right]d\tau + B(s). \end{array} \tag{34}$$

It is possible to evaluate the integrals in (30) and (34) exactly. In general, one may have to resort to numerical integration in order to compute them, however. For example, the Gauss's formula (25.4.30) given in Abramowitz and Stegun [1] may be employed to calculate these integrals numerically, we have the following approximations for the integrals

$$\int_0^1 \exp\left(\pm\sqrt{\frac{s}{\alpha}}x\right)dx \simeq \frac{1}{2}\sum_{i=1}^N w_i \exp\left(\pm\sqrt{\frac{s}{\alpha}}\left[x_i+1\right]\right),$$

$$\int_0^1 x\exp\left(\pm\sqrt{\frac{s}{\alpha}}x\right)dx \simeq \frac{1}{2}\sum_{i=1}^N w_i\left(\frac{1}{2}\left[x_i+1\right]\right)\exp\left(\pm\sqrt{\frac{s}{\alpha}}\left[x_i+1\right]\right)$$

$$\begin{array}{c} \int_0^x \left[G\left(\tau,s\right)+\Phi\left(\tau\right)\right]\sinh\left(\sqrt{\frac{s}{\alpha}}\left[x-\tau\right]\right)d\tau \\ \simeq \frac{x}{2}\sum_{i=1}^N w_i\left[G\left(\frac{x}{2}\left[x_i+1\right];s\right)+\Phi\left(\frac{x}{2}\left[x_i+1\right]\right)\right]\sinh\left(\sqrt{\frac{s}{\alpha}}\left[x-\frac{x}{2}\left[x_i+1\right]\right]\right), \end{array}$$

$$\begin{array}{c} \int_0^1 \left[\left[G\left(\tau,s\right)+\Phi\left(\tau\right)\right]\int_\tau^1 \sinh\left(\sqrt{\frac{s}{\alpha}}\left[x-\tau\right]\right)dx\right]d\tau \\ \simeq \frac{1}{4}\sum_{i=1}^N w_i\left[G\left(\frac{1}{2}\left[x_i+1\right];s\right)+\Phi\left(\frac{1}{2}\left[x_i+1\right]\right)\right]\left(1-\frac{1}{2}\left[x_i+1\right]\right)\times \\ \sum_{i=1}^N w_j\sinh\left(\sqrt{\frac{s}{\alpha}}\left[\frac{1}{2}\left[\left(1-\frac{1}{2}\left[x_i+1\right]\right)x_j+\left(1+\frac{1}{2}\left[x_i+1\right]\right)\right]-\frac{1}{2}\left(x_i+1\right)\right]\right), \end{array}$$

$$\begin{aligned}&\int_0^1 \left[\left[F(\tau, s) + \Phi(\tau) \right] \int_\tau^1 x \sinh\left(\sqrt{\tfrac{s}{\alpha}} \left[x - \tau \right] \right) dx \right] d\tau \\ &\simeq \tfrac{1}{4} \sum_{i=1}^{N} w_i \left[G\left(\tfrac{1}{2} \left[x_i + 1 \right]; s \right) + \Phi\left(\tfrac{1}{2} \left[x_i + 1 \right] \right) \right] \left(1 - \tfrac{1}{2} \left[x_i + 1 \right] \right) \times \\ &\left(\tfrac{1}{2} \left[\left(1 - \tfrac{1}{2} \left[x_i + 1 \right] \right) x_j + \left(1 + \tfrac{1}{2} \left[x_i + 1 \right] \right) \right] \right) \times \sum_{i=1}^{N} w_j \sinh \times \\ &\left(\sqrt{\tfrac{s}{\alpha}} \left[\tfrac{1}{2} \left[\left(1 - \tfrac{1}{2} \left[x_i + 1 \right] \right) x_j + \left(1 + \tfrac{1}{2} \left[x_i + 1 \right] \right) \right] - \tfrac{1}{2} \left(x_i + 1 \right) \right] \right),\end{aligned} \tag{35}$$

where x_i and w_i are the abscissa and weights, defined as

$$x_i : i^{th} \text{ zero of } P_n(x), \omega_i = 2/\left(1 - x_i^2\right) \left[P_n'(x) \right]^2 .$$

Their tabulated values can be found in [1] for different values of N.

4.1. **Numerical inversion of Laplace transform.** Sometimes, an analytical inversion of a Laplace domain solution is difficult to obtain; therefore a numerical inversion method must be used. A nice comparison of four frequently used numerical Laplace inversion algorithms is given by Hassan Hassanzadeh, Mehran Pooladi-Darvish [16]. In this work we use the Stehfest's algorithm [18] it is easy to implement. This numerical technique was first introduced dy Graver [15] and its algorithm then offered by [18]. Stehfest's algorithm approximates the time domain solution as

$$v(x,t) \approx \frac{\ln 2}{t} \sum_{n=1}^{2m} \beta_n V\left(x; \frac{n \ln 2}{t} \right), \tag{36}$$

where, m is the positive integer and

$$\beta_n = (-1)^{n+m} \sum_{k=\left[\frac{n+1}{2}\right]}^{\min(n,m)} \frac{k^m (2k)!}{(m-k)!k!(k-1)!(n-k)!(2k-n)!} \tag{37}$$

where $[q]$ denotes the integer part of the real number q. Then by relations (5)–(6) we obtain the unknown function u.

5. Numerical Examples

In this section, we report some results of numerical computations using Laplace transform method proposed in the previous section. These technique are applied to solve the problem defined by (1)–(4) for particular functions g, Φ, E, M and positive constant α.

Example 1. *We take*

$$\begin{aligned} g(x,t) &= -\exp(-(x+t)),\ 0 < x < 1,\ 0 < t \leq T \text{ and } \alpha = 1, \\ \Phi(x) &= \exp(-x),\ 0 < x < 1, \\ E(t) &= \left(1 - e^{-1}\right) \cosh(t),\ 0 < t \leq T, \\ M(t) &= \left(1 - 2e^{-1}\right) \cosh(t),\ 0 < t \leq T, \end{aligned}$$

in this case exact solution given by

$$v(x,t) = e^{-x} \cosh(t),\ 0 < x < 1,\ 0 < t \leq T.$$

The method of solution is easily implemented on the computer, numerical results obtained by $N = 8$ in (35) and $m = 5$ in (36), then we compared the exact solution with numerical solution. For $t = 0.10$, $x \in [0.10, 0.90]$, we calculate v numerically using the proposed method of solution and compare it with the exact solution in Table 1.

Table 1.

x	0.10	0.30	0.50	0.70	0.90
v exact	0.9093654	0.7445254	0.6095658	0.490703	0.4086042
v numerical	0.9093851	0.7443921	0.6097452	0.500183	0.4080919
error	0.000217	−0.0001790	0.0002943	0.0022295	−0.0012538

Table 2.

x	0.10	0.30	0.50	0.70	0.90
v exact	0.1151730	0.3015269	0.3727078	0.3015270	0.1151730
v numerical	0.1150014	0.3015291	0.3728361	0.3012199	0.1152185
error	−0.0014899	0.0000073	0.0003442	−0.0010185	0.0003951

Table 3.

x	0.10	0.30	0.50	0.70	0.90
v exact	0.7320288	−0.2796101	−0.9048374	−0.2796101	0.7320288
v numerical	0.7324162	−0.2795921	−0.9047562	−0.2795421	0.7321329
error	0.0005292	−0.0000644	−0.0000897	−0.0002432	0.0001422

Example 2. *We take*

$$\begin{aligned} g(x,t) &= 0,\ 0<x<1,\ 0<t\leq T \text{ and } \alpha=1, \\ \Phi(x) &= \sin(\pi x),\ 0<x<1, \\ E(t) &= \frac{2}{\pi}\exp\left(-\pi^2 t\right),\ 0<t\leq T, \\ M(t) &= \frac{1}{\pi}\exp\left(-\pi^2 t\right),\ 0<t\leq T, \end{aligned}$$

in this case exact solution given by

$$v(x,t) = \exp\left(-\pi^2 t\right)\sin(\pi x),\ 0<x<1,\ 0<t\leq T.$$

For $t = 0.10$, $x \in [0.10, 0.90]$, we calculate v numerically using the proposed method of solution and compare it with the exact solution in Table 2.

Example 3. *We take*

$$\begin{aligned} g(x,t) &= -\frac{2\left(x^2+1+t\right)}{(1+t)^3},\ 0<x<1,\ 0<t\leq T \text{ and } \alpha=1, \\ \Phi(x) &= x^2,\ 0<x<1, \\ E(t) &= \frac{1}{3(1+t)^2},\ 0<t\leq T, \\ M(t) &= \frac{1}{4(1+t)^2},\ 0<t\leq T, \end{aligned}$$

in this case exact solution given by

$$v(x,t) = \left(\frac{x}{1+t}\right)^2,\ 0<x<1,\ 0<t\leq T.$$

For $t = 0.1$, $x \in [0.1, 0.9]$, we calculate v numerically using the proposed method of solution and compare it with the exact solution in Table 3.

References

[1] Abramowitz, M. and Stegun. I.A., *Handbook of Mathematical Functions*, Dover, New York, 1972.

[2] Ang, W.T., *A Method of Solution for the One-Dimensional Heat Equation Subject to Nonlocal Conditions*, Southeast Asian Bulletin of Mathematics **26** (2002), 185-191.

[3] Benouar, N.E. and Yurchuk. N.I., *Mixed problem with an integral condition for parabolic equation with the Bessel operator*, Differentsial'nye **27** (1991), 2094-2098.

[4] Beïlin, S.A., *Existence of solutions for one-dimensional wave nonlocal conditions*, Electron. J. Differential Equations **76** (2001), 1-8.

[5] Bouziani, A., *Mixed problem with boundary integral conditions for a certain parabolic equation*, J. Appl. Math. Stochastic Anal. **9** (1996), No. 3, 323-330.

[6] Bouziani, A., *Solution forte d'un problème mixte avec une condition non locale pour une classe d'équations hyperbolique*, Acad. Roy. Belg. Bull. Cl. Sci. **8** (1997), 53-70.

[7] Bouziani, A., *Strong solution for a mixed problem with nonlocal condition for certain pluriparabolic equations*, Hiroshima Math. J. **27** (1997), No. 3, 373-390.

[8] Bouziani, A., *On the solvability of nonlocal pluriparabolic problems*, Electron, J. Differential Equations (2001), 1-16.

[9] Bouziani, A., *Initial-boundary value problem with nonlocal condition for a viscosity equation*, Int. J. Math. And Math. Sci.**30** (2002), No. 6, 327-338.

[10] Bouziani, A., *On the solvabiliy of parabolic and hyperbolic problems with a boundary integral condition*, Internat. J. Math. And Math. Sci. **31** (2002), 435-447.

[11] Bouziani, A., *On the solvability of a class of singular parabolic equations with nonlocal boundary conditions in nonclassical function spaces*, Internat. J. Math. And Math. Sci. **30** (2002), 435-447.

[12] Bouziani, A., *On a classe of nonclassical hyperbolic equations with nonlocal conditions*, J. Appl. Math. Stochastic Anal. **15** (2002), No. 2, 136-153.

[13] Bouziani, A. and Benouar, N., *Mixed problem with integral conditions for a third order parabolic equation*, Kobe J. Math. **15** (1998), No. 1, 47-58.

[14] Ekolin, G., *Finite difference methods for a nonlocal boundary value problem for the heat equation*, BIT **31** (1991), 245-26.

[15] Graver D.P., *Observing stochastic processes and approximate transform inversion*, Oper. Res. **14** (1966), 444-459.

[16] Hassanzadeh Hassan and Pooladi-Darvish Mehran, *Comparison of different numerical Laplace inversion methods for engineering applications*, Appl. Math. Comp. **189** (2007), 1966-1981.

[17] Liu, Y., *Numerical Solution of the Heat Equation With Nonlocal Boundary Conditions*, J.Comput. Appl. Math. (1997), 115-127.

[18] Stehfest, H., *Numerical Inversion of the Laplace Transform*, Comm. ACM **13** (1970),47-49.

[19] Shruti, A.D., *Numerical Solution for Nonlocal Sobolev-type Differential Equations*, Electronic Journal of Differential Equations, Conf. 19 (2010), 75-83.

[20] Tikhonov, A.I. and Samarskii, A.A., *Equations of Mathematical physics*, Edit. Mir, Moscow, 1984.

[21] Yurchuk, N.I., *Mixed problem with an integral condition for certain parabolic equations*, Differents. Uravn. **22** (1986), 2117-2126.

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