TJMM
5 (2013), No. 2, 115-120

HEAT CAPACITY OF THE RELATIVISTIC ELECTRON GAS IN A MAGNETIC FIELD

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Abstract. The heat capacity of degenerate and nondegenerate relativistic electron gases in a magnetic field is presented. In both cases, it comprises a term independent of the magnetic induction and a term varying monotonically with the magnetic induction. In the case of the degenerate relativistic electron gas, a third term with an oscillatory dependence on the magnetic induction appears.

1. Introduction

In the presence of a magnetic field in a body, a great variety of phenomena takes place in it: magnetic phenomena (diamagnetism, paramagnetism, ferromagnetism, antiferromagnetism, ferrimagnetism, helimagnetism, etc.), electromagnetic phenomena (cyclotronic resonance, electron paramagnetic resonance, ferromagnetic resonance, antiferromagnetic resonance, spin wave resonance, Zeeman effect, Faraday effect, Cotton-Mouton effect, etc.), thermomagnetic phenomena (Righi-Leduc effect, Maggie-Righi-Leduc effect, transversal Nernst-Ettingshausen effect, longitudinal Nernst-Ettingshausen effect), galvanomagnetic phenomena (Hall effect, magnetoresistive effect, Ettingshausen effect, Nernst effect) [1].

At low temperatures and high magnetic fields, quantum oscillations are observed in different physical quantities: ultrasound velocity [2, 3], ultrasound absorption coefficient [4, 5], temperature of an adiabatically isolated body [6], magnetic susceptibility (de Haas-van Alphen effect) [7, 8, 9], Knight shift [10], light reflection and transmission coefficients [11, 12, 13, 14], thermal conductibility [15], thermoelectric power [15], electrical resistivity (Shubnikov-de Haas effect) [16, 17, 18, 19], contact potential difference [20, 21, 22, 23].

Usually, in all these phenomena, the electron gas is considered nonrelativistic. However, in high energy physics, high pressure physics, geophysics, astrophysics, and cosmology, we have to do with relativistic electrons.

In Ref. [24] we presented the magnetic susceptibility of the relativistic electron gas. In the following, we expose a theory of the heat capacity of degenerate and nondegenerate relativistic electron gases in a magnetic field.

2. Degenerate relativistic electron gas

The energy eigenvalues for a relativistic electron situated in a constant uniform magnetic field, oriented along the z-axis, are given by the relation

$$\varepsilon_{n\sigma}\left(p_z\right) = c\sqrt{m_0^2c^2 + p_z^2 + 2m_0\mu_B B\left(2n+1+\sigma\right)}, \tag{1}$$

where c is the light velocity in vacuum, m_0 is the electron rest mass, μ_B is the Bohr magneton, p_z is the z-component of the electron linear momentum, B is the magnetic

2010 *Mathematics Subject Classification.* 82D05.

Key words and phrases. Relativistic electron gas, heat capacity, magnetic quantum oscillations.

induction, $n = 0, 1, 2, \ldots$, and $\sigma = \pm 1$. All energy levels, except the level $(n = 0, \sigma = -1)$, are doubly degenerate: the levels $(n, \sigma = +1)$ and $(n + 1, \sigma = -1)$ coincide.

The density of states with a determinate value of the spin projection on the direction of the magnetic field, corresponding to the dispersion law (1), is

$$D_\sigma(\varepsilon) = \frac{V m_0 \mu_B B}{\pi^2 \hbar^3 c} \sum_{n=0}^{\nu} \frac{\varepsilon}{\sqrt{\varepsilon^2 - m_0^2 c^4 - 2 m_0 c^2 \mu_B B (2n + 1 + \sigma)}}, \tag{2}$$

where V is the volume occupied by electrons and ν satisfies the inequalities

$$\nu \leq \frac{\varepsilon^2 - m_0^2 c^4 - 2 m_0 c^2 \mu_B B (1 + \sigma)}{4 m_0 c^2 \mu_B B} < \nu + 1. \tag{3}$$

With the aid of formulas (1), (2), and (3), we can calculate the free energy of the electron gas

$$F = N\mu - k_B T \sum_{\sigma = \pm 1} \int_{\varepsilon_{0\sigma}(0)}^{\infty} D_\sigma(\varepsilon) \ln\left[1 + \exp\left(\frac{\mu - \varepsilon}{k_B T}\right)\right] d\varepsilon, \tag{4}$$

where N is the electron number, μ is the chemical potential of the electron gas, k_B is the Boltzmann constant, and T is the absolute temperature.

Following the calculi performed in Ref. [24], we obtain

$$\begin{aligned} F = {} & \tfrac{V}{8\pi^2 \hbar^3 c^3} \Big[\varepsilon_F \sqrt{\varepsilon_F^2 - m_0^2 c^4} \left(2\varepsilon_F^2 - m_0^2 c^4\right) \\ & - m_0^4 c^8 \ln \tfrac{\varepsilon_F + \sqrt{\varepsilon_F^2 - m_0^2 c^4}}{m_0 c^2} - \tfrac{4\pi^2}{3} \varepsilon_F \sqrt{\varepsilon_F^2 - m_0^2 c^4} (k_B T)^2 \Big] \\ & - \tfrac{V (m_0 \mu_B B)^2 c}{3\pi^2 \hbar^3} \left[\ln \tfrac{\varepsilon_F + \sqrt{\varepsilon_F^2 - m_0^2 c^4}}{m_0 c^2} - \tfrac{\pi^2}{6} \tfrac{2\varepsilon_F^2 - m_0^2 c^4}{\varepsilon_F \left(\varepsilon_F^2 - m_0^2 c^4\right)^{3/2}} (k_B T)^2 \right] \\ & + \tfrac{V\sqrt{2} (m_0 \mu_B B)^{3/2} k_B T}{\pi^2 \hbar^3} \sum_{l=1}^{\infty} \tfrac{\cos\left(\pi l \frac{\varepsilon_F^2 - m_0^2 c^4}{2 m_0 c^2 \mu_B B} - \frac{\pi}{4}\right)}{l^{3/2} \sinh\left(\pi^2 l \frac{\varepsilon_F k_B T}{m_0 c^2 \mu_B B}\right)}, \end{aligned} \tag{5}$$

with ε_F the Fermi energy of the electron gas.

From the equation

$$U = F - T(\partial F / \partial T)_V, \tag{6}$$

we immediately find the internal energy of the electron gas

$$\begin{aligned} U = {} & \tfrac{V}{8\pi^2 \hbar^3 c^3} \Big[\varepsilon_F \sqrt{\varepsilon_F^2 - m_0^2 c^4} \left(2\varepsilon_F^2 - m_0^2 c^4\right) \\ & - m_0^4 c^8 \ln \tfrac{\varepsilon_F + \sqrt{\varepsilon_F^2 - m_0^2 c^4}}{m_0 c^2} + \tfrac{4\pi^2}{3} \varepsilon_F \sqrt{\varepsilon_F^2 - m_0^2 c^4} (k_B T)^2 \Big] \\ & - \tfrac{V (m_0 \mu_B B)^2 c}{3\pi^2 \hbar^3} \left[\ln \tfrac{\varepsilon_F + \sqrt{\varepsilon_F^2 - m_0^2 c^4}}{m_0 c^2} + \tfrac{\pi^2}{6} \tfrac{3\varepsilon_F^2 - m_0^2 c^4}{\varepsilon_F \left(\varepsilon_F^2 - m_0^2 c^4\right)^{3/2}} (k_B T)^2 \right] \\ & + \tfrac{V\sqrt{2} \varepsilon_F \sqrt{m_0 \mu_B B} (k_B T)^2}{\hbar^3 c^2} \sum_{l=1}^{\infty} \tfrac{\coth\left(\pi^2 l \frac{\varepsilon_F k_B T}{m_0 c^2 \mu_B B}\right)}{\sqrt{l} \sinh\left(\pi^2 l \frac{\varepsilon_F k_B T}{m_0 c^2 \mu_B B}\right)} \cos\left(\pi l \tfrac{\varepsilon_F^2 - m_0^2 c^4}{2 m_0 c^2 \mu_B B} - \tfrac{\pi}{4}\right). \end{aligned} \tag{7}$$

The relation

$$C_V = (\partial U / \partial T)_V \tag{8}$$

allows us to determine the heat capacity at constant volume

$$\begin{aligned} C_V = {} & \tfrac{V}{3\hbar^3 c^3} \varepsilon_F \sqrt{\varepsilon_F^2 - m_0^2 c^4} k_B^2 T \\ & - \tfrac{V (m_0 \mu_B B)^2 c}{9\hbar^3} \tfrac{3\varepsilon_F^2 - m_0^2 c^4}{\varepsilon_F \left(\varepsilon_F^2 - m_0^2 c^4\right)^{3/2}} k_B^2 T + \tfrac{V\sqrt{2}\pi^2 \varepsilon_F^2 k_B^3 T^2}{\hbar^3 c^4 \sqrt{m_0 \mu_B B}} \\ & \times \sum_{l=1}^{\infty} \sqrt{l} \tfrac{1 - 2\coth^2\left(\pi^2 l \frac{\varepsilon_F k_B T}{m_0 c^2 \mu_B B}\right) + 2 \frac{m_0 c^2 \mu_B B}{\varepsilon_F k_B T} \coth\left(\pi^2 l \frac{\varepsilon_F k_B T}{m_0 c^2 \mu_B B}\right)}{\sinh\left(\pi^2 l \frac{\varepsilon_F k_B T}{m_0 c^2 \mu_B B}\right)} \cos\left(\pi l \tfrac{\varepsilon_F^2 - m_0^2 c^4}{2 m_0 c^2 \mu_B B} - \tfrac{\pi}{4}\right). \end{aligned} \tag{9}$$

It contains three terms. The first term is independent of the magnetic induction. It corresponds to the heat capacity of the degenerate relativistic electron gas in the absence of a magnetic field [25]. The second term varies monotonically with the magnetic induction. The third term has an oscillatory dependence on the magnetic induction. It describes a quantum oscillation like the ones presented in *Introduction* [1].

Let us introduce the characteristic quantities

$$N_0 = \frac{Vm_0^3c^3}{3\pi^2\hbar^3},\ \ T_0 = \frac{m_0c^2}{k_B},\ \ B_0 = \frac{m_0c^2}{4\mu_B},\ \ C_{V0} = N_0k_B \tag{10}$$

and the dimensionless quantities

$$\nu = N/N_0,\ \ \tau = T/T_0,\ \ \beta = B/B_0. \tag{11}$$

Then, the expression (9) can be rewritten in the dimensionless form

$$\begin{aligned} \frac{C_V}{C_{V0}} &= \pi^2\nu^{1/3}\sqrt{1+\nu^{2/3}}\tau - \frac{\pi^2}{48}\frac{\left(2+3\nu^{2/3}\right)\tau\beta^2}{\nu\sqrt{1+\nu^{2/3}}} + \frac{3\sqrt{2}\pi^2\left(1+\nu^{2/3}\right)\tau^2}{\sqrt{\beta}} \\ &\times \sum_{l=1}^{\infty}\sqrt{l}\frac{1-2\coth^2\frac{4\pi^2l\sqrt{1+\nu^{2/3}}\tau}{\beta}+2\frac{\beta}{4\pi^2l\sqrt{1+\nu^{2/3}}\tau}\coth\frac{4\pi^2l\sqrt{1+\nu^{2/3}}\tau}{\beta}}{\sinh\frac{4\pi^2l\sqrt{1+\nu^{2/3}}\tau}{\beta}}\cos\left(\frac{2\pi l\nu^{2/3}}{\beta}-\frac{\pi}{4}\right). \end{aligned} \tag{12}$$

The graph of the function $C_V/C_{V_0}(B_0/B)$ is given in Fig. 1. It has an oscillatory form with the period $\Delta(1/\beta) = \nu^{2/3}$.

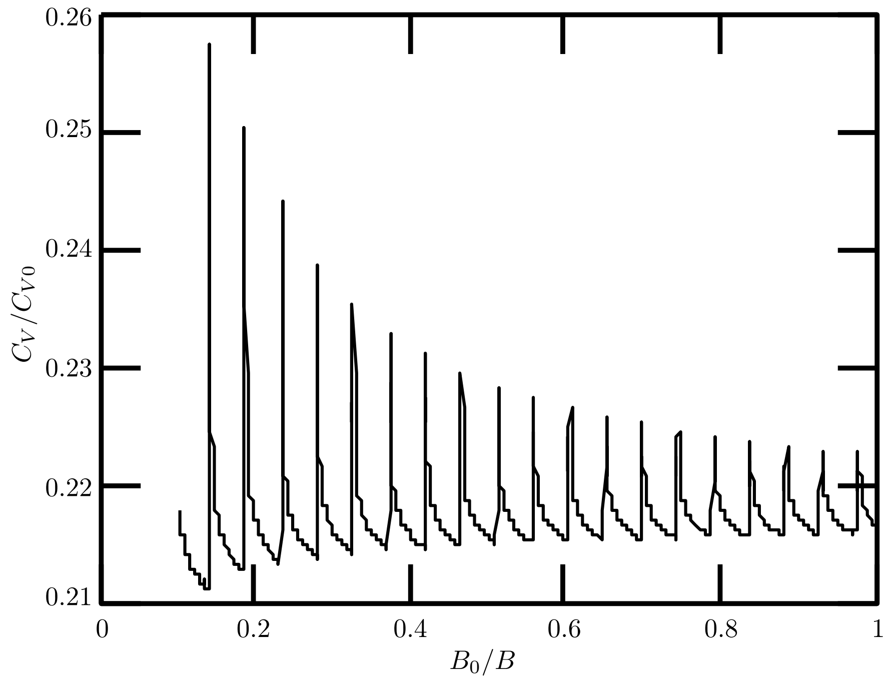

FIGURE 1. Magnetic-induction dependence of the heat capacity for a degenerate relativistic electron gas: $l = 1, 2, \ldots, 100$; $\nu = 10^2$; $\tau = 10^{-3}$.

At nonrelativistic limit, the expression (9) of the heat capacity becomes

$$\begin{aligned} C_V &= \frac{\pi^2}{3}D\left(\varepsilon_{F0}\right)k_B^2T - \frac{V\sqrt{2}m_0^{3/2}(\mu_BB)^2}{18\hbar^3\varepsilon_{F0}^{3/2}}k_B^2T \\ &+\frac{V\sqrt{2}\pi^2m_0^{3/2}k_B^3T^2}{\hbar^3\sqrt{\mu_BB}}\sum_{l=1}^{\infty}\sqrt{l}\frac{1-2\coth^2\left(\pi^2l\frac{k_BT}{\mu_BB}\right)+2\frac{\mu_BB}{k_BT}\coth\left(\pi^2l\frac{k_BT}{\mu_BB}\right)}{\sinh\left(\pi^2l\frac{k_BT}{\mu_BB}\right)}\cos\left(\pi l\frac{\varepsilon_{F0}}{\mu_BB}-\frac{\pi}{4}\right), \end{aligned} \tag{13}$$

where

$$D\left(\varepsilon_{F0}\right)=\frac{V}{2\pi^2}\left(\frac{2m_0}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon_{F0}} \tag{14}$$

is the density of states at the nonrelativistic Fermi energy ε_{F_0}.

3. Nondegenerate relativistic electron gas

In this case, the free energy is calculated with the aid of the formula

$$F=-Nk_BT\ln\left[\frac{e}{N}\sum_{\sigma=\pm1}\int_{\varepsilon_{0\sigma}(0)}^{\infty}D_\sigma\left(\varepsilon\right)\exp\left(\frac{-\varepsilon}{k_BT}\right)d\varepsilon\right]. \tag{15}$$

Performing the calculi, we obtain [24]

$$\begin{gathered}F=-Nk_BT\\ \times\ln\left\{\frac{e}{N}\frac{Vm_0^2ck_BT}{\pi^2\hbar^3}\left[K_0\left(\frac{m_0c^2}{k_BT}\right)+2\frac{k_BT}{m_0c^2}K_1\left(\frac{m_0c^2}{k_BT}\right)+\frac{1}{3}\left(\frac{\mu_BB}{k_BT}\right)^2K_0\left(\frac{m_0c^2}{k_BT}\right)\right]\right\},\end{gathered} \tag{16}$$

where $K_p(z)$ are the MacDonald functions (the Hankel functions of imaginary argument).

These functions have the following properties [26]

$$\begin{gathered}K_2\left(z\right)=K_0\left(z\right)+\frac{2}{z}K_1\left(z\right),\ \ \frac{d}{dz}K_0\left(z\right)=-K_1\left(z\right),\\ \frac{d}{dz}K_1\left(z\right)=-K_0\left(z\right)-\frac{1}{z}K_1\left(z\right),\ \ \frac{d}{dz}K_2\left(z\right)=-K_1\left(z\right)-\frac{2}{z}K_2\left(z\right).\end{gathered} \tag{17}$$

Substituting (16) into (6) and taking into account (17), it is easy to calculate the internal energy of the electron gas

$$\begin{gathered}U=Nk_BT\left[3+\frac{m_0c^2}{k_BT}\frac{K_1\left(m_0c^2/k_BT\right)}{K_2\left(m_0c^2/k_BT\right)}\right]-\frac{N(\mu_BB)^2}{3k_BT}\\ \times\left[4\frac{K_0\left(m_0c^2/k_BT\right)}{K_2\left(m_0c^2/k_BT\right)}-\frac{m_0c^2}{k_BT}\frac{K_1\left(m_0c^2/k_BT\right)}{K_2\left(m_0c^2/k_BT\right)}+\frac{m_0c^2}{k_BT}\frac{K_0\left(m_0c^2/k_BT\right)K_1\left(m_0c^2/k_BT\right)}{K_2^2\left(m_0c^2/k_BT\right)}\right].\end{gathered} \tag{18}$$

It follows from (8), (17), and (18) that the heat capacity comprises two terms:

$$\begin{gathered}C_{V_1}=Nk_B\left\{\left[3-\frac{1}{2}\left(\frac{m_0c^2}{k_BT}\right)^2-\frac{1}{4}\left(\frac{m_0c^2}{k_BT}\right)^4\right]\right.\\ \left.+\frac{1}{2}\left(\frac{m_0c^2}{k_BT}\right)^2\left[3+\left(\frac{m_0c^2}{k_BT}\right)^2\right]\frac{K_0\left(m_0c^2/k_BT\right)}{K_2\left(m_0c^2/k_BT\right)}-\frac{1}{4}\left(\frac{m_0c^2}{k_BT}\right)^4\frac{K_0^2\left(m_0c^2/k_BT\right)}{K_2^2\left(m_0c^2/k_BT\right)}\right\}\end{gathered} \tag{19}$$

and

$$\begin{gathered}C_{V_2}=\frac{1}{3}Nk_B\left(\frac{\mu_BB}{k_BT}\right)^2\left\{-\frac{1}{2}\left(\frac{m_0c^2}{k_BT}\right)^2\left[7+\left(\frac{m_0c^2}{k_BT}\right)^2\right]\right.\\ +\frac{3}{2}\left[8+6\left(\frac{m_0c^2}{k_BT}\right)^2+\left(\frac{m_0c^2}{k_BT}\right)^4\right]\frac{K_0\left(m_0c^2/k_BT\right)}{K_2\left(m_0c^2/k_BT\right)}\\ \left.-\frac{1}{2}\left(\frac{m_0c^2}{k_BT}\right)^2\left[11+3\left(\frac{m_0c^2}{k_BT}\right)^2\right]\frac{K_0^2\left(m_0c^2/k_BT\right)}{K_2^2\left(m_0c^2/k_BT\right)}+\frac{1}{2}\left(\frac{m_0c^2}{k_BT}\right)^4\frac{K_0^3\left(m_0c^2/k_BT\right)}{K_2^3\left(m_0c^2/k_BT\right)}\right\}.\end{gathered} \tag{20}$$

The first term is independent of the magnetic induction. It corresponds to the heat capacity of the nondegenerate relativistic electron gas in the absence of a magnetic field [27]. The second term is a quadratic function of the magnetic induction.

In the nonrelativistic case, we will take into account the asymptotic expansions of the MacDonald functions for $z\gg1$, namely [26]:

$$K_0(z)\sim\sqrt{\frac{\pi}{2z}}e^{-z}\left(1-\frac{1}{8z}+\frac{9}{128z^2}-\frac{225}{3072z^3}+\frac{11025}{98304z^4}+\ldots\right), \tag{21}$$

$$K_2(z)\sim\sqrt{\frac{\pi}{2z}}e^{-z}\left(1+\frac{15}{8z}+\frac{105}{128z^2}-\frac{945}{3072z^3}+\frac{31185}{98304z^4}+\ldots\right). \tag{22}$$

Then, the sum of the terms (19) and (20) becomes

$$C_V = \frac{3}{2} N k_B + \frac{2}{3} N k_B \left(\frac{\mu_B B}{k_B T} \right)^2 . \tag{23}$$

This result is well-known in the theory of the nondegenerate nonrelativistic electron gas. The first term expresses the Dulong and Petit law and the second term is related to the Curie law for the magnetic susceptibility.

4. Conclusions

In this paper we determined the heat capacity of degenerate and nondegenerate relativistic electron gases in a magnetic field. In both cases, it comprises a term independent of the magnetic induction and a term with a quadratic dependence on the magnetic induction. In the case of the degenerate relativistic electron gas, a third term with an oscillatory dependence on the magnetic induction appears. These oscillations have the period dependent on the electron concentration and the amplitude dependent on the electron concentration, temperature, and magnetic induction. The expressions of the heat capacity are very complicated.

The nonrelativistic limit was also studied. In this case, the expressions of the heat capacity are much simpler than in the relativistic case.

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