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A NOTE ON STABILITY CONSTRAINTS FOR ASYMPTOTIC STABILITY OF THE SCALAR EQUATIONS

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Abstract. We consider the scalar equation of the form $x(n+2k)+px(n+k)+qx(n) = 0$. In the case of $p = 1$ or $p = -1$, q constraints have determined for stability of the scalar equation by using the analytical solution. Thus, we have a practical knowledge about the stability of the scalar equation without the need of a computer programming and stability test.

1. Introduction

Scalar equations have extensively used in mathematical expression of circuits in the analysis of movement, in the formation of demand and supply equations in econometry, in the explanation of economic fluctuations or cyclic movements, in the calculation of unemployment rate, in the filter design of spectrum analysis. Therefore this study aims to present a new perspective for optimum use of such equations in disciplines such as Mathematics, Economy, Management, Statistics, Engineering and Economic analysis [3, 4, 5, 6]. In this sense, Levin and May obtained stability constraint for asimtptotically stable of the scalar equation [12]. In a different study, Brooks concerned with the linear stability of a first-order three-dimensional discrete dynamic in evolutionary game theory research [7].

Real-valued homogeneous linear difference equations are named as a scalar equation. The equation

$$x(n+k) + a_{k-1}x(n+k-1) + a_{k-2}x(n+k-2) + \ldots + a_0x(n) = 0$$

is called as k^{th} order scalar equation, where $a_{k-1}, a_{k-2}, \ldots, a_0 \in \mathbb{R}$ [1].

In this paper we consider the following form

$$x(n+2k) + px(n+k) + qx(n) = 0. \tag{1}$$

Eq. (1) is asymtotically stable if and only if $|\lambda_i| < 1, i = 1, 2, \ldots, 2k$. Here, λ_i are called as eigenvalues or characteristic roots which obtained characteristic polynomial of Eq. (1). We write the charecteristic polynomial of Eq. (1) as

$$F(\lambda) = \lambda^{2k} + p\lambda^k + q. \tag{2}$$

Calculation of eigenvalue provides ease in the examination of the stability of Eq. (1). Although the Eq. (1) is possible to solve analytically according to the certain p and q values, we will need to make an examination for each equation which obtained according to the changing values. On the other hand, eigenvalues can be calculated by forming the companion matrix of the Eq. (1), which can be obtained through the characteristic polynomial of the scalar equations. If A_{nxn} matrix is a companion matrix of Eq. (1), then the form $Ax = \lambda x,\ x \neq 0$ is obtained that λ is an *eigenvalue* and x is also called

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as an *eigenvector* corresponding to λ eigenvalue [11]. According to Lyapunov's Theorem: Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of the companion matrix A which corresponds to $\mathbf{n}^{th}$ order a scalar equation. In this case, the necessary and sufficient condition for the scalar equation to be asymptotically stable is that $|\lambda_i| < 1, i = 1, 2, \ldots, n$. Also, the necessary and sufficient condition for scalar equation to be stable is that the number of Jordan blocks corresponding to $|\lambda_k| = 1$ must not be more than 1.

Eq. (1) is called as stable for any $\varepsilon > 0$, if there exist $\delta = \delta(\varepsilon, n_0) > 0$ to be $\|x_n\| < \varepsilon$, such that for all $\|x_0\| < \delta$. If Eq. (1) is stable and there is $\underset{x\to\infty}{Lim} x_n = 0$, then Eq. (1) is *asymptotically stable.*

In a different way, Schur-Cohn criterion has given the necessary conditions for the characteristic roots of Eq. (1) to fall inside the unit circle [13]. Similar studies have given methods used determination of the stability of linear difference equations [8, 9, 10].

Bistritz has presented efficient-preserving (IP) form of the Bistritz Test for determine critical stability constraints of discrete-time linear systems [9]. Jury and Anderson gave the necessary and sufficient conditions for the roots of a real polynomial to be within the unit circle [14]. Brooks, in [7], obtained the linear stability conditions for a first-order three-dimensional discrete dynamic connected to the terms of the trace, determinant and sum of principle minors of the Jacobian evaluated at the equilibrium.

Evaluation of this process is time-consuming to the large-order equations. Other than this, even if a mathematical program is seen more convenient for to find the solutions, we will need to repeat it every time.

In this study have been investigated the stability constrait of q by solving the Eq. (1) analytically for specific values of p. Accordingly, the applications were given for validate the results. More clearly, in the case of $p = 1$ or $p = -1$, we have determine q constraint for stability of the Eq. (1) using the analytical solution. So, we no longer have been a practical knowledge about the stability the Eq. (1) without computing account, stability test and making any analytical solution.

2. Stability Constraints of Eq. (1)

We obtained stability constraint relative to q, which ensured that the Eq. (1) is asymptotically stable when $p = 1$ and $p = -1$. Then, we have the following the theorem.

Theorem 1. *Eq.* (1) *is to be asymptotically stable for* $p = 1$ *and* $p = -1$ *if and only if* $0 < q < 1$.

Proof. Let us now consider characteristic polynomial of Eq. (1) for $t = \lambda^k$, we get that

$$G(t) = t^2 + t + q.$$

If the roots of this polynomial are examined, these are

$$t_1 = \frac{-1 - \sqrt{1-4q}}{2}, \tag{3}$$

$$t_2 = \frac{-1 + \sqrt{1-4q}}{2}. \tag{4}$$

(i) if we consider the root in (3)), we write

$$\lambda^k = \frac{-1 - \sqrt{1-4q}}{2}. \tag{5}$$

By using (5), roots of Eq. (1) can be obtained for λ_j, $j = 0, 1, 2, \ldots, k-1$

$$\lambda_j = \left|\frac{-1 - \sqrt{1-4q}}{2}\right|^{\frac{1}{k}} [\cos(\frac{\theta + 2j\pi}{k}) + i\sin(\frac{\theta + 2j\pi}{k})]. \tag{6}$$

We know that Eq. (1) is asymtotically stable if and only if $|\lambda_j| < 1$, $j = 1, 2, \ldots, 2k$. For $|\cos\theta + i\sin\theta| < 1$, we need to show that

$$\left|\frac{-1-\sqrt{1-4q}}{2}\right|^{\frac{1}{k}} < 1. \tag{7}$$

Case 1: Firstly assume that $1 - 4q = 0$. Then the following inequality confirm.

$$\left|\frac{-1-\sqrt{1-4q}}{2}\right|^{\frac{1}{k}} = (\frac{1}{2})^{\frac{1}{k}} < 1.$$

Case 2: Secondly suppose that $1 - 4q > 0$, as follows

$$\begin{aligned}\left|\frac{-1-\sqrt{1-4q}}{2}\right|^{\frac{1}{k}} &= \left(\frac{1+\sqrt{1-4q}}{2}\right)^{\frac{1}{k}} < 1\\ &\Longleftrightarrow \frac{1+\sqrt{1-4q}}{2} < 1\\ &\Longleftrightarrow \sqrt{1-4q} < 1\\ &\Longleftrightarrow q > 0.\end{aligned}$$

So, we can obtain the following inequality from case 1 and case 2

$$0 < q \leq \frac{1}{4}. \tag{8}$$

Case 3: We now assume that $1 - 4q < 0$, as follows

$$\begin{aligned}\left|\frac{-1-\sqrt{(4q-1)(-1)}}{2}\right|^{\frac{1}{k}} &= \left|\frac{-1-\sqrt{4q-1}i}{2}\right|^{\frac{1}{k}} < 1\\ &\Rightarrow \left(\sqrt{\frac{1}{4}+\frac{4q-1}{4}}\right)^{\frac{1}{k}} = (\sqrt{q})^{\frac{1}{k}} < 1\\ &\Rightarrow q < 1.\end{aligned}$$

Then, we get that

$$\frac{1}{4} < q < 1. \tag{9}$$

By combining (8) and (9), as seen that we have q constraint the following

$$0 < q < 1. \tag{10}$$

(ii) if λ_i is written according to (4), as follows

$$\lambda^k = \frac{-1+\sqrt{1-4q}}{2}. \tag{11}$$

By using (11), roots of Eq. (1) can be obtained for λ_j, $j = 0, 1, 2, \ldots, k-1$

$$\lambda_j = \left|\frac{-1+\sqrt{1-4q}}{2}\right|^{\frac{1}{k}} [\cos(\frac{\theta+2j\pi}{k}) + i\sin(\frac{\theta+2j\pi}{k})]. \tag{12}$$

With the information of $|\cos\theta + i\sin\theta| < 1$ for $\forall\theta$, it is easy to see that $|\lambda_j| < 1$ if and only if

$$\left|\frac{-1+\sqrt{1-4q}}{2}\right|^{\frac{1}{k}} < 1. \tag{13}$$

If $1-4q=0$ is written in (13), λ_j roots of Eq. (1) are to fall inside the unit circle if and only if

$$\left|\frac{-1+\sqrt{1-4q}}{2}\right|^{\frac{1}{k}}=(\frac{1}{2})^{\frac{1}{k}}<1.$$

Then last inequality is provided for

$$q=\frac{1}{4}. \tag{14}$$

If inequality (13) is solved for $1-4q>1$, as follows

$$\begin{aligned}\left|\frac{-1+\sqrt{1-4q}}{2}\right|^{\frac{1}{k}} &= \left(\frac{-1+\sqrt{1-4q}}{2}\right)^{\frac{1}{k}}<1\\ &\Longleftrightarrow \frac{-1+\sqrt{1-4q}}{2}<1\\ &\Longleftrightarrow \sqrt{1-4q}<3\\ &\Longleftrightarrow q>-2.\end{aligned}$$

Then, it can be written as

$$-2<q<0. \tag{15}$$

If (13) is examined for $0<1-4q<1$, we have

$$\begin{aligned}\left|\frac{-1+\sqrt{1-4q}}{2}\right|^{\frac{1}{k}} &= \left(\frac{1-\sqrt{1-4q}}{2}\right)^{\frac{1}{k}}<1\\ &\Longleftrightarrow \frac{1-\sqrt{1-4q}}{2}<1\\ &\Longleftrightarrow \sqrt{1-4q}<-1.\end{aligned}$$

As a result, the last inequality confirms for $\forall q>0$. Then it can obtain as

$$0<q<\frac{1}{4}. \tag{16}$$

Lastly assume that $1-4q<0$ in (13). We get that

$$\begin{aligned}\left|\frac{-1-\sqrt{(4q-1)(-1)}}{2}\right|^{\frac{1}{k}} &= \left|\frac{-1+\sqrt{4q-1}i}{2}\right|^{\frac{1}{k}}<1\\ &\Longleftrightarrow \left(\sqrt{\frac{1}{4}+\frac{4q-1}{4}}\right)^{\frac{1}{k}}=(\sqrt{q})^{\frac{1}{k}}<1\\ &\Longleftrightarrow q<1\end{aligned}$$

Then, it can be writen as

$$\frac{1}{4}<q<1. \tag{17}$$

Now, if (14), (15), (16) and (17) are combined, we can see that

$$-2<q<1. \tag{18}$$

So, we can get the following stability constraints from the intersection of solution (10) and (18), it is seen that

$$0<q<1. \tag{19}$$

□

TABLE 1. $p = 1$

q	$\lambda_j,\ j = 0,\ldots,k-1$	$\lambda_j,\ j = 0,\ldots,k-1$
0.1	$exp\left(\frac{-2.18301+(\pi+2k\pi)i}{k}\right)$	$exp\left(\frac{-0.119574+(\pi+2k\pi)i}{k}\right)$
0.2	$exp\left(\frac{-1.28593+(\pi+2k\pi)i}{k}\right)$	$exp\left(\frac{-0.323507+(\pi+2k\pi)i}{k}\right)$
0.3	$exp\left(\frac{-0.60199+(2.72106+2k\pi)i}{k}\right)$	$exp\left(\frac{-0.60199-(2.72106+2k\pi)i}{k}\right)$
0.4	$exp\left(\frac{-0.45815+(2.48254+2k\pi)i}{k}\right)$	$exp\left(\frac{-0.45815-(2.48254+2k\pi)i}{k}\right)$
0.5	$exp\left(\frac{-0.34657+(2.25620+2k\pi)i}{k}\right)$	$exp\left(\frac{-0.34657-(2.25620+2k\pi)i}{k}\right)$
0.6	$exp\left(\frac{-0.25541+(2.27247+2k\pi)i}{k}\right)$	$exp\left(\frac{-0.25541-(2.27247+2k\pi)i}{k}\right)$
0.7	$exp\left(\frac{-0.17834+(2.21132+2k\pi)i}{k}\right)$	$exp\left(\frac{-0.17834-(2.21132+2k\pi)i}{k}\right)$
0.8	$exp\left(\frac{-0.11157+(2.16400+2k\pi)i}{k}\right)$	$exp\left(\frac{-0.11157-(2.16400+2k\pi)i}{k}\right)$
0.9	$exp\left(\frac{-0.05268+(2.1259242k\pi)i}{k}\right)$	$exp\left(\frac{-0.052680-(2.1259242k\pi)i}{k}\right)$

Similarly, we get inequality (19), for $p = -1$ with the same operations. Moreover, if the roots of scalar equations formed according to the values outside the range of $0 < q < 1$ are calculated, the obtained roots are always $|\lambda_j| > 1$. Namely, Eq. (1) is not asymptotically stable. This completes the proof.

3. Applications

In this section, we will give applications to support Theorem 1. First, we will calculate the roots of the scalar equation obtained for some values of q with Mapple programming; and secondly we will confirm q constraint using Bistritz's stability method.

3.1. **Calculation of the roots of Eq.** (1) **by using Mapple programming for some values of** q**.** Our purpose confirm that the roots of the scalar equation obtained for some values of q when $p = 1$ and $p = -1$ are to fall inside the unit circle with Mapple programming. Accordigly, we can clearly see that each equation formed relative to values of q is asymptotically stable. In other words, the roots obtained according to $p = 1$ and some changing values of q in Eq. (1) are given in the table 1.

It is observed that $|\lambda_j| < 1$, $j = 1, 2, \ldots, 2k$, for some value $0 < q < 1$. Likewise, the values obtained according to $p = -1$ and some changing values of q in Eq. (1) are given below in table 2.

As is seen from the Table 2, it is observed that $|\lambda_j| < 1$, $i = 1, 2, \ldots, 2k$ for $0 < q < 1$. Moreover, if the roots of the scalar equation formed according to the values outside the range of $0 < q < 1$ are calculated, they are always $|\lambda_j| > 1$. Namely, Eq. (1) is not asymptotically stable.

3.2. **Calculation of stability constraints of Eq.** (1) **according to Bistritz's stability method.** We will revaluation stability constraints of Eq. (1) which has specific an order with the Bistritz stability test. We examine the examples the following

1) For $p = 1$ and $k = 2$, we write

$$x(n+4) + x(n+2) + qx(n) = 0 \tag{20}$$

Table 2. $p = -1$

q	$\lambda_j,\ j = 0, \cdots, k-1$	$\lambda_j,\ j = 0, \cdots, k-1$
0.1	$exp\left(\frac{-0.11957+k\pi i}{k}\right)$	$exp\left(\frac{-2.18301+k\pi i}{k}\right)$
0.2	$exp\left(\frac{-0.32351+k\pi i}{k}\right)$	$exp\left(\frac{-1.28593+k\pi i}{k}\right)$
0.3	$exp\left(\frac{-0.60199+(0.42053+2k\pi)i}{k}\right)$	$exp\left(\frac{-0.60199-(0.42053+2k\pi)i}{k}\right)$
0.4	$exp\left(\frac{-0.45815+(0.65906+2k\pi)i}{k}\right)$	$exp\left(\frac{-0.45815-(0.65906+2k\pi)i}{k}\right)$
0.5	$exp\left(\frac{-0.34657+(0.78540+2k\pi)i}{k}\right)$	$exp\left(\frac{-0.34657-(0.78540+2k\pi)i}{k}\right)$
0.6	$exp\left(\frac{-0.25541+(0.86912+2k\pi)i}{k}\right)$	$exp\left(\frac{-0.25541-(0.86912+2k\pi)i}{k}\right)$
0.7	$exp\left(\frac{-0.17834+(0.93027+2k\pi)i}{k}\right)$	$exp\left(\frac{-0.17834-(0.93027+2k\pi)i}{k}\right)$
0.8	$exp\left(\frac{-0.11157+(0.97760+2k\pi)i}{k}\right)$	$exp\left(\frac{-0.11157-(0.97760+2k\pi)i}{k}\right)$
0.9	$exp\left(\frac{-0.05268+(1.0156842k\pi)i}{k}\right)$	$exp\left(\frac{-0.05268-(1.0156842k\pi)i}{k}\right)$

Table 3.

$-\infty$	-2	$-2/3$	0	1	∞
$4+2q$	$-$	$+$	$+$	$+$	$+$
$4-4q$	$+$	$+$	$+$	$+$	$-$
$-3q^2+q+2$	$-$	$-$	$+$	$+$	$-$
$\left(q-q^2\right)(1-q)$	$-$	$-$	$-$	$+$	$+$

is obtained. Let's prove the stability constraint of this equation using Bistritz's stability method [10]. By this method, the following symmetrical polynomial series are obtained

$$\begin{aligned}
T_4(z) &= (q+1)z^4 + 2z^2 + (q+1)\\
T_3(z) &= (1-q)z^3 + (1-q)z^2 + (1-q)z + (1-q)\\
T_2(z) &= (1-q^2)z^2 + (q-q^2)z + (1-q^2)\\
T_1(z) &= (1-q)(q-q^2)z + (1-q)(q-q^2)\\
T_0(z) &= q^3 - 2q^2 + q.
\end{aligned}$$

By this way, if the process is continued, we have

$$\begin{aligned}
T_4(1,z) &= 4 + 2q > 0\\
T_3(1,z) &= 4 - 4q > 0\\
T_2(1,z) &= -3q^2 + q + 2 > 0\\
T_1(1,z) &= 2(-3q^2 + q + 2) > 0\\
T_0(1,z) &= (q-q^2)(1-q) > 0.
\end{aligned}$$

If signal table is used, we can obtain table 3.

It is observed clearly from table 3 that $0 < q < 1$. If roots of Eq. (20) are calculated for some changing values of q with the Mapple, we get table 4.

From Table 4, it is observed that $|\lambda_i| < 1$, $i = 1, 2, 3, 4$.

2) For $p = -1$ and $k = 2$, we write

$$x(n+4) - x(n+2) + qx(n) = 0. \tag{21}$$

TABLE 4.

q	λ_1	λ_2	λ_3	λ_4
0.1	-0.33571	0.33571	$-0.94197i$	$0.94197i$
0.2	$-0.52573i$	$0.52573i$	$-0.85065i$	$0.85065i$
0.3	$-0.15447-0.72378i$	$-0.15447+0.72378i$	$0.15447-0.72378i$	$0.15447+0.72378i$
0.4	$-0.25735-0.75248i$	$-0.257345+0.75248i$	$0.25735+0.75248i$	$0.25735-0.75248i$
0.5	$-0.32180-0.77689i$	$-0.32180+0.77689i$	$0.32180+0.77689i$	$0.32180-0.77689i$
0.6	$-0.37054-0.79831i$	$-0.37054+0.79831i$	$0.37054+0.79831i$	$0.37054-0.79831i$
0.7	$-0.41028-0.81752i$	$-0.41028+0.81752i$	$0.41028+0.81752i$	$0.41028-0.81752i$
0.8	$-0.44409-0.83499i$	$-0.44409+0.83499i$	$0.44409-0.83499i$	$0.44409+0.83499i$
0.9	$-0.47365-0.85108i$	$-0.47365+0.85108i$	$0.47365-0.85108i$	$0.47365+0.85108i$

TABLE 5.

$-\infty$	-3	$-4/3$	0	1	∞
$2q$	-	-	-	+	+
$4-4q$	+	+	+	+	-
$-3q^2-q+4$	-	-	+	+	-
$(1-q)\left(-2q^2-4q+6\right)$	-	+	+	+	+
$\left(-q^2-2q+3\right)\left(-q^2+q\right)$	+	-	-	+	+

The q constraint is confirmed by using Bistritz's stability criterion for the above equation.

If $D_z = D_4(z) = z^4 - z^2 + q$ and $D^*(z) = qz^4 - z^2 + 1$, we obtain

$$\begin{aligned}
T_4(1,z) &= 2q \\
T_3(1,z) &= 4-4q \\
T_2(1,z) &= -3q^2-q+4 \\
T_1(1,z) &= (1-q)(-2q^2-4q+6) \\
T_0(1,z) &= (-q^2-2q+3)(-q^2+q).
\end{aligned}$$

By this way, if the process is continued, we have

$$\begin{aligned}
T_4(z) &= (q+1)z^4 - 2z^2 + (q+1) > 0 \\
T_3(z) &= (1-q)z^3 + (1-q)z^2 + (1-q)z + (1-q) > 0 \\
T_2(z) &= (1-q^2)z^2 + (-q^2-q+2)z + (1-q^2) > 0 \\
T_1(z) &= (1-q)(-q^2-2q+3)z + (1-q)(-q^2-2q+3) > 0 \\
T_0(z) &= (-q^2-2q+3)(-q^2+q) > 0.
\end{aligned}$$

it is observed that $0 < q < 1$ the following table 5.

The roots of the Eq. (20) according to some changing q values are given in the table below 6.

It is obvious from Table 4 that each $|\lambda_i| < 1$, $i = 1,2,3,4$.

4. CONCLUSION

This study demonstrate the stability constraint of q, which is asymptotic stability of Eq. (1) for $p = 1$ and $p = -1$. This constraint obtained with analitical investigation under some values determine whether Eq. (1) was stable or not. Related examples were

TABLE 6.

q	λ_1	λ_2	λ_3	λ_4
0.1	0.94197	-0.94197	0.33571	-0.33571
0.2	0.85065	-0.85065	-0.52573	0.52573
0.3	$0.72378 + 0.15447i$	$0.72378 - 0.15447i$	$-0.72378 + 0.15447i$	$-0.72378 - 0.15447i$
0.4	$0.75248 + 0.25735i$	$0.75248 - 0.25735i$	$-0.75248 - 0.25735i$	$-0.75248 + 0.25735i$
0.5	$0.77689 + 0.32180i$	$0.77689 - 0.32180i$	$-0.77689 + 0.32180i$	$-0.77689 - 0.32180i$
0.6	$-0.79831 + 0.37054i$	$0.79831 - 0.37054i$	$-0.79831 - 0.37054i$	$-0.79831 + 0.37054i$
0.7	$0.81752 + 0.41028i$	$0.81752 - 0.41028i$	$-0.81752 - 0.41028i$	$-0.81752 + 0.41028i$
0.8	$0.83499 - 0.44409i$	$0.83499 + 0.44409i$	$-0.83499 - 0.44409i$	$-0.83499 + 0.44409i$
0.9	$0.85108 + 0.47365i$	$0.85108 - 0.47365i$	$-0.85108 - 0.47365i$	$-0.85108 + 0.47365i$

given by using Mapple programming and Bistritz test, which is a favourable stability test. As a result, we obtained the information about the characteristics of the scalar equation of type (1) without ever needing to apply a stability test or using computer.

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