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NEW FRACTIONAL INEQUALITIES OF OSTROWSKI TYPE

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ABSTRACT. In this work, we establish a new weighted Montgomery identity for Riemann-Liouville fractional integrals. Then using this new fractional Montgomery identity, we obtain some new fractional inequalities of Ostrowski type.

1. INTRODUCTION

The theory of fractional calculus has known an intensive development over the last few decades. It is shown that derivatives and integrals of fractional type provide an adequate mathematical modeling of real objects and processes see [3, 8, 12]. Therefore, the study of fractional differential equations need more developmental of inequalities of fractional type. The main aim of this work is to develop new weighted Montgomery identity for Riemann-Liouville fractional integrals that will be used to establish new weighted Ostrowski inequalities. Let us begin by introducing this type of inequality. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function such that its derivative f' be integrable on $[a, b]$ and let $w : [a, b] \to [0, +\infty)$ be a probability density function that satisfies $\int_a^b w(t) = 1$. Set $W(t) = \int_a^t w(x)dx$ for $t \in [a, b], W(t) = 0$ if $t < a$ and $W(t) = 1$ if $t > b$.

The weighted generalization of the Montgomery identity, given by Mitrinović et al [10] is the following:

$$f(x) = \int_a^b w(t)f(t)dt + \int_a^b P_w(x, t)f'(t)dt \tag{1}$$

where the weighted Peano kernel is

$$P_w(x, t) = \begin{cases} W(t), & a \le t \le x \\ W(t) - 1, & x \le t \le b \end{cases} \tag{2}$$

In [1], Anastassiou et al obtained the weighted Montgomery identity for fractional integrals

$$\begin{gathered} f(x) = (b - x)^{1-\alpha}\Gamma(\alpha)J_a^\alpha(w(b)f(b)) \\ -J_a^{\alpha-1}(Q_w(x, b)f(b)) + J_a^\alpha(Q_w(x, b)f'(b)) \end{gathered} \tag{3}$$

where $\alpha \ge 1$, the weighted fractional Peano kernel is

$$Q_w(x, t) = \begin{cases} (b - x)^{1-\alpha}\Gamma(\alpha)W(t), & a \le t \le x \\ (b - x)^{1-\alpha}\Gamma(\alpha)(W(t) - 1), & x \le t \le b \end{cases} \tag{4}$$

J_a^α denotes the Riemann-Liouville integral operator of order $\alpha > 0$ with $a \ge 0$ defined by $J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t)\, dt$ and $J_a^0 f(x) = f(x)$, and Γ is Gamma function. Then

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the authors derive the following interesting fractional integral inequality:

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha)J_a^{\alpha}(f(b)) + J_a^{\alpha-1}(P_2(x,b)f(b)) \right| \\ &\leq \tfrac{M}{\alpha(\alpha+1)} \left[(b-x)\left(2\alpha\left(\tfrac{b-x}{b-a}\right) - \alpha - 1\right) + (b-a)^{\alpha}(b-x)^{1-\alpha} \right] \end{aligned} \tag{5}$$

under the assumption that $|f'(x)| \leq M$, for any $x \in [a,b]$ and here the Peano kernel P_2 is defined by

$$P_2(x,t) = \begin{cases} \frac{t-a}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & a \leq t \leq x \\ \frac{t-b}{b-a}(b-x)^{1-\alpha}\Gamma(\alpha), & x \leq t \leq b \end{cases} \tag{6}$$

More results and properties of fractional integrals can be found in [8]. We refer the reader to [4, 5, 6, 7, 8, 9, 11], for more results on Ostrowski type inequalities.

Motived by the above work, we establish new weighted Ostrowski inequalities and new weighted Montgomery identity for Riemann-Liouville fractional integrals.

2. Weighted Montgomery identity for fractional integrals

Let $\varphi : [0,1] \to \mathbb{R}$ be a differentiable function such $\varphi(0) = 0, \varphi(1) \neq 0$ and $\varphi' \in L^1[0,1]$. Our first result is the following.

Theorem 1. *The generalization of weighted Montgomery identity for fractional integrals is:*

$$\begin{aligned} &f(x) = \tfrac{1}{\varphi(1)}(b-x)^{1-\alpha}\Gamma(\alpha)J_a^{\alpha}(w(b)\varphi'(1)f(b)) \\ &-\tfrac{1}{\varphi(1)}J_a^{\alpha-1}(Q_{w,\varphi}(x,b)f(b)) + \tfrac{1}{\varphi(1)}J_a^{\alpha}(Q_{w,\varphi}(x,b)f'(b)) \end{aligned} \tag{7}$$

where the weighted fractional Peano kernel is

$$Q_{w,\varphi}(x,t) = \begin{cases} (b-x)^{1-\alpha}\Gamma(\alpha)\varphi(W(t)), & a \leq t \leq x \\ (b-x)^{1-\alpha}\Gamma(\alpha)(\varphi(W(t)) - \varphi(1)), & x \leq t \leq b \end{cases} \tag{8}$$

Proof. Using (8) and properties of fractional integrals we get

$$\begin{aligned} &J_a^{\alpha}(Q_{w,\varphi}(x,b)f'(b)) = \tfrac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}Q_{w,\varphi}(x,t)f'(t)dt \\ &= (b-x)^{1-\alpha}\left[\int_a^x (b-t)^{\alpha-1}\varphi(W(t))f'(t)dt + \int_x^b (b-t)^{\alpha-1}(\varphi(W(t)) - \varphi(1))f'(t)dt\right] \end{aligned} \tag{9}$$

that can be written as

$$\begin{aligned} J_a^{\alpha}(Q_{w,\varphi}(x,b)f'(b)) = (b-x)^{1-\alpha}\Bigg[&\int_a^b (b-t)^{\alpha-1}\varphi(W(t))f'(t)dt \\ &- \int_x^b (b-t)^{\alpha-1}\varphi(1)f'(t)dt\Bigg] \end{aligned} \tag{10}$$

Taking in consideration that $W(a) = 0$ and $W(b) = 1$, the first term in the right hand side of (10) is equal to

$$\begin{aligned} &\int_a^b (b-t)^{\alpha-1}\varphi(W(t))f'(t)dt = -\Gamma(\alpha)J_a^{\alpha}(w(b)\varphi'(1)f(b)) \\ &+(\alpha-1)\int_a^b (b-t)^{\alpha-2}\varphi(W(t))f(t)dt \end{aligned} \tag{11}$$

the second term in the right hand side of (10) gives

$$\int_x^b (b-t)^{\alpha-1}\varphi(1)f'(b)dt = -\varphi(1)(b-x)^{\alpha-1}f(x) + (\alpha-1)\varphi(1)\int_x^b (b-t)^{\alpha-2}f(t)dt \tag{12}$$

Substituting (11) and (12) in (10), we obtain

$$J_a^\alpha(Q_{w,\varphi}(x,b)f'(b)) = \varphi(1)f(x) - \Gamma(\alpha)(b-x)^{1-\alpha}J_a^\alpha(w(b)\varphi'(1)f(b)) \\ +J_a^{\alpha-1}(Q_{w,\varphi}(x,b)f(b))$$

that implies

$$f(x) = \tfrac{1}{\varphi(1)}(b-x)^{1-\alpha}\Gamma(\alpha)J_a^\alpha(w(b)\varphi'(1)f(b)) \\ -\tfrac{1}{\varphi(1)}J_a^{\alpha-1}(Q_{w,\varphi}(x,b)f(b)) + \tfrac{1}{\varphi(1)}J_a^\alpha(Q_{w,\varphi}(x,b)f'(b)) \tag{13}$$

The proof is complete. □

3. An Ostrowski type fractional inequalities

In this section we give and prove our second results. first using the Peano kernel (4) and the weighted Montgomery identity (3), we get a new Ostrowski fractional inequality:

Theorem 2. *Assume that the function f is a differential on $[a,b]$ such that $|f'(x)| \leq M$, for any $x \in [a,b]$. Then the following Ostrowski fractional inequality holds:*

$$\left|f(x) - (b-x)^{1-\alpha}\Gamma(\alpha)J_a^\alpha(w(b)f(b)) + J_a^{\alpha-1}(Q_w(x,b)f(b))\right| \\ \leq \tfrac{M(b-x)}{\alpha}(1-2W(x)) + M(b-x)^{1-\alpha}\Gamma(\alpha)J_a^{\alpha+1}(w(b)) \tag{14}$$

Proof. Using Montgomery identity (3) and assumptions on the function f we obtain

$$\frac{1}{\Gamma(\alpha)}\left|\int_a^b (b-t)^{\alpha-1}Q_w(x,t)f'(t)dt\right| \leq \frac{M}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}\left|Q_w(x,t)\right| dt \tag{15}$$

Consequently

$$\tfrac{M}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}\left|Q_w(x,t)\right|dt \leq \tfrac{M}{\alpha}(b-x)(1-2W(x)) + \tfrac{M}{\alpha}(b-x)^{1-\alpha}\left(\int_a^b (b-t)^\alpha w(t)dt\right) \\ = \tfrac{M}{\alpha}(b-x)(1-2W(x)) + M(b-x)^{1-\alpha}\Gamma(\alpha)J_a^{\alpha+1}(w(b))$$

which achieves the proof. □

Now using the new weighted Montgomery identity for fractional integrals (7) and the corresponding weighted fractional Peano kernel (8), we derive a new Ostrowski inequality of Fractional type.

Theorem 3. *Let f be a differentiable function on $[a,b]$ and $|f'(x)| \leq M$ for any $x \in [a,b]$ and assume that φ is an increasing differentiable function on $[0,1]$. Then the following Ostrowski fractional inequality holds:*

$$\left|f(x) - \tfrac{1}{\varphi(1)}(b-x)^{1-\alpha}\Gamma(\alpha)J_a^\alpha(w(b)\varphi'(1)f(b)) + \tfrac{1}{\varphi(1)}J_a^{\alpha-1}(Q_{w,\varphi}(x,b)f(b))\right| \\ \leq \tfrac{M(b-x)}{\alpha}(1-\tfrac{2}{\varphi(1)}\varphi(W(x))) + \tfrac{M}{\varphi(1)}(b-x)^{1-\alpha}\Gamma(\alpha)J_a^{\alpha+1}(\varphi'(W(b))w(b)) \tag{16}$$

Proof. From Theorem 3 we have

$$\left|f(x) - \tfrac{1}{\varphi(1)}(b-x)^{1-\alpha}\Gamma(\alpha)J_a^\alpha(w(b)\varphi'(1)f(b)) + \tfrac{1}{\varphi(1)}J_a^{\alpha-1}(Q_{w,\varphi}(x,b)f(b))\right| \\ = \tfrac{1}{\Gamma(\alpha)\varphi(1)}\left|\int_a^b (b-t)^{\alpha-1}Q_{w,\varphi}(x,t)f'(t)dt\right|$$

Taking into account the assumptions on the function f and since φ is increasing, it yields

$$\left|f(x) - \tfrac{1}{\varphi(1)}(b-x)^{1-\alpha}\Gamma(\alpha)J_a^\alpha(w(b)\varphi'(1)f(b)) + \tfrac{1}{\varphi(1)}J_a^{\alpha-1}(Q_{w,\varphi}(x,b)f(b))\right| \\ \leq \tfrac{M(b-x)^{1-\alpha}}{\varphi(1)}\left[\int_a^x (b-t)^{\alpha-1}\varphi(W(t))dt + \int_x^b (b-t)^{\alpha-1}(\varphi(1)-\varphi(W(t)))dt\right] \tag{17}$$

Noting the left hand side of (17) by I_1 then integrating by parts the right hand side of (17), we obtain

$$I_1 \le \frac{M(b-x)^{1-\alpha}}{\varphi(1)} \Big[-\frac{(b-x)^\alpha}{\alpha}\varphi(W(x)) - \frac{(b-x)^\alpha}{\alpha}\varphi(W(x)) + \frac{(b-x)^\alpha}{\alpha}\varphi(1) + \frac{1}{\alpha}\left(\int_a^x (b-t)^\alpha \varphi'(W(t))w(t)dt - \int_x^b (b-t)^\alpha \varphi'(W(t))w(t)dt \right) \Big]$$

that can be written as

$$I_1 \le \frac{M}{\alpha}(b-x)\left(1 - \frac{2}{\varphi(1)}\varphi(W(x))\right) + \frac{M}{\alpha\varphi(1)}(b-x)^{1-\alpha} \left(2\int_a^x (b-t)^\alpha \varphi'(W(t))w(t)dt - \int_a^b (b-t)^\alpha \varphi'(W(t))w(t)dt \right)$$

Consequently

$$I_1 \le \frac{M}{\alpha}(b-x)\left(1 - \frac{2}{\varphi(1)}\varphi(W(x))\right) + \frac{M}{\varphi(1)}(b-x)^{1-\alpha}\Gamma(\alpha)J_a^{\alpha+1}(\varphi'(W(b))w(b))$$

This completes the proof. □

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