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OSTROWSKI INEQUALITY FOR FRACTIONAL INTEGRALS AND RELATED FRACTIONAL INEQUALITIES

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Abstract. In this paper, we obtain some new generalizations for Ostrowski inequality by using Riemann-Liouville fractional integral.

1. Introduction

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b) and assume $|f^{'}(x)| \leq M$ for all $x \in (a, b)$.Then the following holds:

$$|f(x) - \mathcal{M}(f; a, b)| \leq \frac{M}{b-a} \cdot \frac{(b-x)^2 + (x-a)^2}{2} \tag{1}$$

for all $x \in [a, b]$. Where $\mathcal{M}(f; a, b) = \frac{1}{b-a} \int_a^b f(x)dx$.

Above inequality is well known in the literature as Ostrowski inequality [1]. The inequality (1) has evoked the interest of many researchers and numerous generalizations, variants and extensions have appeared in the literature, to mention a few, see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and the references cited therein.

The main aim of this paper is to establish some new generalizations for (1) by using Riemann-Liouville fractional integral. In the following, we will give some necessary definitions and preliminaries which are used further in this paper.

Definition 1 ([18])**.** *Let $f \in L^1[a, b]$. The Riemann-Liouville integrals $J^{\alpha}_{a_+} f$ and $J^{\alpha}_{b_-} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^{\alpha}_{a_+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b_-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (b-t)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Where, $\Gamma(\alpha)$ is the Gamma function and $J^0_{a_+} f(x) = J^0_{b_-} f(x) = f(x)$.

Grüss [2] proved the following inequality:

$$|\mathcal{M}(fg; a, b) - \mathcal{M}(f; a, b)\mathcal{M}(g; a, b)| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2), \tag{2}$$

provided that f and g are two integral function on $[a, b]$ satisfying the condition $m_1 \leq f \leq M_1$ and $m_2 \leq g \leq M_2$ for all $x \in [a, b]$, where $m_1, m_2, M_1, M_2 \in \mathbb{R}$. The constant $\frac{1}{4}$ is the best possible. So, we call (2) the Grüss inequality.

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Korkine's identity [3] state that if f and g are two integral function on $[a,b]$, then

$$\mathcal{M}(fg;a,b)-\mathcal{M}(f;a,b)\mathcal{M}(g;a,b)=\frac{1}{2(b-a)^2}\int_a^b\int_a^b\left(f\left(t\right)-f\left(s\right)\right)\left(g\left(t\right)-g\left(s\right)\right)dsdt.$$

2. Main Results

Theorem 1. *Let f be a differentiable function on $[a,b]$ and $|f'(x)|\le M$ for any $x\in[a,b]$. Then the following fractional inequality holds:*

$$\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{\Gamma(\alpha+1)}f(x)-J_{x_-}^\alpha f(a)-J_{x_+}^\alpha f(b)\right|\le\frac{M\left((x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right)}{\Gamma(\alpha+2)}$$

for any $x\in[a,b]$ and $\alpha\ge 0$.

Proof. From Definition 1 and using integration by parts, we have

$$\begin{aligned}J_{x_-}^{\alpha+1}f'(a)&=\frac{1}{\Gamma(\alpha+1)}\int_a^x(t-a)^\alpha f'(t)dt\\&=\frac{1}{\Gamma(\alpha+1)}(t-a)^\alpha f(t)\Big|_a^x-\frac{1}{\Gamma(\alpha+1)}\int_a^x\alpha(t-a)^{\alpha-1}f(t)dt\\&=\frac{(x-a)^\alpha}{\Gamma(\alpha+1)}f(x)-J_{x_-}^\alpha f(a),\end{aligned}\tag{3}$$

and

$$J_{x_+}^{\alpha+1}f'(b)=\frac{-(b-x)^\alpha}{\Gamma(\alpha+1)}f(x)+J_{x_+}^\alpha f(b).\tag{4}$$

By (3) and (4) we get

$$\begin{aligned}&\frac{(x-a)^\alpha+(b-x)^\alpha}{\Gamma(\alpha+1)}f(x)-J_{x_-}^\alpha f(a)-J_{x_+}^\alpha f(b)\\=&\ \frac{1}{\Gamma(\alpha+1)}\int_a^x(t-a)^\alpha f'(t)dt-\frac{1}{\Gamma(\alpha+1)}\int_x^b(b-t)^\alpha f'(t)dt.\end{aligned}\tag{5}$$

Therefore we obtain

$$\begin{aligned}&\left|\frac{(x-a)^\alpha+(b-x)^\alpha}{\Gamma(\alpha+1)}f(x)-J_{x_-}^\alpha f(a)-J_{x_+}^\alpha f(b)\right|\\\le&\ \frac{1}{\Gamma(\alpha+1)}\left(\int_a^x(t-a)^\alpha|f'(t)|dt+\int_x^b(b-t)^\alpha|f'(t)|dt\right)\\\le&\ \frac{M}{\Gamma(\alpha+1)}\left(\int_a^x(t-a)^\alpha dt+\int_x^b(b-t)^\alpha dt\right)\\=&\ \frac{M\left((x-a)^{\alpha+1}+(b-x)^{\alpha+1}\right)}{\Gamma(\alpha+2)}.\end{aligned}\tag{6}$$

This completes the proof. □

Remark 1. *Letting $\alpha=1$, formula (6) reduces Ostrowski inequality:*

$$|f(x)-\mathcal{M}(f;a,b)|\le\frac{M}{b-a}\cdot\frac{(x-a)^2+(b-x)^2}{2}.\tag{7}$$

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping and $f'\in L^2[a,b]$. If f' is bounded on $[a,b]$ with $m\le f'(x)\le M$, then we have*

$$\left|\frac{\alpha f(x)+f(a)}{\Gamma(\alpha)(\alpha+1)}(x-a)^{\alpha-1}-\frac{\alpha}{x-a}J^{\alpha}_{x_-}f(a)+\frac{\alpha f(x)+f(b)}{\Gamma(\alpha)(\alpha+1)}(b-x)^{\alpha-1}-\frac{\alpha}{b-x}J^{\alpha}_{x_+}f(b)\right|$$
$$\le\sqrt{\frac{1}{2\alpha+1}-\frac{1}{(\alpha+1)^2}}\cdot\frac{(x-a)^{\alpha}K_1+(b-x)^{\alpha}K_2}{\Gamma(\alpha)}$$
$$\le\sqrt{\frac{1}{2\alpha+1}-\frac{1}{(\alpha+1)^2}}\cdot\frac{(x-a)^{\alpha}+(b-x)^{\alpha}}{2\Gamma(\alpha)}(M-m) \tag{8}$$

for all $x\in[a,b]$ and $\alpha\ge 1$. Where

$$K_1{}^2=\mathcal{M}(f'^2;a,x)-\mathcal{M}^2(f';a,x),$$

and

$$K_2{}^2=\mathcal{M}(f'^2;x,b)-\mathcal{M}^2(f';x,b).$$

Proof. From (3) and (4) we have

$$\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}f(x)-\frac{\alpha}{x-a}J^{\alpha}_{x_-}f(a)=\frac{1}{x-a}\int_a^x\frac{(t-a)^{\alpha}}{\Gamma(\alpha)}f'(t)dt. \tag{9}$$

and

$$\frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)}f(x)-\frac{\alpha}{b-x}J^{\alpha}_{x_+}f(b)=\frac{1}{b-x}\int_x^b-\frac{(b-t)^{\alpha}}{\Gamma(\alpha)}f'(t)dt. \tag{10}$$

Therefore we obtain

$$\frac{\alpha f(x)+f(a)}{\Gamma(\alpha)(\alpha+1)}(x-a)^{\alpha-1}-\frac{\alpha}{x-a}J^{\alpha}_{x_-}f(a)+\frac{\alpha f(x)+f(b)}{\Gamma(\alpha)(\alpha+1)}(b-x)^{\alpha-1}-\frac{\alpha}{b-x}J^{\alpha}_{x_+}f(b)$$
$$=\frac{1}{x-a}\int_a^x\frac{(t-a)^{\alpha}}{\Gamma(\alpha)}f'(t)dt-\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)(\alpha+1)}\int_a^x f'(t)dt$$
$$+\frac{1}{b-x}\int_x^b-\frac{(b-t)^{\alpha}}{\Gamma(\alpha)}f'(t)dt+\frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)(\alpha+1)}\int_x^b f'(t)dt. \tag{11}$$

Furthermore, by Korkine's identity, we have

$$\frac{\alpha f(x)+f(a)}{\Gamma(\alpha)(\alpha+1)}(x-a)^{\alpha-1}-\frac{\alpha}{x-a}J^{\alpha}_{x_-}f(a)+\frac{\alpha f(x)+f(b)}{\Gamma(\alpha)(\alpha+1)}(b-x)^{\alpha-1}-\frac{\alpha}{b-x}J^{\alpha}_{x_+}f(b)$$
$$=\frac{1}{2(x-a)^2\Gamma(\alpha)}\int_a^x\int_a^x\left((t-a)^{\alpha}-(s-a)^{\alpha}\right)\left(f'(t)-f'(s)\right)dsdt$$
$$+\frac{1}{2(b-x)^2\Gamma(\alpha)}\int_x^b\int_x^b\left((b-s)^{\alpha}-(b-t)^{\alpha}\right)\left(f'(t)-f'(s)\right)dsdt. \tag{12}$$

Using the Cauchy-Schwarz inequality for double integrals, we have

$$\left|\frac{1}{2(x-a)^2\Gamma(\alpha)}\int_a^x\int_a^x\left((t-a)^{\alpha}-(s-a)^{\alpha}\right)\left(f'(t)-f'(s)\right)dsdt\right|$$
$$\le\left(\frac{1}{2(x-a)^2\Gamma(\alpha)}\int_a^x\int_a^x\left((t-a)^{\alpha}-(s-a)^{\alpha}\right)^2dsdt\right)^{\frac{1}{2}}$$
$$\left(\frac{1}{2(x-a)^2\Gamma(\alpha)}\int_a^x\int_a^x\left(f'(t)-f'(s)\right)^2dsdt\right)^{\frac{1}{2}}. \tag{13}$$

However

$$\int_a^x \int_a^x \left((t-a)^\alpha - (s-a)^\alpha\right) dsdt = (\frac{2}{2\alpha+1} - \frac{2}{(\alpha+1)^2})(x-a)^{2\alpha+2}, \tag{14}$$

and

$$\int_a^x \int_a^x \left(f'(t) - f'(s)\right)^2 dsdt = 2(x-a)\int_a^x |f'|^2 dt - 2(f(x)-f(a))^2. \tag{15}$$

By (13)–(15), we obtain

$$\left|\frac{1}{2(x-a)^2\Gamma(\alpha)} \int_a^x \int_a^x \left((t-a)^\alpha - (s-a)^\alpha\right)\left(f'(t) - f'(s)\right) dsdt\right|$$
$$\leq \frac{(x-a)^\alpha}{\Gamma(x)} \sqrt{\frac{1}{2\alpha+1} - \frac{1}{(\alpha+1)^2}} \left(\mathcal{M}(f'^2; a, x) - \mathcal{M}^2(f'; a, x)\right)^{\frac{1}{2}}. \tag{16}$$

Similarly, we have

$$\left|\frac{1}{2(b-x)^2\Gamma(\alpha)} \int_x^b \int_x^b \left((b-s)^\alpha - (b-t)^\alpha\right)\left(f'(t) - f'(s)\right) dsdt\right|$$
$$\leq \frac{(b-x)^\alpha}{\Gamma(x)} \sqrt{\frac{1}{2\alpha+1} - \frac{1}{(\alpha+1)^2}} \left(\mathcal{M}(f'^2; x, b) - \mathcal{M}^2(f'; x, b)\right)^{\frac{1}{2}}. \tag{17}$$

Using (12),(16) and (17), we deduce the (8) inequality. Moreover, if $m \leq f'(t) \leq M$ on $[a, b]$, then by Grüss inequality, we have

$$0 \leq \frac{1}{x-a}\|f'\|^2_{L^2[a,x]} - \left(\mathcal{M}(f'; a, x)\right)^2 \leq \frac{1}{4}(M-m)^2, \tag{18}$$

$$0 \leq \frac{1}{b-x}\|f'\|^2_{L^2[x,b]} - \left(\mathcal{M}(f'; x, b)\right)^2 \leq \frac{1}{4}(M-m)^2, \tag{19}$$

which prove the last inequality of (8). □

Corollary 1. *Under the assumptions of Theorem 2 with $\alpha = 1$. Then the following inequality holds*

$$\left|f(x) + \frac{f(a)+f(b)}{2} - \frac{1}{x-a}\int_a^x f(t)dt - \frac{1}{b-x}\int_x^b f(t)dt\right| \leq \frac{b-a}{4\sqrt{3}}(M-m) \tag{20}$$

for all $x \in [a, b]$.

Remark 2. *If we take $x = \frac{a+b}{2}$ in (20), it follows that*

$$\left|\frac{1}{2}\left(f(\frac{a+b}{2}) + \frac{f(a)+f(b)}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{8\sqrt{3}}(M-m). \tag{21}$$

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$$\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2}\int_a^b f(x)dx \int_a^b g(x)dx,$$

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