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SOME INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR EXTENDED (s,m)-CONVEX FUNCTIONS

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ABSTRACT. In the paper, the authors establish some new integral inequalities of Hermite-Hadamard type for extended (s,m)-convex functions.

1. INTRODUCTION

Throughout this paper, we use the following notations

$$\mathbb{R}=(-\infty,\infty),\quad \mathbb{R}_0=[0,\infty)\quad\text{and}\quad \mathbb{R}_+=(0,\infty). \tag{1}$$

We first recite some definitions of various convex functions.

Definition 1. *A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to be convex if*

$$f(\lambda x+(1-\lambda)y)\le \lambda f(x)+(1-\lambda)f(y) \tag{2}$$

holds for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 2 ([7])**.** *Let $s\in(0,1]$. A function $f:\mathbb{R}_0\to\mathbb{R}_0$ is said to be s-convex in the second sense if*

$$f(\lambda x+(1-\lambda)y)\le \lambda^s f(x)+(1-\lambda)^s f(y) \tag{3}$$

holds for all $x,y\in I$ and $\lambda\in[0,1]$.

Definition 3 ([17])**.** *For $f:[0,b]\to\mathbb{R}$ and $m\in(0,1]$, if*

$$f(\lambda x+m(1-\lambda)y)\le \lambda f(x)+m(1-\lambda)f(y) \tag{4}$$

is valid for all $x,y\in[0,b]$ and $\lambda\in[0,1]$, then we say that $f(x)$ is a m-convex function on $[0,b]$.

Definition 4 ([12])**.** *For $f:[0,b]\to\mathbb{R}$ and $(\alpha,m)\in(0,1]^2$, if*

$$f(\lambda x+m(1-\lambda)y)\le \lambda^\alpha f(x)+m\big(1-\lambda^\alpha\big)f(y) \tag{5}$$

is valid for all $x,y\in[0,b]$ and $\lambda\in[0,1]$, then we say that $f(x)$ is a (α,m)-convex function on $[0,b]$.

Definition 5 ([21])**.** *For some $s\in[-1,1]$, a function $f:I\subseteq\mathbb{R}_0\to\mathbb{R}_0$ is said to be extended s-convex if*

$$f(\lambda x+(1-\lambda)y)\le \lambda^s f(x)+(1-\lambda)^s f(y) \tag{6}$$

holds for all $x,y\in I$ and $\lambda\in(0,1)$.

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Definition 6 ([13])**.** *For some $(s,m)\in(0,1]^2$, a function $f:[0,b]\to\mathbb{R}_0$ is said to be (s,m)-convex if*

$$f(\lambda x+m(1-\lambda)y)\le\lambda^s f(x)+m(1-\lambda)^s f(y) \tag{7}$$

holds for all $x,y\in I$ and $\lambda\in(0,1)$.

The following theorems are some inequalities of Hermite-Hadamard type for the above mentioned convex functions.

Theorem 1 ([6, Theorem 2.2])**.** *Let $f:I^\circ\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping and $a,b\in I^\circ$ with $a<b$. If $|f'(x)|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{(b-a)}{8}\bigl(|f'(a)|+|f'(b)|\bigr). \tag{8}$$

Theorem 2 ([10, Theorems 2.3 and 2.4])**.** *Let $f:I\subseteq\mathbb{R}_0\to\mathbb{R}$ be differentiable on I° and $a,b\in I$ with $a<b$. If $|f'(x)|^p$ is s-convex on $[a,b]$ for some $s\in(0,1]$ and $p>1$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\bigl(|f'(a)|+|f'(b)|\bigr), \tag{9}$$

and

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|&\le\frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}\Bigl\{\bigl[|f'(a)|^{p/(p-1)}\\&\quad+3|f'(b)|^{p/(p-1)}\bigr]^{1-1/p}+\bigl[3|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}\bigr]^{1-1/p}\Bigr\}.\end{aligned} \tag{10}$$

Theorem 3 ([11, Theorem 3])**.** *Let $f:I\subseteq\mathbb{R}_0\to\mathbb{R}$ be differentiable on I°, $a,b\in I$ with $a<b$, and $f'\in L[a,b]$. If $|f'(x)|^q$ is s-convex on $[a,b]$ for some $s\in(0,1]$ and $q>1$, then*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|&\le\frac{b-a}{2}\left[\frac{q-1}{2(2q-1)}\right]^{1/p}\left(\frac{1}{s+1}\right)^{1/q}\\&\quad\times\left\{\left[|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\}.\end{aligned} \tag{11}$$

Theorem 4 ([4, Theorem 2.2])**.** *Let $I\supseteq\mathbb{R}_0$ be an open interval and let $f:I\to\mathbb{R}$ be a differentiable function such that $f'\in L[a,b]$ for $0\le a<b<\infty$. If $|f'(x)|^q$ is m-convex on $[a,b]$ for some $m\in(0,1]$ and $q\ge1$, then*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|&\le\frac{b-a}{4}\\&\quad\times\min\left\{\left(\frac{|f'(a)|^q+m|f'(b/m)|^q}{2}\right)^{1/q},\left(\frac{m|f'(a/m)|^q+|f'(b)|^q}{2}\right)^{1/q}\right\}.\end{aligned} \tag{12}$$

Theorem 5 ([15, Theorem 4])**.** *Let $I\subseteq\mathbb{R}$ be an open interval, $a,b\in I$ with $a<b$, $f:I\to\mathbb{R}$ be a twice differentiable mapping such that $f''(x)$ is integrable. If $|f''(x)|$ is a convex function on $[a,b]$ and $0\le\lambda\le1$, then*

$$\left|(\lambda-1)f\left(\frac{a+b}{2}\right)-\lambda\frac{f(a)+f(b)}{2}+\int_a^b f(x)\,\mathrm{d}x\right|$$

$$\leq \begin{cases} \dfrac{(b-a)^2}{24}\left\{\left[\lambda^4+(1+\lambda)(1-\lambda)^3+\dfrac{5\lambda-3}{4}\right]|f''(a)| \right. \\ \qquad \left. +\left[\lambda^4+(2-\lambda)\lambda^3+\dfrac{1-3\lambda}{4}\right]|f''(b)|\right\}, \quad 0\leq\lambda\leq\dfrac{1}{2}; \\ \dfrac{(b-a)^2}{48}(3\lambda-1)\big(|f''(a)|+|f''(b)|\big), \quad \dfrac{1}{2}\leq\lambda\leq 1. \end{cases} \tag{13}$$

Theorem 6 ([15, Theorem 5])**.** *Let $I\subseteq\mathbb{R}$ be an open interval, $a,b\in I$ with $a<b$, $f:I\to\mathbb{R}$ be a twice differentiable mapping such that $f''(x)$ is integrable, and $0\leq\lambda\leq 1$. If $|f''(x)|^q$ is a convex function on $[a,b]$ for $q\geq 1$, then*

$$\left|(\lambda-1)f\left(\frac{a+b}{2}\right)-\lambda\frac{f(a)+f(b)}{2}+\int_a^b f(x)\,\mathrm{d}x\right|\leq\frac{(b-a)^2}{2}$$

$$\times\begin{cases} \left(\dfrac{\lambda^3}{3}+\dfrac{1-3\lambda}{24}\right)^{1-1/q}\left\{\left[\left(\dfrac{\lambda^4}{6}+\dfrac{3-8\lambda}{3\times 2^6}\right)|f''(a)|^q\right.\right. \\ \qquad \left.+\left(\dfrac{(2-\lambda)\lambda^3}{6}+\dfrac{5-16\lambda}{3\times 2^6}\right)|f''(b)|^q\right]^{1/q} \\ \qquad +\left[\left(\dfrac{(1+\lambda^4)(1-\lambda)^3}{6}+\dfrac{48\lambda-27}{3\times 2^6}\right)|f''(a)|^q\right. \\ \qquad \left.\left.+\left(\dfrac{\lambda^4}{6}+\dfrac{3-8\lambda}{3\times 2^6}\right)|f''(b)|^q\right]^{1/q}\right\}, \quad 0\leq\lambda\leq\dfrac{1}{2}; \\ \left(\dfrac{3\lambda-1}{24}\right)^{1-1/q}\left\{\left[\left(\dfrac{8\lambda-3}{3\times 2^6}\right)|f''(a)|^q+\left(\dfrac{16\lambda-5}{3\times 2^6}\right)|f''(b)|^q\right]^{1/q}\right. \\ \qquad \left.+\left[\left(\dfrac{16\lambda-5}{3\times 2^6}\right)|f''(a)|^q+\left(\dfrac{8\lambda-3}{3\times 2^6}\right)|f''(b)|^q\right]^{1/q}\right\}, \quad \dfrac{1}{2}\leq\lambda\leq 1. \end{cases}$$

Theorem 7 ([13, Theorem 2.1])**.** *Let $I\supseteq\mathbb{R}_0$ be an open interval and let $f:I\to\mathbb{R}$ be a differentiable function such that $f'\in L[a,b]\cap L\big[\frac{a}{m},\frac{b}{m}\big]$ for $0\leq a<b<\infty$. If $|f'(x)|$ is (s,m)-convex on $[a,b]$ for some numbers $(s,m)\in(0,1]^2$, then*

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\leq\frac{b-a}{4}\min\left\{\frac{|f'(a)|+m|f'(b/m)|}{s+1},\frac{m|f'(a/m)|+|f'(b)|}{s+1}\right\}.$$

For more information on this topic, please refer to [1, 2, 3, 5, 8, 9, 14, 16, 18, 19, 20, 22, 23, 24, 25] and closely related references therein.

In this paper, we establish some new integral inequalities of Hermite-Hadamard type for extended (s,m)-convex functions.

2. A new definition and a lemma

In this section, we define the concept "extended (s,m)-convex function".

Definition 7. *For some $(s,m)\in[-1,1]\times(0,1]$, a function $f:[0,b]\to\mathbb{R}_0$ is said to be extended (s,m)-convex if*

$$f(\lambda x+m(1-\lambda)y)\leq\lambda^s f(x)+m(1-\lambda)^s f(y) \tag{14}$$

holds for all $x,y\in I$ and $\lambda\in(0,1)$.

For establishing our new integral inequalities of Hermite-Hadamard type for extended (s,m)-convex functions, we need the following integral identity.

Lemma 1. *Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function and $a,b\in I$ with $a<b$. If $f''\in L[a,b]$ and $\lambda\in\mathbb{R}$, then*

$$
\begin{aligned}
&(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\\
&=\frac{(b-a)^2}{16}\int_0^1(1-t)(t+1-2\lambda)\left[f''\left(ta+(1-t)\frac{a+b}{2}\right)+f''\left(tb+(1-t)\frac{a+b}{2}\right)\right]\mathrm{d}t.
\end{aligned}
$$

Proof. Integrating by part and changing variables of definite integral yield

$$
\begin{aligned}
&\frac{1}{16}\int_0^1(1-t)(t+1-2\lambda)\left[f''\left(ta+(1-t)\frac{a+b}{2}\right)+f''\left(tb+(1-t)\frac{a+b}{2}\right)\right]\mathrm{d}t\\
&=\frac{1}{8(b-a)}\left\{(1-t)(t+1-2\lambda)\left[f'\left(tb+(1-t)\frac{a+b}{2}\right)-f'\left(ta+(1-t)\frac{a+b}{2}\right)\right]\Big|_0^1\right.\\
&\quad\left.+\int_0^1(-2t+2\lambda)\left[f'\left(ta+(1-t)\frac{a+b}{2}\right)-f'\left(tb+(1-t)\frac{a+b}{2}\right)\right]\mathrm{d}t\right\}\\
&=\frac{1}{4(b-a)}\int_0^1(-t+\lambda)\left[f'\left(ta+(1-t)\frac{a+b}{2}\right)-f'\left(tb+(1-t)\frac{a+b}{2}\right)\right]\mathrm{d}t\\
&=\frac{1}{2(b-a)^2}\left\{(-t+\lambda)\left[-f\left(ta+(1-t)\frac{a+b}{2}\right)-f\left(tb+(1-t)\frac{a+b}{2}\right)\right]\Big|_0^1\right.\\
&\quad\left.-\int_0^1\left[f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)\right]\mathrm{d}t\right\}\\
&=\frac{1}{(b-a)^2}\left[(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int_a^b f(x)\,\mathrm{d}x\right].
\end{aligned}
$$

Lemma 1 is proved. ∎

3. Some new integral inequalities of Hermite-Hadamard type

We are now in a position to establish some new integral inequalities of Hermite-Hadamard type for differentiable and extended (s,m)-convex functions.

Theorem 8. *Let $0\le\lambda\le1$ and $f:\mathbb{R}_0\to\mathbb{R}_0$ be a twice differentiable function, $0\le a<b<\infty$, $f''\in L\left[a,\frac{b}{m}\right]$, such that $|f''(x)|^q$ for $q\ge1$ is extended (s,m)-convex on $\left[0,\frac{b}{m}\right]$ for some $(s,m)\in(-1,1]\times(0,1]$. Then*

$$
\begin{aligned}
&\left|(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le\frac{(b-a)^2}{16}\left[\frac{1}{(1+s)(2+s)(3+s)}\right]^{1/q}\big[S_1(\lambda)\big]^{1-1/q}\\
&\quad\times\left\{\left[S_2(\lambda)|f''(a)|^q+mS_3(\lambda)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right.\\
&\quad\left.+\left[S_2(\lambda)|f''(b)|^q+mS_3(\lambda)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right\},
\end{aligned}
$$

where

$$
S_1(\lambda)=\begin{cases}\dfrac{2-3\lambda}{3}, & 0\le\lambda\le\dfrac12,\\[2mm] \dfrac{6-21\lambda+24\lambda^2-8\lambda^3}{3}, & \dfrac12\le\lambda\le1,\end{cases}
$$

$$S_2(\lambda)=\begin{cases}4+2s-2\lambda(3+s), & 0\le\lambda\le\dfrac{1}{2},\\ 2[3+s-\lambda(4+s)]+4[2+s-\lambda(1+s)](2\lambda-1)^{2+s}, & \dfrac{1}{2}\le\lambda\le 1,\end{cases}$$

$$S_3(\lambda)=\begin{cases}(1+s)[4+s-2\lambda(3+s)], & 0\le\lambda\le\dfrac{1}{2},\\ (1+s)\big[2\lambda(3+s)-3-s+2(2-2\lambda)^{3+s}\big], & \dfrac{1}{2}\le\lambda\le 1.\end{cases}$$

Proof. By Lemma 1 and Hölder's inequality, we have

$$\begin{aligned}&\left|(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le\frac{(b-a)^2}{16}\left\{\int_0^1(1-t)|t+1-2\lambda|\left[\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|\right.\right.\\
&\quad\left.\left.+\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|\right]\mathrm{d}t\right\}\\
&\le\frac{(b-a)^2}{16}\left[\int_0^1(1-t)|t+1-2\lambda|\,\mathrm{d}t\right]^{1-1/q}\\
&\quad\times\left\{\left[\int_0^1(1-t)|t+1-2\lambda|\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right]^{1/q}\right.\\
&\quad\left.+\left[\int_0^1(1-t)|t+1-2\lambda|\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right]^{1/q}\right\}.\end{aligned}$$

Since $|f''(x)|^q$ is extended (s,m)-convex on $\left[0,\frac{b}{m}\right]$, we have

$$\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q\le t^s|f''(a)|^q+m(1-t)^s\left|f''\left(\frac{a+b}{2m}\right)\right|^q$$

and

$$\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q\le t^s|f''(b)|^q+m(1-t)^s\left|f''\left(\frac{a+b}{2m}\right)\right|^q.$$

When $0\le\lambda\le\frac{1}{2}$, we have

$$\int_0^1(1-t)|t+1-2\lambda|\,\mathrm{d}t=\frac{2}{3}-\lambda,$$
$$\int_0^1(1-t)|t+1-2\lambda|t^s\,\mathrm{d}t=\frac{4+2s-2\lambda(3+s)}{(1+s)(2+s)(3+s)},$$
$$\int_0^1(1-t)|t+1-2\lambda|(1-t)^s\,\mathrm{d}t=\frac{4+s-2\lambda(3+s)}{(2+s)(3+s)}.$$

When $\frac{1}{2}\le\lambda\le 1$, we have

$$\int_0^1(1-t)|t+1-2\lambda|\,\mathrm{d}t=2-7\lambda+8\lambda^2-\frac{8}{3}\lambda^3,$$
$$\int_0^1(1-t)|t+1-2\lambda|t^s\,\mathrm{d}t=\frac{2[3+s-\lambda(4+s)]+4[2+s-\lambda(1+s)](2\lambda-1)^{2+s}}{(1+s)(2+s)(3+s)},$$
$$\int_0^1(1-t)|t+1-2\lambda|(1-t)^s\,\mathrm{d}t=\frac{2\lambda(3+s)-3-s+2(2-2\lambda)^{3+s}}{(2+s)(3+s)}.$$

Combining the above inequalities and equalities leads to the required inequality in Theorem 8. The proof is completed. □

Corollary 1. *Under conditions of Theorem 8,*

(1) when $q=1$, we have

$$\left|(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{(b-a)^2}{16}\times\frac{1}{(1+s)(2+s)(3+s)}\left\{S_2(\lambda)\left[|f''(a)|+|f''(b)|\right]+2mS_3(\lambda)\left|f''\left(\frac{a+b}{2m}\right)\right|\right\};$$

(2) when $q=1$ and $m=1$, we have

$$\begin{aligned}&\left|(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\&\le\frac{(b-a)^2}{16(1+s)(2+s)(3+s)}\left\{S_2(\lambda)\left[|f''(a)|+|f''(b)|\right]+2S_3(\lambda)\left|f''\left(\frac{a+b}{2}\right)\right|\right\}\\&\le\frac{(b-a)^2}{16(1+s)(2+s)(3+s)}\left[S_2(\lambda)+2^{1-s}S_3(\lambda)\right]\left[|f''(a)|+|f''(b)|\right];\end{aligned}$$

where $S_2(\lambda)$ and $S_3(\lambda)$ are defined as in Theorem 8.

Corollary 2. *Under conditions of Theorem 8, when $m=1$, we have*

$$\begin{aligned}&\left|(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\&\le\frac{(b-a)^2}{16}\left[\frac{1}{(1+s)(2+s)(3+s)}\right]^{1/q}\left[S_1(\lambda)\right]^{1-1/q}\\&\quad\times\left\{\left[S_2(\lambda)|f''(a)|^q+S_3(\lambda)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.\\&\quad\left.+\left[S_3(\lambda)\left|f''\left(\frac{a+b}{2}\right)\right|^q+S_2(\lambda)|f'(b)|^q\right]^{1/q}\right\}\\&\le\frac{(b-a)^2}{16}\left[\frac{1}{2^s(1+s)(2+s)(3+s)}\right]^{1/q}\left[S_1(\lambda)\right]^{1-1/q}\\&\quad\times\left\{\left[(2^sS_2(\lambda)+S_3(\lambda))|f''(a)|^q+S_3(\lambda)|f''(b)|^q\right]^{1/q}\right.\\&\quad\left.+\left[S_3(\lambda)|f''(a)|^q+(2^sS_2(\lambda)+S_3(\lambda))|f''(b)|^q\right]^{1/q}\right\},\end{aligned}$$

where $S_1(\lambda)$, $S_2(\lambda)$, and $S_3(\lambda)$ are defined as in Theorem 8.

Theorem 9. *Let $0\le\lambda\le1$ and $f:\mathbb{R}_0\to\mathbb{R}_0$ be a twice differentiable function, $0\le a<b<\infty$, $f''\in L\left[a,\frac{b}{m}\right]$, such that $|f''(x)|^q$ for $q>1$ is extended (s,m)-convex on $\left[0,\frac{b}{m}\right]$ for some $(s,m)\in(-1,1]\times(0,1]$. Then*

$$\begin{aligned}&\left|(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\&\le\frac{(b-a)^2}{16}\left[\frac{q-1}{(3q-2)(2q-1)}S_4(\lambda)\right]^{1-1/q}\left[\frac{1}{(1+s)(2+s)}\right]^{1/q}\end{aligned}$$

$$\times\left\{\left[|f''(a)|^q+m(1+s)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}+\left[|f''(b)|^q+m(1+s)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right\},$$

where

$$S_4(\lambda)=\begin{cases}(q-1)(2-2\lambda)^{(3q-2)/(q-1)}+[3-4q\\ \quad+2\lambda(q-1)](1-2\lambda)^{(2q-1)/(q-1)}, & 0\le\lambda\le\frac{1}{2};\\ (q-1)(2-2\lambda)^{(3q-2)/(q-1)}+[4q-3\\ \quad-2\lambda(q-1)](2\lambda-1)^{(2q-1)/(q-1)}, & \frac{1}{2}\le\lambda\le 1.\end{cases}\tag{15}$$

Proof. Using Lemma 1, the extended (s,m)-convexity of $|f''(x)|^q$ on $\left[0,\frac{b}{m}\right]$, and Hölder's integral inequality, we have

$$\begin{aligned}&\left|(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ &\le\frac{(b-a)^2}{16}\left\{\int_0^1(1-t)|t+1-2\lambda|\left[\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|\right.\right.\\ &\quad\left.\left.+\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|\right]\mathrm{d}t\right\}\\ &\le\frac{(b-a)^2}{16}\left(\int_0^1(1-t)|t+1-2\lambda|^{q/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\\ &\quad\times\left\{\left[\int_0^1(1-t)\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right]^{1/q}\right.\\ &\quad\left.+\left[\int_0^1(1-t)f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right]^{1/q}\right\}\\ &\le\frac{(b-a)^2}{16}\left(\int_0^1(1-t)|t+1-2\lambda|^{q/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\\ &\quad\times\left\{\left[\int_0^1(1-t)\left(t^s|f''(a)|^q+m(1-t)^s\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right.\\ &\quad\left.+\left[\int_0^1(1-t)\left(t^s|f''(b)|^q+m(1-t)^s\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right\}.\end{aligned}$$

Furthermore, a straightforward computation gives

$$\int_0^1(1-t)t^s\,\mathrm{d}t=\frac{1}{(1+s)(2+s)}\quad\text{and}\quad\int_0^1(1-t)^{s+1}\,\mathrm{d}t=\frac{1}{2+s}.\tag{16}$$

If $0\le\lambda\le\frac{1}{2}$, we have

$$\begin{aligned}\int_0^1(1-t)|t+1-2\lambda|^{q/(q-1)}\,\mathrm{d}t&=\frac{q-1}{(3q-2)(2q-1)}\\ &\quad\times\left\{(q-1)(2-2\lambda)^{(3q-2)/(q-1)}+[3-4q+2\lambda(q-1)](1-2\lambda)^{(2q-1)/(q-1)}\right\}.\end{aligned}$$

If $\frac{1}{2}\le\lambda\le 1$, we have

$$\int_0^1 (1-t)|t+1-2\lambda|^{q/(q-1)}\,\mathrm{d}t = \frac{q-1}{(3q-2)(2q-1)} \times \big\{(q-1)(2-2\lambda)^{(3q-2)/(q-1)} + [4q-3-2\lambda(q-1)](2\lambda-1)^{(2q-1)/(q-1)}\big\}.$$

Substituting the above equalities into the above inequality results in the required inequality. The proof of Theorem 9 is complete. □

Corollary 3. *Under conditions of Theorem 9, when $m=1$, we have*

$$\begin{aligned}
&\left|(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le \frac{(b-a)^2}{16}\left[\frac{q-1}{(3q-2)(2q-1)}S_4(\lambda)\right]^{1-1/q}\left[\frac{1}{(1+s)(2+s)}\right]^{1/q} \\
&\quad\times\left\{\left[|f''(a)|^q+(1+s)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right. \\
&\quad\left.+\left[|f''(b)|^q+(1+s)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\} \\
&\le \frac{(b-a)^2}{16}\left[\frac{q-1}{(3q-2)(2q-1)}S_4(\lambda)\right]^{1-1/q}\left[\frac{1}{2^s(1+s)(2+s)}\right]^{1/q} \\
&\quad\times\big\{\big[(1+s+2^s)|f''(a)|^q+(1+s)|f''(b)|^q\big]^{1/q} \\
&\quad+\big[(1+s)|f''(a)|^q+(1+s+2^s)|f''(b)|^q\big]^{1/q}\big\},
\end{aligned}$$

where $S_4(\lambda)$is defined as in Theorem 9.

Theorem 10. *If $f:\mathbb{R}_0\to\mathbb{R}_0$ is a twice differentiable function, $0\le a<b<\infty$, $f''\in L\big[a,\frac{b}{m}\big]$, and $0\le\lambda\le1$, such that $|f''(x)|^q$ for $q>1$ is extended (s,m)-convex on $\big[0,\frac{b}{m}\big]$ for some $(s,m)\in(-1,1]\times(0,1]$, then*

$$\begin{aligned}
&\left|(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le \frac{(b-a)^2}{16}\left[\frac{q-1}{(3q-2)(2q-1)}S_5(\lambda)\right]^{1-1/q}\left[\frac{1}{(1+s)(2+s)}\right]^{1/q} \\
&\quad\times\left\{\left[S_6(\lambda)|f''(a)|^q+mS_7(\lambda)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right. \\
&\quad\left.+\left[S_6(\lambda)|f''(b)|^q+mS_7(\lambda)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right\},
\end{aligned}$$

where

$$S_5(\lambda)=\begin{cases}4q-3+2\lambda(2-3q), & 0\le\lambda\le\dfrac12,\\ 3-4q+2\lambda(3q-2)+2(q-1)(2-2\lambda)^{(3q-2)/(q-1)}, & \dfrac12\le\lambda\le1,\end{cases} \tag{17}$$

$$S_6(\lambda)=\begin{cases}3+2s-2\lambda(2+s), & 0\le\lambda\le\dfrac12,\\ 3+2s-2\lambda(2+s)+2(2\lambda-1)^{2+s}, & \dfrac12\le\lambda\le1,\end{cases} \tag{18}$$

$$S_7(\lambda)=\begin{cases}3+s-2\lambda(2+s), & 0\le\lambda\le\dfrac{1}{2},\\ 2\lambda(2+s)-3-s+2(2-2\lambda)^{2+s}, & \dfrac{1}{2}\le\lambda\le 1.\end{cases}\tag{19}$$

Proof. By Lemma 1, the extended (s,m)-convexity of $|f''(x)|^q$ on $\left[0,\frac{b}{m}\right]$, and Hölder's integral inequality, we have

$$\begin{aligned}&\left|(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le\frac{(b-a)^2}{16}\left\{\int_0^1(1-t)|t+1-2\lambda|\left[\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|\right.\right.\\
&\quad\left.\left.+\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|\right]\mathrm{d}t\right\}\\
&\le\frac{(b-a)^2}{16}\left\{\left[\int_0^1(1-t)^{q/(q-1)}|t+1-2\lambda|\,\mathrm{d}t\right]^{1-1/q}\right.\\
&\quad\times\left[\int_0^1|t+1-2\lambda|\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right]^{1/q}\\
&\quad\left.+\left[\int_0^1|t+1-2\lambda|\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right]^{1/q}\right\}\\
&\le\frac{(b-a)^2}{16}\left\{\left[\int_0^1(1-t)^{q/(q-1)}|t+1-2\lambda|\,\mathrm{d}t\right]^{1-1/q}\right.\\
&\quad\times\left\{\left[\int_0^1|t+1-2\lambda|\left(t^s|f''(a)|^q+m(1-t)^s\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right.\\
&\quad\left.\left.+\left[\int_0^1|t+1-2\lambda|\left(t^s|f''(b)|^q+m(1-t)^s\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right\}.\end{aligned}$$

If $0\le\lambda\le\frac{1}{2}$, we have

$$\int_0^1(1-t)^{q/(q-1)}|t+1-2\lambda|\,\mathrm{d}t=\frac{q-1}{(3q-2)(2q-1)}[4q-3+2\lambda(2-3q)],$$

$$\int_0^1|t+1-2\lambda|t^s\,\mathrm{d}t=\frac{3+2s-2\lambda(2+s)}{(1+s)(2+s)},$$

$$\int_0^1|t+1-2\lambda|(1-t)^s\,\mathrm{d}t=\frac{3+s-2\lambda(2+s)}{(1+s)(2+s)}.$$

If $\frac{1}{2}\le\lambda\le 1$, we have

$$\begin{aligned}\int_0^1(1-t)^{q/(q-1)}|t+1-2\lambda|\,\mathrm{d}t&=\frac{q-1}{(3q-2)(2q-1)}\big[3-4q+2\lambda(3q-2)\\
&\quad+2(q-1)(2-2\lambda)^{(3q-2)/(q-1)}\big],\end{aligned}$$

$$\int_0^1|t+1-2\lambda|t^s\,\mathrm{d}t=\frac{3+2s-2\lambda(2+s)+2(2\lambda-1)^{2+s}}{(1+s)(2+s)},$$

$$\int_0^1|t+1-2\lambda|(1-t)^s\,\mathrm{d}t=\frac{2\lambda(2+s)-3-s+2(2-2\lambda)^{2+s}}{(1+s)(2+s)}.$$

Substituting these inequalities into the above inequality leads to the required inequality. Theorem 10 is proved. □

Corollary 4. *Under conditions of Theorem 10,*

(1) when $q=1$, we have

$$\left|(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le \frac{(b-a)^2}{16}\times\left[\frac{1}{(1+s)(2+s)}\right]\left\{S_6(\lambda)\big[|f''(a)|+|f''(b)|\big]+2mS_7(\lambda)\left|f''\left(\frac{a+b}{2m}\right)\right|\right\}. \tag{20}$$

(2) when $q=1$ and $m=1$, we have

$$\begin{aligned}&\left|(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ &\le \frac{(b-a)^2}{16}\left[\frac{1}{(1+s)(2+s)}\right]\left\{S_6(\lambda)\big[|f''(a)|+|f''(b)|\big]+2S_7(\lambda)\left|f''\left(\frac{a+b}{2}\right)\right|\right\}\\ &\le \frac{(b-a)^2}{16}\left[\frac{1}{(1+s)(2+s)}\right]\big[S_6(\lambda)+2^{1-s}S_7(\lambda)\big]\big[|f''(a)|+|f''(b)|\big],\end{aligned}$$

where $S_2(\lambda)$ and $S_3(\lambda)$ is defined as in Theorem 8.

Corollary 5. *Under conditions of Theorem 8, when $m=1$, we have*

$$\begin{aligned}&\left|(1-\lambda)\frac{f(a)+f(b)}{2}+\lambda f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ &\le \frac{(b-a)^2}{16}\left[\frac{q-1}{(3q-2)(2q-1)}S_5(\lambda)\right]^{1-1/q}\left[\frac{1}{(1+s)(2+s)}\right]^{1/q}\\ &\quad\times\left\{\left[S_6(\lambda)|f''(a)|^q+S_7(\lambda)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.\\ &\quad\left.+\left[S_6(\lambda)|f''(b)|^q+S_7(\lambda)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\}\\ &\le \frac{(b-a)^2}{16}\left[\frac{q-1}{(3q-2)(2q-1)}S_5(\lambda)\right]^{1-1/q}\left[\frac{1}{(1+s)(2+s)}\right]^{1/q}\\ &\quad\times\left\{\big[(2^sS_6(\lambda)+S_7(\lambda))|f''(a)|^q+S_7(\lambda)|f''(b)|^q\big]^{1/q}\right.\\ &\quad\left.+\left[S_7(\lambda)|f''(a)|^q+(2^sS_2(\lambda)+S_7(\lambda))|f''(b)|^q\right]^{1/q}\right\},\end{aligned}$$

where $S_5(\lambda)$, $S_6(\lambda)$, and $S_7(\lambda)$ are defined as in Theorem 10.

Theorem 11. *Let $f:\mathbb{R}_0\to\mathbb{R}_0$ be a twice differentiable function, $0\le a<b<\infty$, $f''\in L\big[a,\frac{b}{m}\big]$, and $0\le\lambda\le1$. If $|f''(x)|^q$ is extended (s,m)-convex on $\big[0,\frac{b}{m}\big]$, $(s,m)\in(-1,1]\times(0,1]$, $q>1$, and $q\ge p\ge0$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\ &\le\frac{(b-a)^2}{16}\left[\frac{(q-1)(4q-p-3)}{(3q-p-2)(2q-p-1)}\right]^{1-1/q}\left[\frac{1}{(1+s+p)(2+s+p)}\right]^{1/q}\\ &\times\left\{\left[(1+s+p)(3+2s+p)B(1+s,1+p)|f''(a)|^q+m(3+s+p)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right.\end{aligned}$$

$$+\left[(1+s+p)(3+2s+p)B(1+s,1+p)|f''(b)|^q+m(3+s+p)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\Bigg\},$$

where $B(\alpha,\beta)$ denotes the well known Beta function which may be defined by

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t,\quad \alpha,\beta>0. \tag{21}$$

Proof. By Lemma 1, the extended (s,m)-convexity of $|f''(x)|^q$ on $\left[0,\frac{b}{m}\right]$, and Hölder's integral inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le\frac{(b-a)^2}{16}\Bigg\{\int_0^1(1-t)(1+t)\bigg[\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|\\
&\quad+\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|\bigg]\,\mathrm{d}t\Bigg\}\\
&\le\frac{(b-a)^2}{16}\Bigg\{\left[\int_0^1(1+t)(1-t)^{(q-p)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\\
&\quad\times\left[\int_0^1(1+t)(1-t)^p\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right]^{1/q}\\
&\quad+\left[\int_0^1(1+t)(1-t)^p\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}t\right]^{1/q}\Bigg\}\\
&\le\frac{(b-a)^2}{16}\left[\int_0^1(1+t)(1-t)^{(q-p)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\\
&\quad\times\Bigg\{\left[\int_0^1(1+t)(1-t)^p\left(t^s|f''(a)|^q+m(1-t)^s\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\\
&\quad+\left[\int_0^1(1+t)(1-t)^p\left(t^s|f''(b)|^q+m(1-t)^s\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\Bigg\}\\
&=\frac{(b-a)^2}{16}\left[\frac{(q-1)(4q-p-3)}{(3q-p-2)(2q-p-1)}\right]^{1-1/q}\left[\frac{1}{(1+s+p)(2+s+p)}\right]^{1/q}\\
&\quad\times\Bigg\{\left[(1+s+p)(3+2s+p)B(1+s,1+p)|f'(a)|^q+m(3+s+p)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\\
&\quad+\left[(1+s+p)(3+2s+p)B(1+s,1+p)|f''(b)|^q+m(3+s+p)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\Bigg\}.
\end{aligned}$$

Theorem 11 is thus proved. □

Corollary 6. *Under conditions of Theorem 11, when $m=1$, we have*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}+-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le\frac{(b-a)^2}{16}\left[\frac{(q-1)(4q-p-3)}{(3q-p-2)(2q-p-1)}\right]^{1-1/q}\left[\frac{1}{(1+s+p)(2+s+p)}\right]^{1/q}\\
&\times\Bigg\{\left[(1+s+p)(3+2s+p)B(1+s,1+p)|f''(a)|^q+(3+s+p)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}
\end{aligned}$$

$$+\left[(1+s+p)(3+2s+p)B(1+s,1+p)|f''(b)|^q+(3+s+p)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\Bigg\}.$$

Theorem 12. *Let $f:\mathbb{R}_0\to\mathbb{R}_0$ be a twice differentiable function, $0\le a<b<\infty$, $f''\in L\left[a,\frac{b}{m}\right]$. If $|f''(x)|^q$ is extended (s,m)-convex on $\left[0,\frac{b}{m}\right]$, $(s,m)\in(-1,1]\times(0,1]$, $q>1$, $q\ge p$, and $\ell\ge 0$, then*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\&\le\frac{(b-a)^2}{16}\left[B\left(\frac{2q-p-1}{q-1},\frac{2q-\ell-1}{q-1}\right)\right]^{1-1/q}\\&\quad\times\left\{\left[B(1+s+\ell,1+p)|f''(a)|^q+mB(1+s+p,1+\ell)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right.\\&\quad+\left.\left[B(1+s+\ell,1+p)|f''(b)|^q+mB(1+s+p,1+\ell)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right\}.\end{aligned}$$

Proof. By Lemma 1, the extended (s,m)-convexity of $|f''(x)|^q$ on $\left[0,\frac{b}{m}\right]$, and Hölder's integral inequality, we have

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\&\le\frac{(b-a)^2}{16}\left\{\int_0^1 t(1-t)\left[\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|+\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|\right]\mathrm{d}t\right\}\\&\le\frac{(b-a)^2}{16}\left\{\left[\int_0^1 t^{(q-\ell)/(q-1)}(1-t)^{(q-p)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\right.\\&\quad\times\left\{\left[\int_0^1 t^{\ell}(1-t)^p\left(t^s|f''(a)|^q+m(1-t)^s\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right.\\&\quad+\left.\left[\int_0^1 t^{\ell}(1-t)^p\left(t^s|f''(b)|^q+m(1-t)^s\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right\}\\&=\frac{(b-a)^2}{16}\left[B\left(\frac{2q-p-1}{q-1},\frac{2q-\ell-1}{q-1}\right)\right]^{1-1/q}\\&\quad\times\left\{\left[B(1+s+\ell,1+p)|f''(a)|^q+mB(1+s+p,1+\ell)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right.\\&\quad+\left.\left[B(1+s+\ell,1+p)|f''(b)|^q+mB(1+s+p,1+\ell)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right\}.\end{aligned}$$

Theorem 12 is thus proved. □

Corollary 7. *Under conditions of Theorem 12, when $m=1$, we have*

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\&\le\frac{(b-a)^2}{16}\left[B\left(\frac{2q-p-1}{q-1},\frac{2q-\ell-1}{q-1}\right)\right]^{1-1/q}\\&\quad\times\left[\left[B(1+s+\ell,1+p)|f''(a)|^q+B(1+s+p,1+\ell)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.\end{aligned}$$

$$+\left[B(1+s+\ell,1+p)|f''(b)|^q+B(1+s+p,1+\ell)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\Bigg\}.$$

Theorem 13. *Let* $0\le\lambda\le1$ *and* $f:\mathbb{R}_0\to\mathbb{R}_0$ *be a twice differentiable function,* $0\le a<b<\infty$, *such that* $f''\in L\big[a,\frac{b}{m}\big]$. *If* $|f''(x)|^q$ *is extended* (s,m)*-convex on* $\big[0,\frac{b}{m}\big]$, $(s,m)\in(-1,1]\times(0,1]$, $q>1$, *and* $2q\ge p\ge0$, *then*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|&\le\frac{(b-a)^2}{16}\left(\frac{q-1}{3q-p-1}\right)^{1-1/q}\left(\frac{1}{1+s+p}\right)^{1/q}\\&\times\left\{\left[(1+s+p)B(1+s,1+p)|f''(a)|^q+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right.\\&\left.+\left[(1+s+p)B(1+s,1+p)|f''(b)|^q+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right\}.\end{aligned}\tag{22}$$

Proof. By Lemma 1, the extended (s,m)-convexity of $|f''(x)|^q$ on $\big[0,\frac{b}{m}\big]$, and Hölder's integral inequality, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{(b-a)^2}{16}\\&\times\left\{\int_0^1(1-t)^2\left[\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|+\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|\right]\mathrm{d}t\right\}\\&\le\frac{(b-a)^2}{16}\left\{\left[\int_0^1(1-t)^{(2q-p)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\right.\\&\times\left\{\left[\int_0^1(1-t)^p\left(t^s|f''(a)|^q+m(1-t)^s\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right.\\&\left.+\left[\int_0^1(1-t)^p\left(t^s|f''(b)|^q+m(1-t)^s\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right\}\\&=\frac{(b-a)^2}{16}\left(\frac{q-1}{3q-p-1}\right)^{1-1/q}\left(\frac{1}{1+s+p}\right)^{1/q}\\&\times\left\{\left[(1+s+p)B(1+s,1+p)|f''(a)|^q+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right.\\&\left.+\left[(1+s+p)B(1+s,1+p)|f''(b)|^q+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right\}.\end{aligned}$$

Theorem 13 is thus proved. □

Corollary 8. *Under conditions of Theorem 13, when* $m=1$, *we have*

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{(b-a)^2}{16}\left(\frac{q-1}{3q-p-1}\right)^{1-1/q}\left(\frac{1}{s+p+1}\right)^{1/q}\\&\times\left\{\left[(1+s+p)B(1+s,1+p)|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.\\&\left.+\left[(1+s+p)B(1+s,1+p)|f''(b)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\}\\&\le\frac{(b-a)^2}{16}\left(\frac{q-1}{3q-p-1}\right)^{1-1/q}\left[\frac{1}{2^s(1+s+p)}\right]^{1/q}\end{aligned}$$

$$\times\left\{\left[\left(1+2^s(1+s+p)B(1+s,1+p)\right)|f''(a)|^q+|f''(b)|^q\right]^{1/q}\right.$$
$$\left.+\left[|f''(a)|^q+\left(1+2^s(1+s+p)B(1+s,1+p)\right)|f''(b)|^q\right]^{1/q}\right\}.$$

Theorem 14. *Let $f:\mathbb{R}_0\to\mathbb{R}_0$ be a twice differentiable function, $0\le a<b<\infty$, $f''\in L\left[a,\frac{b}{m}\right]$. If $|f''(x)|^q$ is extended $(-1,m)$-convex on $\left[0,\frac{b}{m}\right]$, $m\in(0,1]$, $q\ge1$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{(b-a)^2}{2^{4+1/q}}\left\{\left[|f''(a)|^q+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}+\left[|f''(b)|^q+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right\}.$$

Proof. By Lemma 1, the extended $(-1,m)$-convexity of $|f''(x)|^q$ on $\left[0,\frac{b}{m}\right]$, and Hölder's integral inequality, we have

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{(b-a)^2}{16}\left\{\int_0^1 t(1-t)\left[\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|+\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|\right]\mathrm{d}t\right\}$$
$$\le\frac{(b-a)^2}{16}\left[\int_0^1 1\,\mathrm{d}t\right]^{1-1/q}$$
$$\times\left\{\left[\int_0^1 t(1-t)\left(t^{-1}|f''(a)|^q+m(1-t)^{-1}\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right.$$
$$\left.+\left[\int_0^1 t(1-t)\left(t^{-1}|f''(b)|^q+m(1-t)^{-1}\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right\}$$
$$=\frac{(b-a)^2}{2^{4+1/q}}\left\{\left[|f''(a)|^q+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}+\left[|f''(b)|^q+m\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right\}.$$

Theorem 14 is thus proved. □

Theorem 15. *Let $f:\mathbb{R}_0\to\mathbb{R}_0$ be a twice differentiable function, $0\le a<b<\infty$, $f''\in L\left[a,\frac{b}{m}\right]$. If $|f''(x)|^q$ is extended $(-1,m)$-convex on $\left[0,\frac{b}{m}\right]$, $m\in(0,1]$, $q>1$, $q\ge p$, and $\ell>0$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{(b-a)^2}{16}\left[B\left(\frac{2q-p-1}{q-1},\frac{2q-\ell-1}{q-1}\right)\right]^{1-1/q}$$
$$\times\left\{\left[B(\ell,1+p)|f''(a)|^q+mB(p,1+\ell)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[B(\ell,1+p)|f''(b)|^q+mB(p,1+\ell)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right\}.$$

Proof. By Lemma 1, the extended $(-1,m)$-convexity of $|f''(x)|^q$ on $\left[0,\frac{b}{m}\right]$, and Hölder's integral inequality, we have

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$

$$\leq \frac{(b-a)^2}{16}\left\{\int_0^1 t(1-t)\left[\left|f''\left(ta+(1-t)\frac{a+b}{2}\right)\right|+\left|f''\left(tb+(1-t)\frac{a+b}{2}\right)\right|\right]\mathrm{d}t\right\}$$

$$\leq \frac{(b-a)^2}{16}\left\{\left[\int_0^1 t^{(q-\ell)/(q-1)}(1-t)^{(q-p)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\right.$$

$$\times\left\{\left[\int_0^1 t^{\ell}(1-t)^p\left(t^{-1}|f''(a)|^q+m(1-t)^{-1}\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right.$$

$$\left.+\left[\int_0^1 t^{\ell}(1-t)^p\left(t^{-1}|f''(b)|^q+m(1-t)^{-1}\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}t\right]^{1/q}\right\}$$

$$=\frac{(b-a)^2}{16}\left[B\left(\frac{2q-p-1}{q-1},\frac{2q-\ell-1}{q-1}\right)\right]^{1-1/q}$$

$$\times\left\{\left[B(\ell,1+p)|f''(a)|^q+mB(p,1+\ell)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right.$$

$$\left.+\left[B(\ell,1+p)|f''(b)|^q+mB(p,1+\ell)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right\}.$$

Theorem 15 is thus proved. □

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